Introduction to Elliptic Cryptosystems

An invitation to Elliptic curves

Journée d'Aritmétique

Universitè de Cocody – UFR Matématique et Informatique

Abidjan *Juillet 24, 2014*,

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Università Roma Tre

Proto–History (from WIKIPEDIA)

Giulio Carlo, Count Fagnano, and Marquis de Toschi (December 6, 1682 – September 26, 1766) was an Italian mathematician. He was probably the first to direct attention to the theory of *elliptic integrals*. Fagnano was born in Senigallia.

He made his higher studies at the *Collegio Clementino* in Rome and there won great distinction, except in mathematics, to which his aversion was extreme. Only after his college course he took up the study of mathematics.

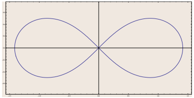
Later, without help from any teacher, he mastered mathematics from its foundations.

Some of His Achievements:

- $\pi = 2i \log \frac{1-i}{1+1}$
- Length of *Lemniscate*

Carlo Fagnano

Collegio Clementino



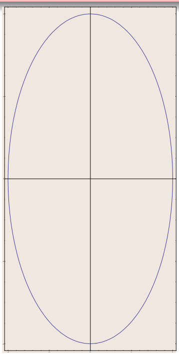
Lemniscate

$(x^2 + y^2)^2 = 2a^2(x^2 - y^2)$

$\ell = 4 \int_0^a \frac{a^2 dr}{\sqrt{a^4 - r^4}} = \frac{a\sqrt{\pi}\Gamma(\frac{5}{4})}{\Gamma(\frac{3}{4})}$

Length of Ellipses

$$\mathcal{E} : \frac{x^2}{4} + \frac{y^2}{16} = 1$$



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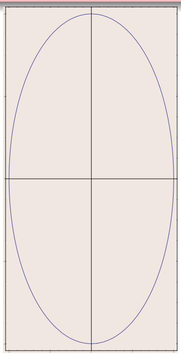

The length of the arc of a plane curve $y = f(x)$, $f : [a, b] \to \mathbb{R}$ is:

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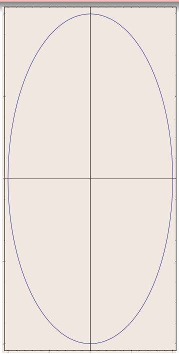

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Applying this formula to $\mathcal{E}$:

$$\ell(\mathcal{E}) = 4 \int_0^4 \sqrt{1 + \left(\frac{d\sqrt{16(1 - t^2/4)}}{dt} \right)^2}\, dt$$

$$= 4 \int_0^1 \sqrt{\frac{1 + 3x^2}{1 - x^2}}\, dx \qquad x = t/2$$

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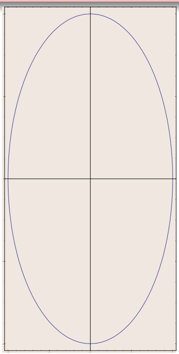

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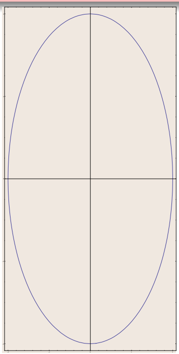

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Apply the invertible change of variables:

$$\begin{cases} x = 1 - 2/t \\ y = \frac{u}{t-1} \end{cases}$$

Arrive to

$$u^2 = t^3 - 4t^2 + 6t - 3$$

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Reasons to study them

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6. represent a mathematical world in itself ... Each of them does!!

The (general) Weierstraß Equation

An elliptic curve E over a $\mathbb{F}_q$ (finite field) is given by an equation

$$E : y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$$

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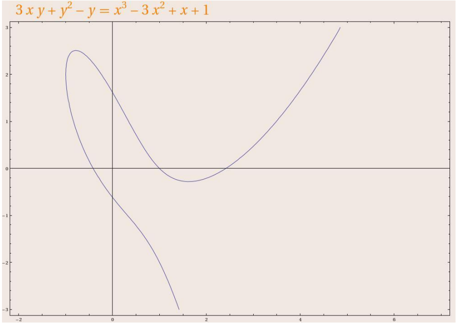

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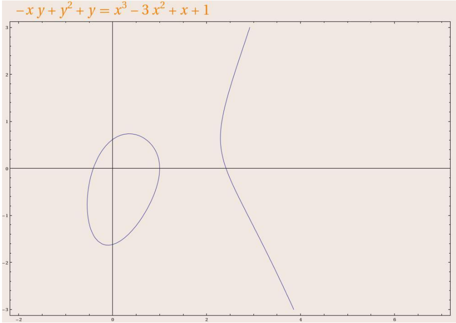

