Workshop in Number Theory
Vietnam
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September 5, 2015

Riemann Hypothesis, History and Ideas

A **conjecture** *(from latin coniectūra, verb conīcere, or also "interpret, infer, conclude")* is a statement or a judgment based on intuition, considered probably true, but not proven.

An **hypothesis** *(From the ancient greek hypothesis, it is composed with of hypo, "under" and thesis, "position", or assumption)* is the premise underlying a reasoning or a proof.

Some conjectures regarding prime numbers: 1/5

Twin primes Conjecture.

There exists infinitely many primes p such that $p+2$ is prime

for example:

3 and 5,

11 and 13,

17 and 19,

101 and 103,

$\vdots$

$\vdots$

$10^{100} + 35737$ and $10^{100} + 35739$,

$\vdots$

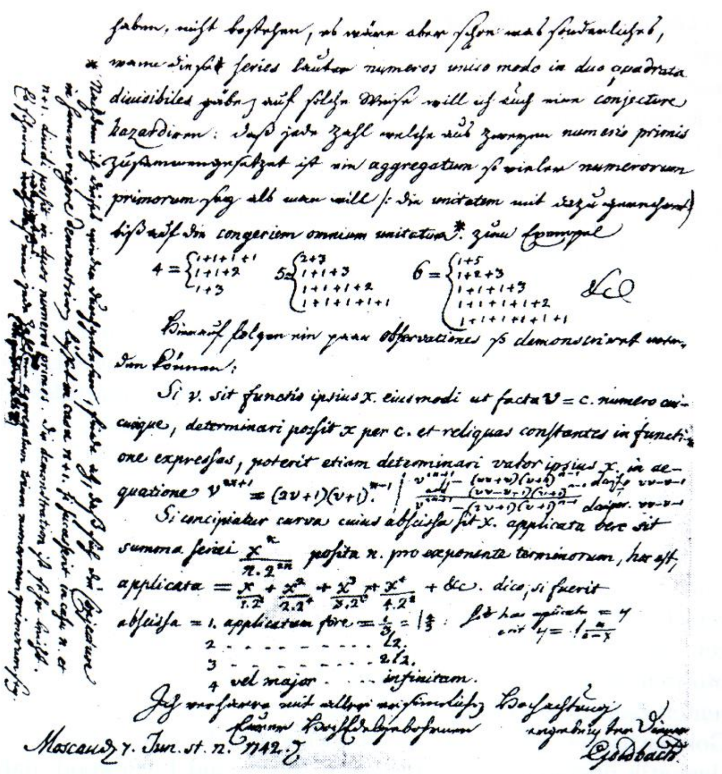
$\vdots$ $3756801695685 \cdot 2^{666669} \pm 1$,

$\vdots$.

Some conjectures regarding prime numbers: 2/5

Goldbach conjecture

Every even number (except for 2) can be written as the sum of two primes



for example:

$42 = 5 + 37,$

$1000 = 71 + 929,$

$888888 = 601 + 888287,$

$\vdots$

Some conjectures regarding prime numbers: 3/5

Hardy-Littlewood Conjecture.

$\exists$ infinitely many primes p s.t. $p-1$ is a prefect square

for example:

$5 = 2^2 + 1,$

$17 = 4^2 + 1,$

$37 = 6^2 + 1,$

$101 = 10^2 + 1,$

$\vdots$

$677 = 26^2 + 1,$

$\vdots$

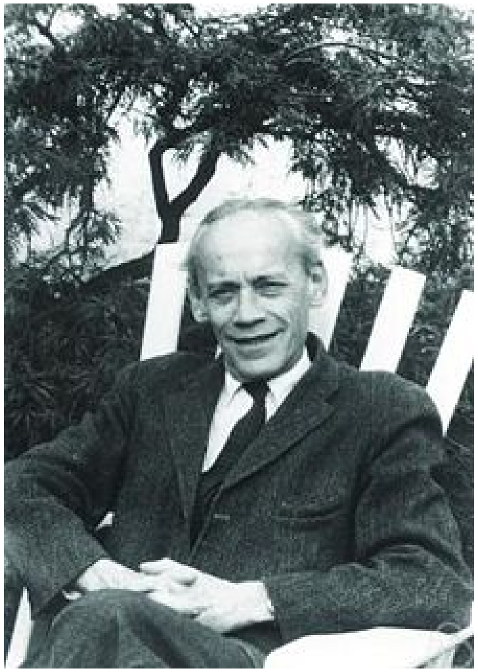
$10^{100} + 420 \cdot 10^{50} + 42437 = (10^{50} + 206)^2 + 1$

$\vdots$

Some conjectures regarding prime numbers: 4/5

Artin Conjecture.