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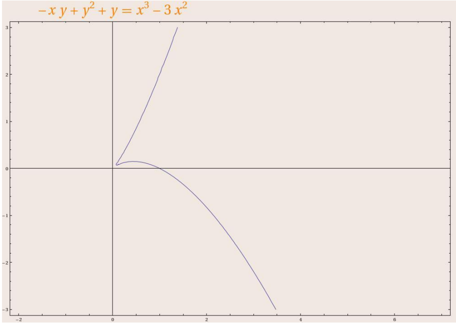

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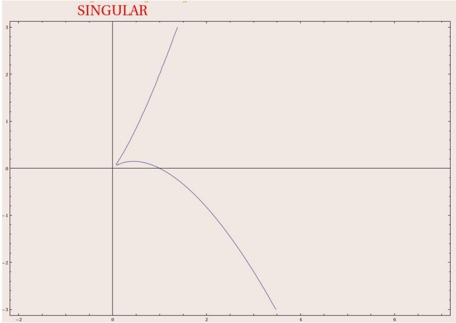

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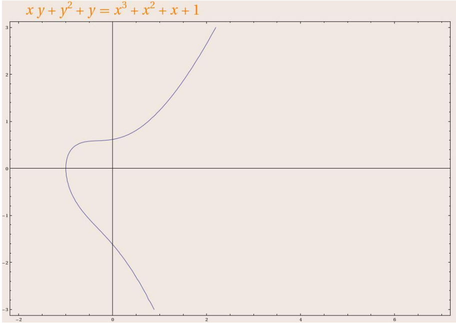

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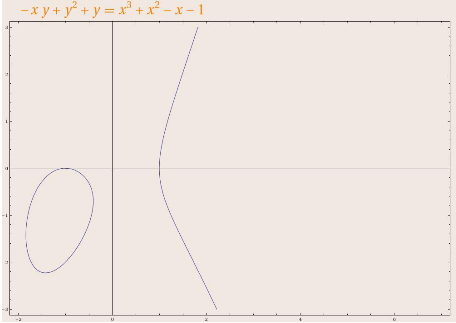

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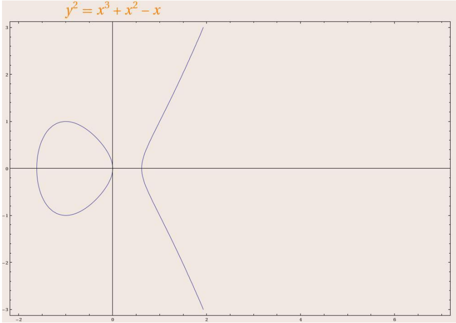

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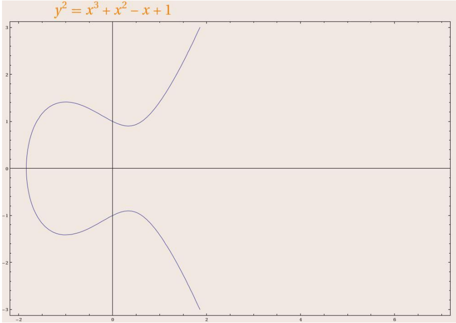

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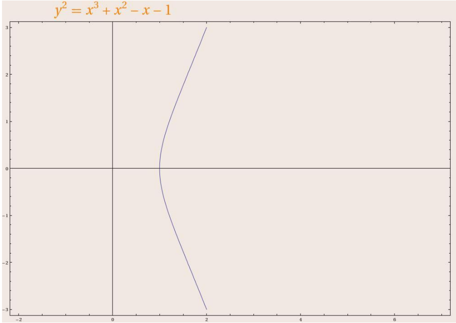

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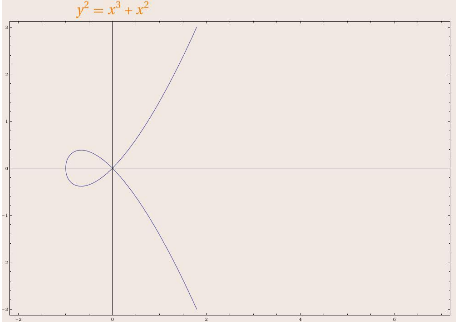