The period of $1/p$ has length $p-1$ for infinitely many primes p



for example:

$\frac{1}{7} = 0.\overline{142857},$

$\frac{1}{17} = 0,\overline{0588235294117647},$

$\frac{1}{19} = 0.\overline{052631578947368421},$

$\vdots$

$\frac{1}{47} = 0.\overline{0212765957446808510638297872340425531914893617}\cdots$

primes with this property: $7, 17, 19, 23, 29, 47, 59, 61, 97, 109, 113, 131, 149, 167, 179, 181, 193, \ldots$

Some conjectures regarding prime numbers: 5/5

Riemann Hypothesis. $\zeta(\sigma + it) = 0, \sigma \in (0,1) \Rightarrow \sigma = \frac{1}{2}$

Georg Friedrich Bernhard Riemann

Birth: 17.09.1826 Breselenz / Königreich Hannover

Death: 20.07.1866 Selasca / Italia

for example:

$$s_1 = \tfrac{1}{2} + 14.135 \cdots i,$$
$$s_2 = \tfrac{1}{2} + 21.022 \cdots i,$$
$$s_3 = \tfrac{1}{2} + 25.011 \cdots i,$$
$$s_4 = \tfrac{1}{2} + 30.425 \cdots i,$$
$$s_5 = \tfrac{1}{2} + 32.935 \cdots i,$$
$$\vdots$$
$$s_{126} = \tfrac{1}{2} + 279.229 \cdots i,$$
$$s_{127} = \tfrac{1}{2} + 282.455 \cdots i,$$
$$\vdots$$

The enumerating function of primes

- **Problem:** rapidly produce primes $p \approx 10^{150}$;
- it is crucial to understand how primes are distributed;
- $\pi(x) = \#\{p \le x \text{ s.t. } p \text{ is prime}\}$;
- That is $\pi(x)$ is the number of prime numbers up to x;
- Examples: $\pi(10) = 4 \qquad \pi(100) = 25 \qquad \pi(1{,}000) = 168$

The enumerating function of primes

$$\boxed{\pi(x) = \#\{p \leq x \text{ such that } p \text{ is prime}\}}$$

That is $\pi(x)$ is the number of prime numbers up to x

for example:

$\pi(10) = 4,$

$\pi(100) = 25,$

$\pi(1,000) = 168$

$\ldots$

$\pi(104729) = 10^5$

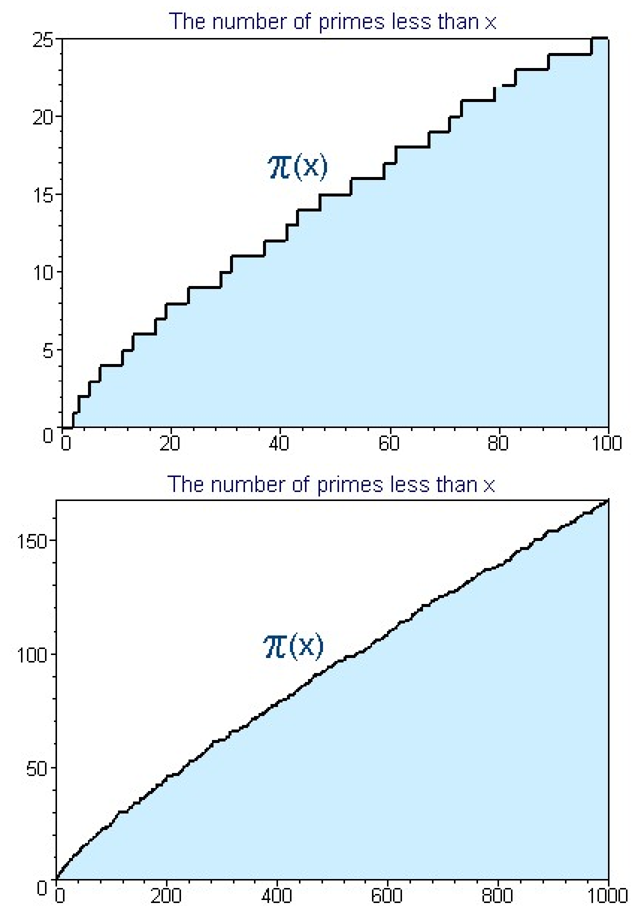
$\ldots$

$\pi(10^{24}) = 18435599767349200867866.$

$\ldots$

x	$\pi(x)$
10000	1229
100000	9592
1000000	78498
10000000	664579
100000000	5761455
1000000000	50847534
10000000000	455052511
100000000000	4118054813
1000000000000	37607912018
10000000000000	346065536839
100000000000000	3204941750802
1000000000000000	29844570422669
10000000000000000	279238341033925
100000000000000000	2623557157654233
1000000000000000000	24739954287740860
10000000000000000000	234057667276344607
100000000000000000000	2220819602560918840
1000000000000000000000	21127269486018731928
10000000000000000000000	201467286689315906290