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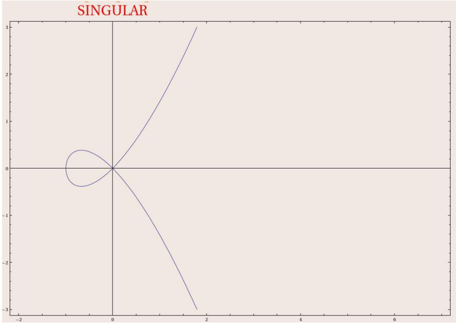

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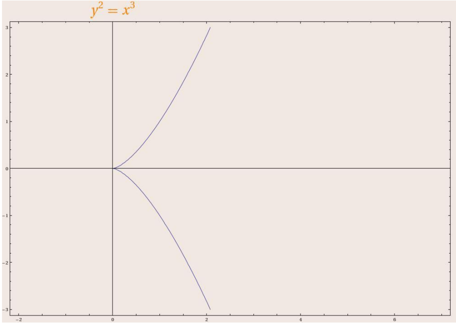

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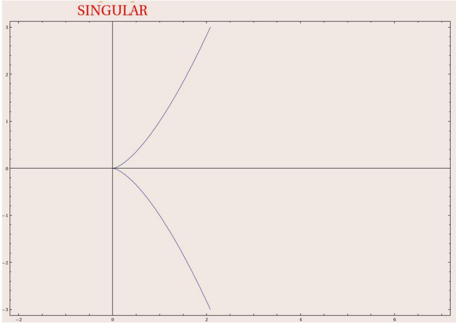

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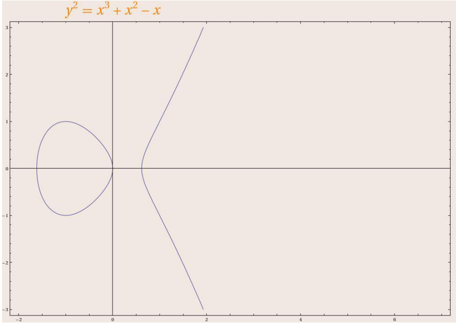

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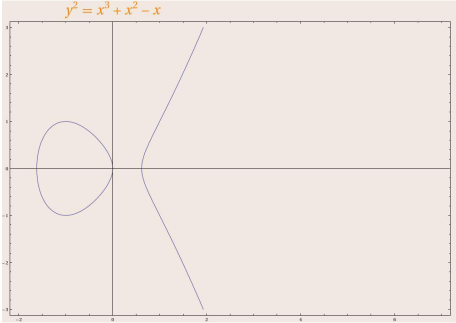

The equation should not be *singular*

The Discriminant of an Equation

The condition of absence of singular points in terms of a_1, a_2, a_3, a_4, a_6

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Definition (The discriminant of a Weierstraßequation is the following quantity)

$$\Delta'_E := \frac{1}{2^4 3^3} \left(-a_1^5 a_3 a_4 - 8a_1^3 a_2 a_3 a_4 - 16 a_1 a_2^2 a_3 a_4 + 36 a_1^2 a_3^2 a_4 \right.$$
$$- a_1^4 a_4^2 - 8 a_1^2 a_2 a_4^2 - 16 a_2^2 a_4^2 + 96 a_1 a_3 a_4^2 + 64 a_4^3 +$$
$$a_1^6 a_6 + 12 a_1^4 a_2 a_6 + 48 a_1^2 a_2^2 a_6 + 64 a_2^3 a_6 - 36 a_1^3 a_3 a_6$$
$$\left. -144 a_1 a_2 a_3 a_6 - 72 a_1^2 a_4 a_6 - 288 a_2 a_4 a_6 + 432 a_6^2 \right)$$

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Definition

Two Weierstraß equations over $\mathbb{F}_q$ are said (affinely) equivalent if there exists a (affine) of the following form

$$\begin{cases} x \longleftarrow u^2 x + r \\ y \longleftarrow u^3 y + u^2 s x + t \end{cases} \quad r, s, t, u \in \mathbb{F}_q$$

The Weierstraß equation

Classification of simplified forms

After applying a suitable affine transformation we can always assume that $E/\mathbb{F}_q (q = p^n)$ has a Weierstraß equation of the following form

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Example (Classification)

E	p	Δ_E
$y^2 = x^3 + Ax + B$	≥ 5	$4A^3 + 27B^2$
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$y^2 + a_3y = x^3 + a_4x + a_6$	2	a_3^4
$y^2 = x^3 + Ax^2 + Bx + C$	3	$4A^3C - A^2B^2 - 18ABC$ $+4B^3 + 27C^2$

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Note: If $p \geq 3, \Delta_E \neq 0 \Leftrightarrow x^3 + Ax^2 + Bx + C$ has no double root

Elliptic curves over $\mathbb{F}_2$

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Fact:

There are 5 affinely inequivalent elliptic curves over $\mathbb{F}_2$

Elliptic curves in characteristic 3

Via a suitable transformation ($x \to u^2x + r, y \to u^3y + u^2sx + t$) over $\mathbb{F}_3$, 8 inequivalent elliptic curves over $\mathbb{F}_3$ are found:

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Weierstraß equations over $\mathbb{F}_3$

1. $y^2 = x^3 + x$
2. $y^2 = x^3 - x$
3. $y^2 = x^3 - x + 1$
4. $y^2 = x^3 - x - 1$
5. $y^2 = x^3 + x^2 + 1$
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Elliptic curves in characteristic 3

Via a suitable transformation ($x \to u^2x + r, y \to u^3y + u^2sx + t$) over $\mathbb{F}_3$, 8 inequivalent elliptic curves over $\mathbb{F}_3$ are found:

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Observations

1. Over $\mathbb{F}_5$ there are 12 elliptic curves
2. Over $\mathbb{F}_p$ there are approximately $2p$

The definition of $E(\mathbb{F}_q)$

Let $E/\mathbb{F}_q$ elliptic curve, $\infty := [0,1,0]$. Set

$E(\mathbb{F}_q) = \{[X,Y,Z] \in \mathbb{P}_2(\mathbb{F}_q) : \ Y^2Z + a_1XYZ + a_3YZ^2 = X^3 + a_2X^2Z + a_4XZ^2 + a_6Z^3\}$

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- $r_{\infty,\infty} \cap E(\mathbb{F}_q) = \{\infty, \infty, \infty\}$
- $r_{P,Q} : aX + bZ = 0$ (vertical) $\Rightarrow \infty = [0,1,0] \in r_{P,Q}$

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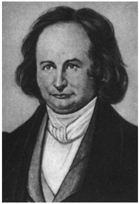

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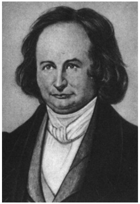

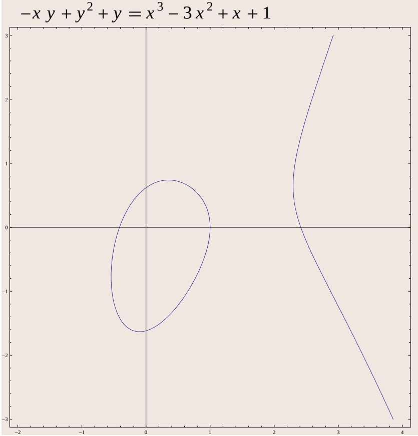

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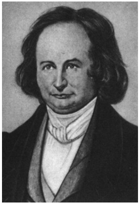

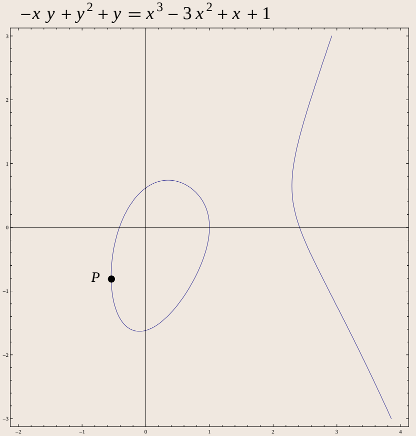

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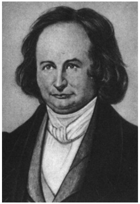

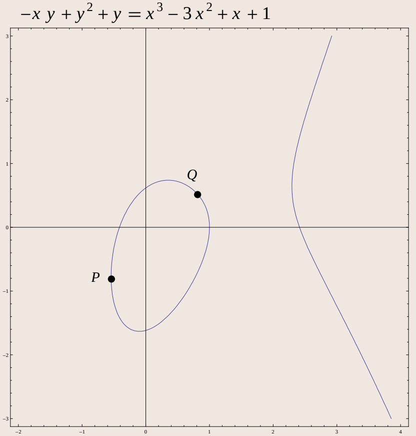

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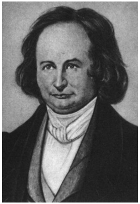