The plot of $\pi(x)$



The School of Athens (Raffaello Sanzio)

Euclid of Alessandria

Birth: 325 B.C. (approximately)
Death: 265 B.C. (approximately)

There exists infinitely many primes: $\pi(x) \to \infty$ if $x \to \infty$

The sieve to count primes

220AC Greeks (Herathostenes from Cirene)

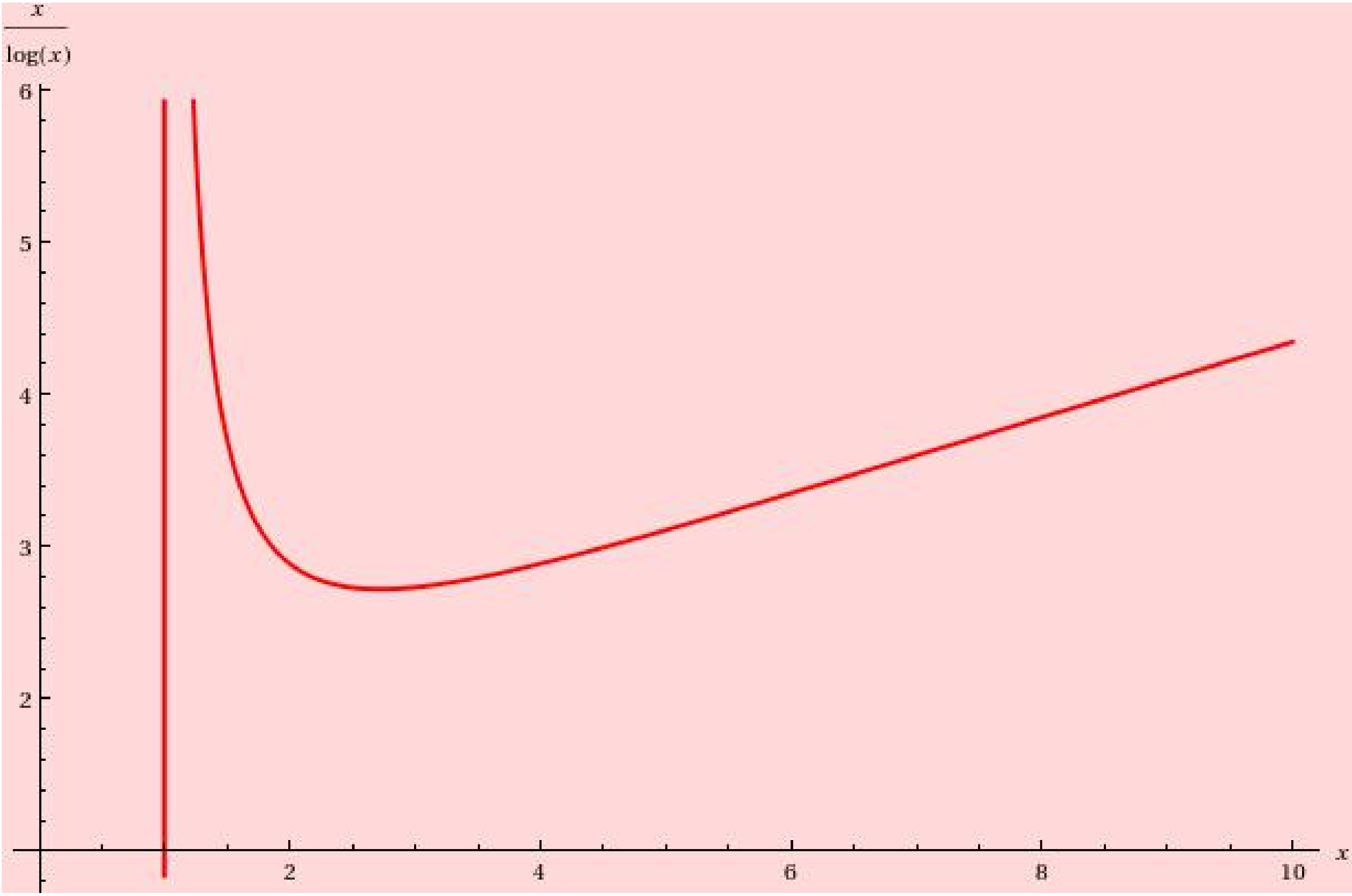
Legendre's Intuition

Adrien-Marie Legendre 1752-1833

$\pi(x)$ **is approximately** $\dfrac{x}{\log x}$

$\log x$ is the natural logarithm

The function $x/\log x$



$$f(x) = x/\log x$$

$\pi(x)$ is approximately $\frac{x}{\log x}$

that is

$$\lim_{x\to\infty} \frac{\pi(x)}{x/\log x} = 1 \qquad \text{and it is writtes as} \qquad \pi(x) \sim \frac{x}{\log x}$$