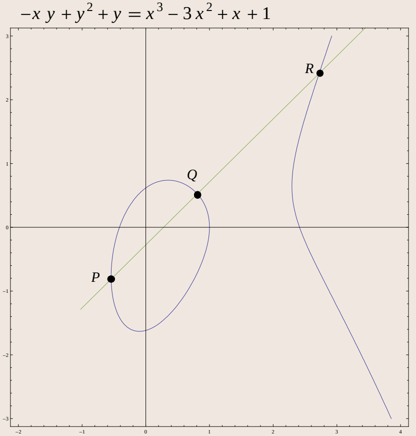

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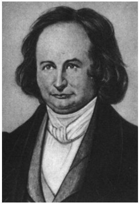

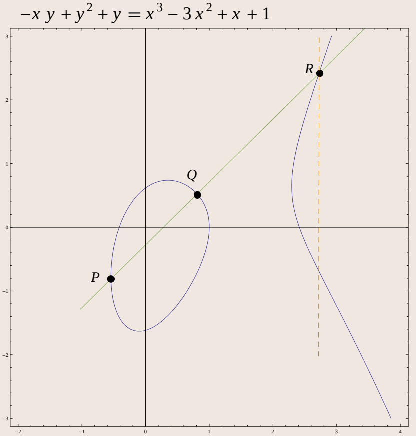

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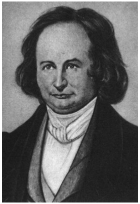

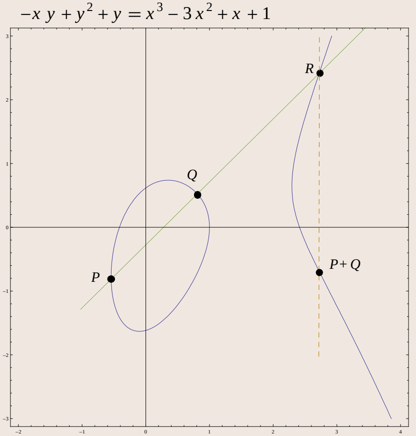

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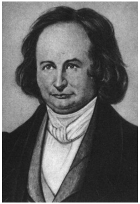

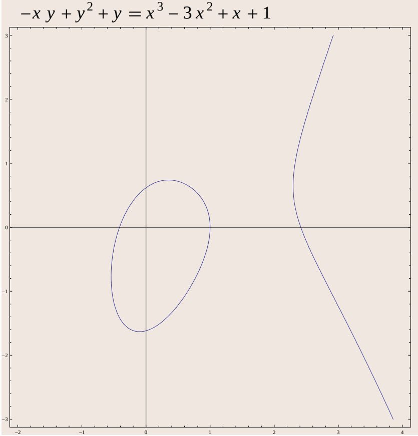

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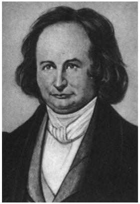

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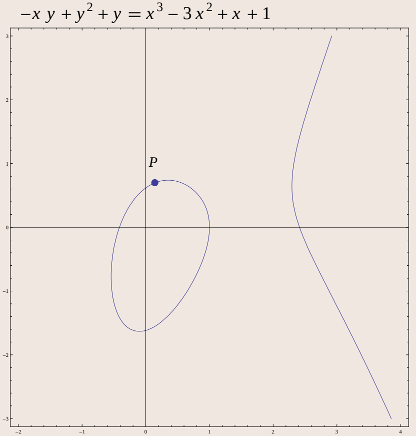

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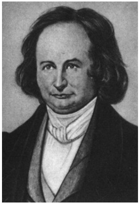

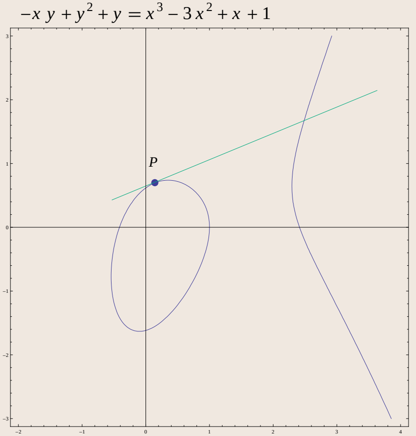

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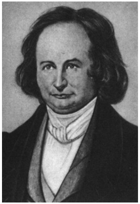

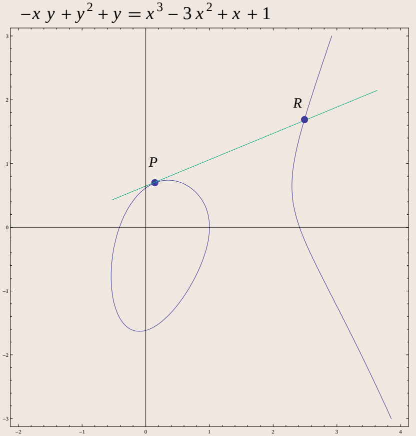

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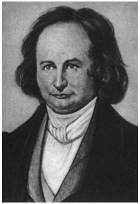

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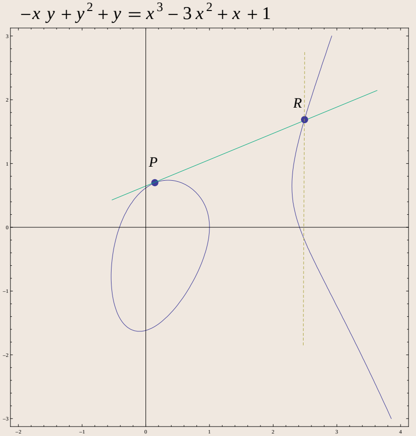

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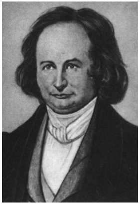

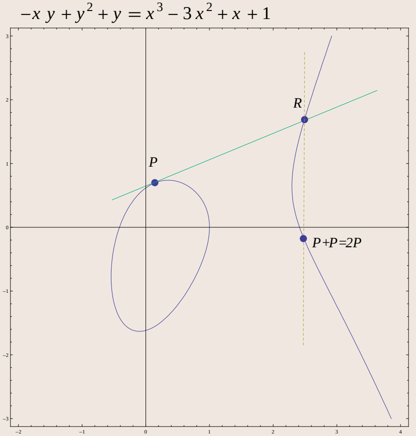

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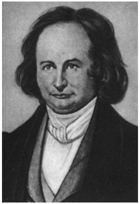

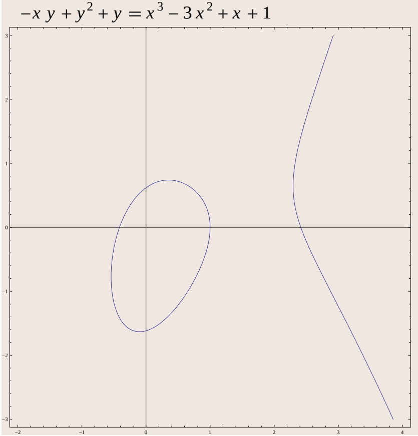

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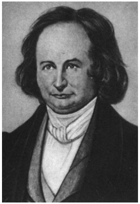

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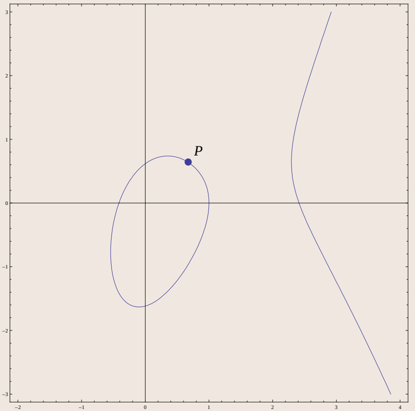

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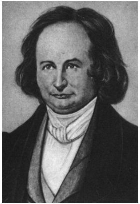

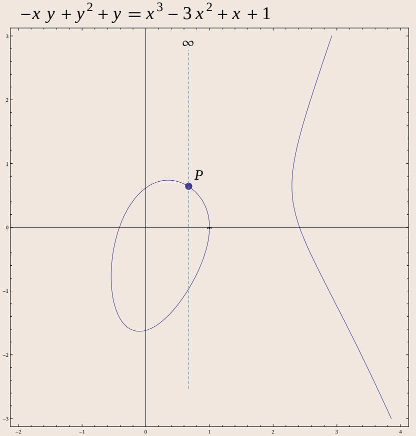

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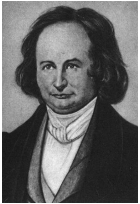

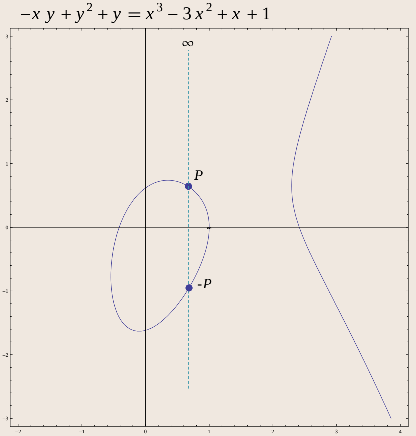

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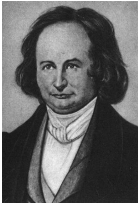