x	$\pi(x)$	$\frac{x}{\log x}$
1000	168	145
10000	1229	1086
100000	9592	8686
1000000	78498	72382
10000000	664579	620420
100000000	5761455	5428681
1000000000	50847534	48254942
10000000000	455052511	434294482
100000000000	4118054813	3948131654
1000000000000	37607912018	36191206825
10000000000000	346065536839	334072678387
100000000000000	3204941750802	3102103442166
1000000000000000	29844570422669	28952965460217
10000000000000000	279238341033925	271434051189532
100000000000000000	2623557157654233	2554673422960305
1000000000000000000	24739954287740860	24127471216847324
10000000000000000000	234057667276344607	228576043106974646
100000000000000000000	2220819602560918840	2171472409516259138

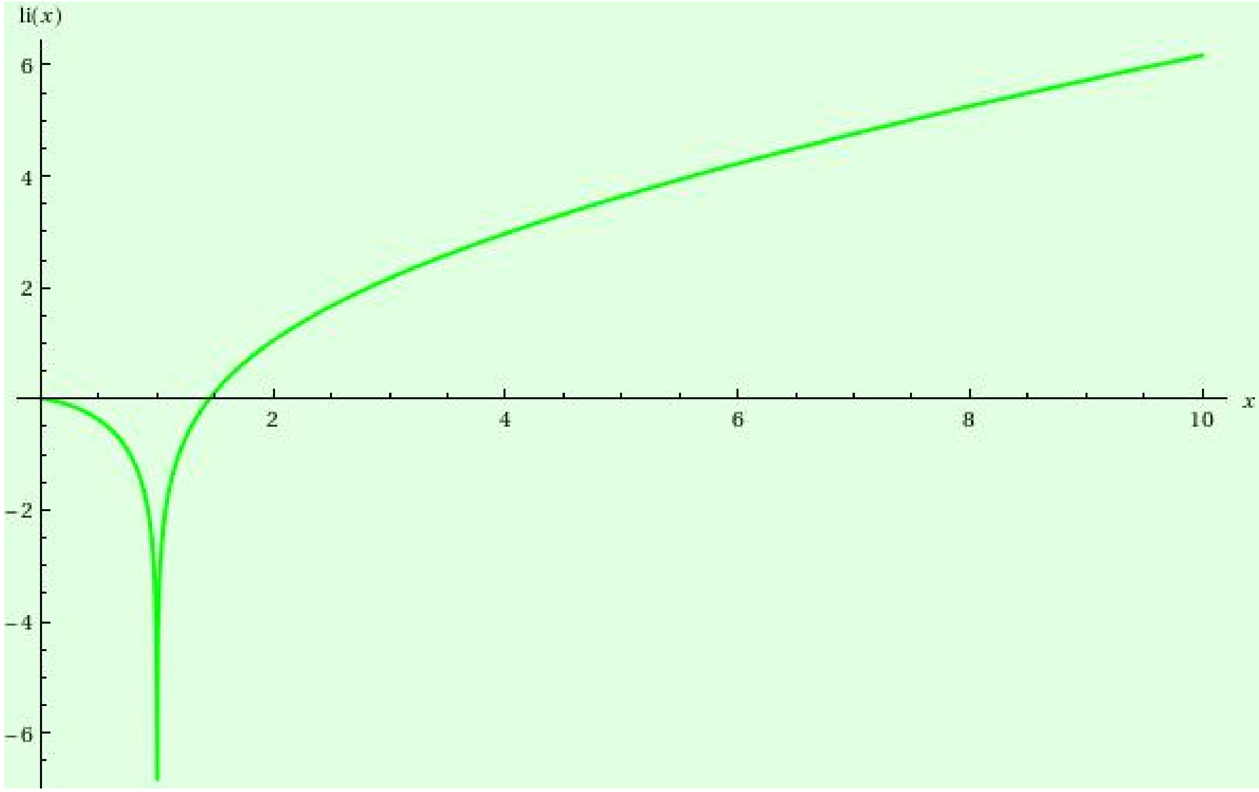
Gauß Conjecture

Johann Carl Friedrich Gauß(1777 - 1855)

$$\boxed{\pi(x) \sim \int_0^x \frac{du}{\log u}}$$

The function Logarithmic Integral

We set $\mathrm{li}(x) = \int_0^x \frac{du}{\log u}$, the function Logarithmic Integral. Here is the plot:



$\mathrm{li}(x)$

More recent photo of Gauß

Johann Carl Friedrich Gauß(1777 - 1855)

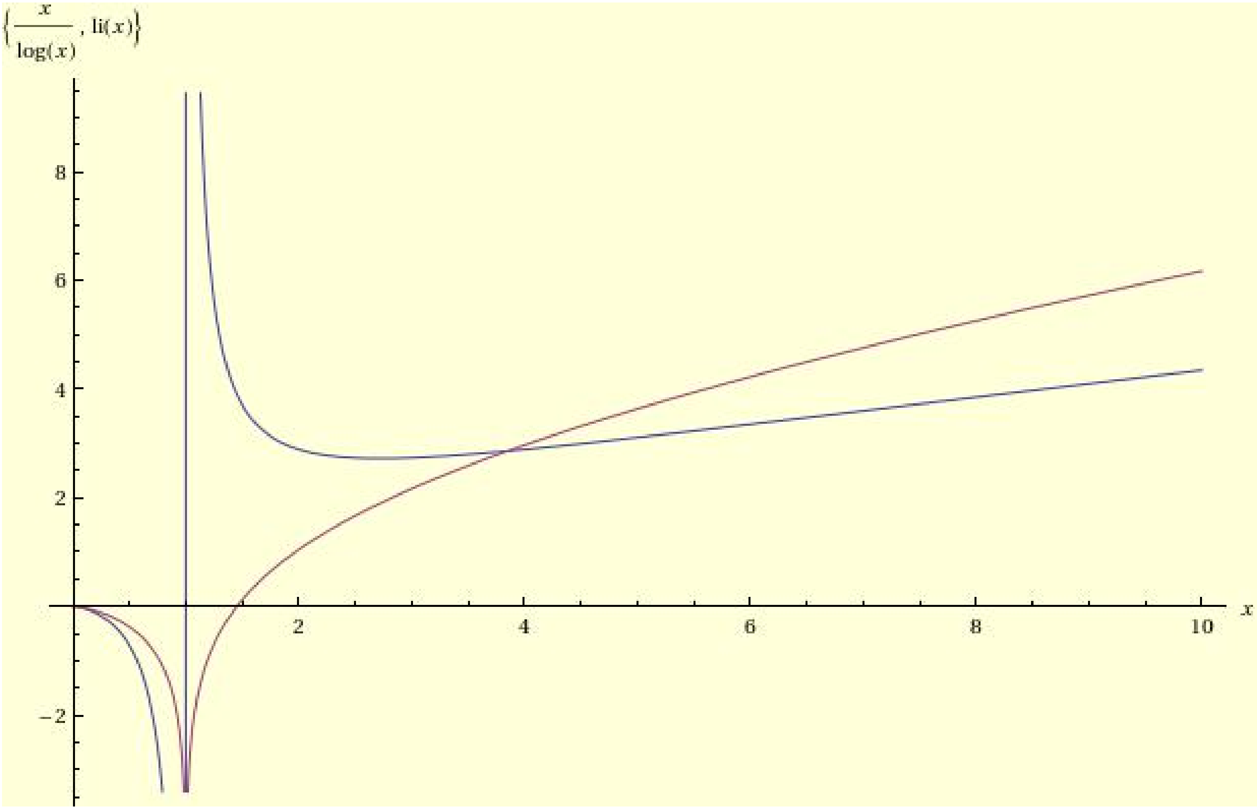
$$\boxed{\pi(x) \sim \mathrm{li}(x) := \int_0^x \frac{du}{\log u}}$$

The function "logarithmic integral" of Gauß

$$\mathrm{li}(x) = \int_0^x \frac{du}{\log u}$$

x	$\pi(x)$	$\mathrm{li}(x)$	$\frac{x}{\log x}$
1000	168	178	145
10000	1229	1246	1086
100000	9592	9630	8686
1000000	78498	78628	72382
10000000	664579	664918	620420
100000000	5761455	5762209	5428681
1000000000	50847534	50849235	48254942
10000000000	455052511	455055614	434294482
100000000000	4118054813	4118066401	3948131654
1000000000000	37607912018	37607950281	36191206825
10000000000000	346065536839	346065645810	334072678387
100000000000000	3204941750802	3204942065692	3102103442166
1000000000000000	29844570422669	29844571475288	28952965460217
10000000000000000	279238341033925	279238344248557	271434051189532
100000000000000000	2623557157654233	2623557165610822	2554673422960305
1000000000000000000	24739954287740860	24739954309690415	24127471216847324
10000000000000000000	234057667276344607	234057667376222382	228576043106974646
100000000000000000000	2220819602560918840	2220819602783663484	2171472409516259138