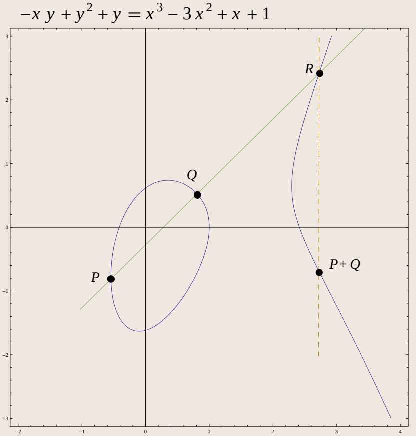

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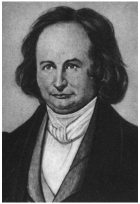

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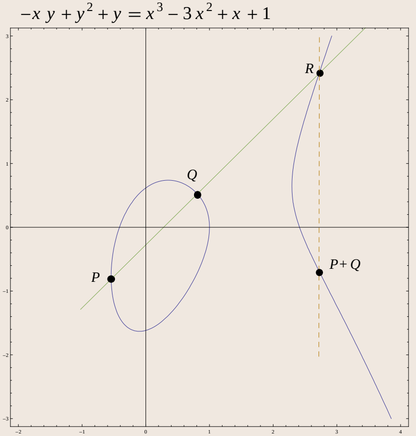

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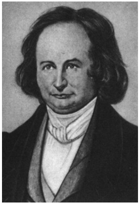

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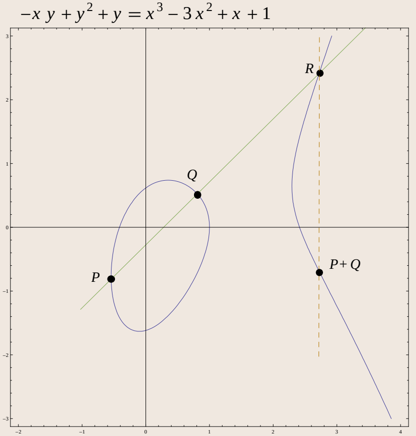

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$$P_1 +_E P_2 = (\lambda^2 - a_1\lambda - a_2 - x_1 - x_2, -\lambda^3 - a_1^2\lambda + (\lambda + a_1)(a_2 + x_1 + x_2) - a_3 - \nu)$$

Formulas for Addition on E (Summary for special equation)

$$E : y^2 = x^3 + Ax + B$$

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- If $P_1 = P_2$
  - $y_1 = 0$ $\Rightarrow$ $P_1 +_E P_2 = 2P_1 = \infty$
  - $y_1 \neq 0$

$$\lambda = \frac{3x_1^2 + A}{2y_1}, \nu = -\frac{x_1^3 - Ax_1 - 2B}{2y_1}$$

Then

$$P_1 +_E P_2 = (\lambda^2 - x_1 - x_2, -\lambda^3 + \lambda(x_1 + x_2) - \nu)$$

EXAMPLE: Elliptic curves over $\mathbb{F}_2$

From our previous list:

Groups of points

E	$E(\mathbb{F}_2)$	$\lvert E(\mathbb{F}_2)\rvert$
$y^2+xy=x^3+x^2+1$	$\{\infty,(0,1)\}$	2
$y^2+xy=x^3+1$	$\{\infty,(0,1),(1,0),(1,1)\}$	4
$y^2+y=x^3+x$	$\{\infty,(0,0),(0,1),(1,0),(1,1)\}$	5
$y^2+y=x^3+x+1$	$\{\infty\}$	1
$y^2+y=x^3$	$\{\infty,(0,0),(0,1)\}$	3

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Definition

2–torsion points

$$E[2] = \{P \in E : 2P = \infty\}.$$

In conclusion

$$E[2] \cong \begin{cases} C_2 \oplus C_2 & \text{if } p > 2 \\ C_2 & \text{if } p = 2, E : y^2 + xy = x^3 + a_4 x + a_6 \\ \{\infty\} & \text{if } p = 2, E : y^2 + a_3 y = x^3 + a_2 x^2 + a_6 \end{cases}$$

Elliptic curves over $\mathbb{F}_2, \mathbb{F}_3$ and $\mathbb{F}_5$

Each curve $/\mathbb{F}_2$ has cyclic $E(\mathbb{F}_2)$.