The function $\mathrm{li}(x)$ vs $\frac{x}{\log x}$



$$\mathrm{li}(x) = \frac{x}{\log x} + \int_0^x \frac{dt}{\log^2 t} \sim \frac{x}{\log x}$$

via integration by parts

Chebyshev Contribution

Pafnuty Lvovich Chebyshev

1821 - 1894

Chebyshev's Theorems

- $\frac{7}{8} \le \dfrac{\frac{\pi(x)}{x}}{\log x} \le \frac{9}{8}$
- $\liminf\limits_{x\to\infty} \dfrac{\pi(x)}{x/\log x} \le 1$
- $\limsup\limits_{x\to\infty} \dfrac{\pi(x)}{x/\log x} \ge 1$
- $\forall n,\ \exists p,\ n < p < 2n$

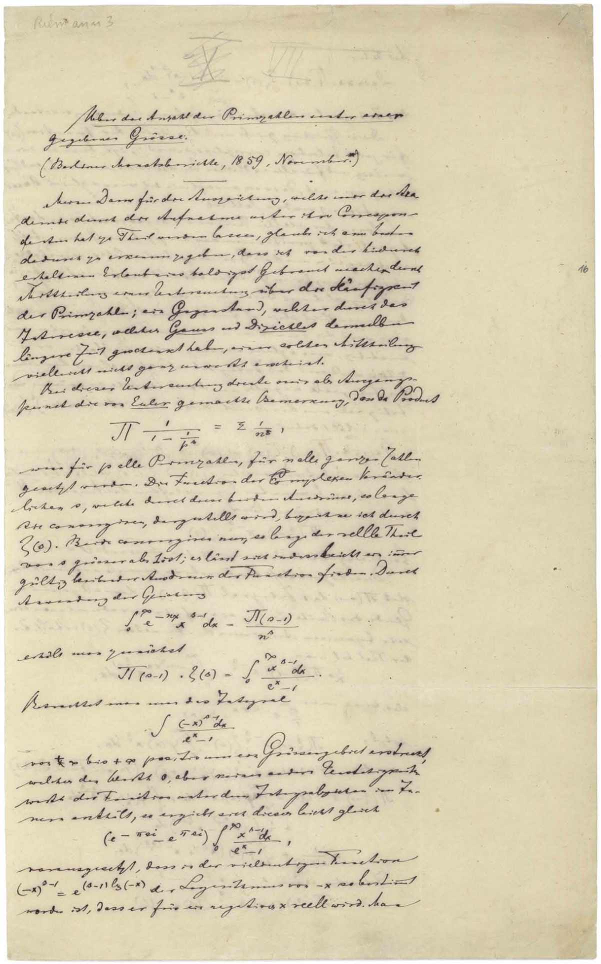
  (Bertrand Postulate)

Great problem of the end of 800:

- Prove the Legendre – Gauß Conjecture $\pi(x) \sim \dfrac{x}{\log x}$ if $x \to \infty$
- that is: $\left|\dfrac{\pi(x)}{\frac{x}{\log x}} - 1\right| \to 0$ if $x \to \infty$
- that is: $\left|\pi(x) - \dfrac{x}{\log x}\right|$ is "much smaller" than $\dfrac{x}{\log x}$ if $x \to \infty$
- that is: $\left|\pi(x) - \dfrac{x}{\log x}\right| = o\left(\dfrac{x}{\log x}\right)$ if $x \to \infty$
- that is (to say it at the Gauß way): $|\pi(x) - \mathrm{li}(x)| = o(\mathrm{li}(x))$ if $x \to \infty$

This statement is historically referred to as The Prime Number Theorem.

Riemann's paper 1859



(Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse.) Monatsberichte der Berliner Akademie,1859

Riemann Hypothesis:

$$|\pi(x) - \mathrm{li}(x)| \ll \sqrt{x}\log x$$

Revolutionary Idea:

Use the function:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

and complex analysis.