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  $E_5(\mathbb{F}_5) \cong C_2 \oplus C_4$ and $E_6(\mathbb{F}_5) \cong C_8$

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Note

- $\psi_3(x) := 3x^4 + 6Ax^2 + 12Bx - A^2$ the 3rd *division* polynomial

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$$P \text{ has order } 3 \iff 3P = \infty \iff 2P = -P$$

So, if $p > 3$ and $E : y^2 = x^2 + Ax + B$

$$2P = (x_{2P}, y_{2P}) = 2(x_1, y_1) = (\lambda^2 - 2x_1, -\lambda^3 + 2\lambda x_1 - \nu)$$

where $\lambda = \frac{3x_1^2 + A}{2y_1}, \nu = -\frac{x_1^3 - Ax_1 - 2B}{2y_1}$.

$$P \text{ has order } 3 \iff x_{2P} = x_1$$

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- $\psi_3(x) := 3x^4 + 6Ax^2 + 12Bx - A^2$ the 3rd *division* polynomial
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Hence

$$E[3] = \left\{\infty, (2, \pm 1), (1, \pm\sqrt{3}), (1 \pm 2\sqrt{3}, \pm(1 \pm \sqrt{3}))\right\}$$

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Determining points of order 3 (continues)

Inequivalent curves $/\mathbb{F}_7$ with $\#E(\mathbb{F}_7) = 9$.

E	$\psi_3(x)$	$E[3] \cap E(\mathbb{F}_7)$	$E(\mathbb{F}_7) \cong$
$y^2 = x^3 + 2$	$x(x+1)(x+2)(x+4)$	$\left\{\begin{array}{l}\infty, (0, \pm 3), (-1, \pm 1), \\ (5, \pm 1), (3, \pm 1)\end{array}\right\}$	$C_3 \oplus C_3$
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Can one count the number of inequivalent $E/\mathbb{F}_q$ with $\#E(\mathbb{F}_q) = r$?

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Determining points of order (dividing) m

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Let E/K and let $\bar{K}$ an *algebraic closure of* K.

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$$E/\mathbb{F}_p \text{ is called } \begin{cases} \textit{ordinary} & \text{if } E[p] \cong C_p \\ \textit{supersingular} & \text{if } E[p] = \{\infty\} \end{cases}$$

Group Structure of $E(\mathbb{F}_q)$

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Let $E/\mathbb{F}_q$ *and* $n, k \in \mathbb{N}$ *s.t.* $E(\mathbb{F}_q) \cong C_n \oplus C_{nk}$. *Then* $n \mid q - 1$.

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So $\#E(\mathbb{F}_q) \in [(\sqrt{q}-1)^2, (\sqrt{q}+1)^2]$ the *Hasse interval* $\mathcal{I}_q$

Example (Hasse Intervals)

q	$\mathcal{I}_q$
2	$\{1,2,3,4,5\}$
3	$\{1,2,3,4,5,6,7\}$
4	$\{1,2,3,4,5,6,7,8,9\}$
5	$\{2,3,4,5,6,7,8,9,10\}$
7	$\{3,4,5,6,7,8,9,10,11,12,13\}$
8	$\{4,5,6,7,8,9,10,11,12,13,14\}$
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Suppose N is a possible order of an elliptic curve $/\mathbb{F}_q$, $q = p^n$. Write

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- If $q = p^{2n}$ and $\#E(\mathbb{F}_q) = q + 1 \pm 2\sqrt{q} = (p^n \pm 1)^2$, then $E(\mathbb{F}_q) \cong C_{p^n \pm 1} \oplus C_{p^n \pm 1}$.

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$$E[m] \cong C_m \oplus C_m$$

We set

$$\mu_m := \{x \in \bar{K} : x^m = 1\}$$

μ_m is a cyclic group with m elements(since $p \nmid m$)

Theorem (Existence of Weil Pairing)

There exists a pairing $e_m : E[m] \times E[m] \to \mu_m$ *called Weil Pairing, s.t.* $\forall P, Q \in E[m]$

1. $e_m(P +_E Q, R) = e_m(P, R)e_m(Q, R)$ *(bilinearity)*
2. $e_m(P, R) = 1 \forall R \in E[m] \Rightarrow P = \infty$ *(non degeneracy)*
3. $e_m(P, P) = 1$
4. $e_m(P, Q) = e_m(Q, P)^{-1}$
5. $e_m(\sigma P, \sigma Q) = \sigma e_m(P, Q) \ \forall \sigma \in \mathrm{Gal}(\bar{K}/K)$
6. $e_m(\alpha(P), \alpha(Q)) = e_m(P, Q)^{\deg \alpha} \ \forall \alpha$ *separable endomorphism*

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The last one needs to be discussed further!!!

Further Reading...

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