Summery:

- Riemann Hypothesis (1859) $\boxed{|\pi(x) - \mathrm{li}(x)| \ll \sqrt{x}\log x}$
- Riemann doe not complete the proof of the Prime Number Theorem but he suggests the right path.
- The idea to use the ζ function as a complex variable function
- Hadamard and de la Vallée Poussin (1897) add the missing piece to Riemann's program and prove the Prime Number Theorem
$$\boxed{|\pi(x) - \mathrm{li}(x)| \ll x \exp\left(-\sqrt{\log x}\right).}$$
- The use of ζ to study primes had already been suggested by Euler!!
- Schoenfeld (1976), Riemann Hypothesis is equivalent to
$$\boxed{|\pi(x) - \mathrm{li}(x)| < \tfrac{1}{8\pi}\sqrt{x}\log(x) \text{ if } x \geq 2657}$$

Prime Number Theorem finally proven (1896)

Jacques Salomon Hadamard 1865 - 1963

Charles Jean Gustave Nicolas
Baron de la Vallée Poussin 1866 - 1962

$$|\pi(x) - \mathrm{li}(x)| \ll x \exp(-a\sqrt{\log x}) \quad \exists a > 0$$

Euler Contribution

Leonhard Euler (1707 - 1783)

$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$ **is related to prime numbers**

Euler Contribution

Leonhard Euler (1707 - 1783)

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right)^{-1}$$

The beautiful formula of Riemann

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \pi^{\frac{s}{2}} \frac{\dfrac{1}{s(s-1)} + \displaystyle\int_1^{\infty} \left(x^{\frac{s}{2}-1} + x^{-\frac{s+1}{2}}\right) \left(\sum_{n=1}^{\infty} e^{-n^2 \pi x}\right) dx}{\displaystyle\int_0^{\infty} e^{-u} u^{\frac{s}{2}-1} \frac{du}{u}}$$

Exercise

Show that, if $\sigma, t \in \mathbb{R}$ are such that

$$\begin{cases} \displaystyle\int_1^{\infty} \frac{\{x\}}{x^{\sigma+1}} \cos(t \log x) dx = \frac{\sigma}{(\sigma-1)^2 + t^2} \\ \displaystyle\int_1^{\infty} \frac{\{x\}}{x^{\sigma+1}} \sin(t \log x) dx = \frac{t}{(\sigma-1)^2 + t^2} \end{cases}$$

Then $\sigma = \frac{1}{2}$.

(Here $\{x\}$ is the fractional part of $x \in \mathbb{R}$.)

Explicit distribution of prime numbers

Theorem. (Rosser - Schoenfeld) if $x \ge 67$

$$\frac{x}{\log x - 1/2} < \pi(x) < \frac{x}{\log x - 3/2}$$

Hence

$$\frac{10^{100}}{\log(10^{100})-1/2} < \pi(10^{100}) < \frac{10^{100}}{\log(10^{100})-3/2}$$

$$43523959267026440185153109567281075805591550920049791753399377550746551916373349269826109730287059.61758148$$

$$< \pi(10^{100}) <$$

$$43714220863853254827942128416877119789366015267226917261629640806806895897149988858712131777940942.89031$$

The five conjectures today – any News?

☞ **twin primes.** There existes infinitely many primes p such that $p+2$ is prime; that is

$$\boxed{\liminf_{n\to\infty}(p_{n+1}-p_n)=2}$$

$p_1 = 2,\ p_2 = 3, p_3 = 5, \cdots, p_n$ is the n–th prime

☞ Enrico Bombieri and Harald Davenport in 1966;

$$\boxed{\liminf_{n\to\infty}\frac{p_{n+1}-p_n}{\log p_n} < 0.46\cdots}$$

in other words, for infinitely many n, $(p_{n+1}-p_n) < 0,46\cdots\log p_n$

✌ Daniel Goldston, János Pintz and Cem Yildirim in 2005;

$$\boxed{\liminf_{n\to\infty}\frac{p_{n+1}-p_n}{\sqrt{\log p_n}\,\log\log p_n} = 0}$$

✌ Yitang Zhang on May 14, 2013;

$$\boxed{\liminf_{n\to\infty}(p_{n+1}-p_n)\le 7\cdot 10^7}$$

Zhang Contribution

Yitang Zhang

(http://en.wikipedia.org/wiki/Yitang_Zhang)

May 14, 2013: $\liminf_{n\to\infty} (p_{n+1} - p_n) \leq 70.000.000$

The race of the summer 2013 started on May 14,

http://michaelnielsen.org/polymath1/index.php?title=Timeline_of_prime_gap_bounds

May 14, 2013: $\liminf_{n\to\infty} (p_{n+1} - p_n) \leq 70.000.000$

In this table, infinitesimal losses in δ, ϖ are ignored.

Date	ϖ or (ϖ, δ)	k_0	H	Comments
14 May	1/1,168 (Zhang)	3,500,000 (Zhang)	70,000,000 (Zhang)	All subsequent work is based on Zhang's breakthrough paper.
21 May			63,374,611 (Lewko)	Optimises Zhang's condition $\pi(H) - \pi(k_0) > k_0$; can be reduced by 1 by parity considerations
28 May			59,874,594 (Trudgian)	Uses $(p_{m+1}, \dots, p_{m+k_0})$ with $p_{m+1} > k_0$
30 May			59,470,640 (Morrison) 58,885,998? (Tao) 59,093,364 (Morrison) 57,554,086 (Morrison)	Uses $(p_{m+1}, \dots, p_{m+k_0})$ and then $(\pm 1, \pm p_{m+1}, \dots, \pm p_{m+k_0/2-1})$ following [HR1973], [HR1973b], [R1974] and optimises in m
31 May		2,947,442 (Morrison) 2,618,607 (Morrison)	48,112,378 (Morrison) 42,543,038 (Morrison) 42,342,946 (Morrison)	Optimizes Zhang's condition $\omega > 0$, and then uses an improved bound on δ_2
1 Jun			42,342,924 (Tao)	Tiny improvement using the parity of k_0

The race of the summer 2013 started on May 14,

http://michaelnielsen.org/polymath1/index.php?title=Timeline_of_prime_gap_bounds

June 15, 2013: $\liminf_{n\to\infty}(p_{n+1} - p_n) \le 60.764$

Jun 10		23,283? (Harcos/v08ltu)	253,118 (xfxie) 386,532* (Sutherland) 253,048 (Sutherland) 252,990 (Sutherland) 252,976 (Sutherland)	More efficient control of the κ error using the fact that numbers with no small prime factor are usually coprime
Jun 11			252,804 (Sutherland) 2,345,896* (Sutherland)	More refined local "adjustment" optimizations, as detailed here An issue with the k_0 computation has been discovered, but is in the process of being repaired.
Jun 12		22,951 (Tao/v08ltu) 22,949 (Harcos)	249,180 (Castryck) 249,046 (Sutherland) 249,034 (Sutherland)	Improved bound on k_0 avoids the technical issue in previous computations.
Jun 13			248,970 (Sutherland) 248,910 (Sutherland)	
Jun 14			248,898 (Sutherland)	
Jun 15	$348\varpi + 68\delta < 1$? (Tao)	6,330? (v08ltu) 6,329? (Harcos)	60,830? (Sutherland) 60,812? (Sutherland) 60,764 (xfxie)	Taking more advantage of the α convolution in the Type III sums

The race of the summer 2013 started on May 14,

http://michaelnielsen.org/polymath1/index.php?title=Timeline_of_prime_gap_bounds

July 27, 2013: $\liminf_{n\to\infty} (p_{n+1} - p_n) \le 4.680$

Jun 27	$108\varpi + 30\delta < 1$? (Tao)	902? (Hannes)	6,966? (Engelsma)	slight improvements to the Type II sums. Tuples page is now accepting submissions.
Jul 1	$(93 + \frac{1}{3})\varpi + (26 + \frac{2}{3})\delta < 1$? (Tao)	873? (Hannes) ~~872? (xfxie)~~	6,712? (Sutherland) ~~6,696? (Engelsma)~~	Refactored the final Cauchy-Schwarz in the Type I sums to rebalance the off-diagonal and diagonal contributions
Jul 5	$(93 + \frac{1}{3})\varpi + (26 + \frac{2}{3})\delta < 1$ (Tao)	720 (xfxie/Harcos)	5,414 (Engelsma)	Weakened the assumption of x^δ-smoothness of the original moduli to that of double x^δ-dense divisibility
Jul 10	7/600? (Tao)			An in principle refinement of the van der Corput estimate based on exploiting additional averaging
Jul 19	$(85 + \frac{5}{7})\varpi + (25 + \frac{5}{7})\delta < 1$? (Tao)			A more detailed computation of the Jul 10 refinement
Jul 20				Jul 5 computations now confirmed
Jul 27		633? (Tao) 632? (Harcos)	4,686? (Engelsma) 4,680? (Engelsma)	
Jul 30	$168\varpi + 48\delta < 1$**? (Tao)	1,788**? (Tao)	14,994**? (Sutherland)	Bound obtained without using Deligne's theorems.
Aug 17		1,783**? (xfxie)	14,950**? (Sutherland)	

Legend:

The race of the summer 2013 started on May 14,

http://michaelnielsen.org/polymath1/index.php?title=Timeline_of_prime_gap_bounds

January 6, 2014: $\liminf_{n\to\infty} (p_{n+1} - p_n) \leq 270$

Dec 28			474,296? [EH] [m=4] (Sutherland) 4,137,854? [EH] [m=5] (Sutherland)	
Jan 2 2014			474,290? [EH] [m=4] (Sutherland)	
Jan 6		54# (Nielsen)	270# (Clark-Jarvis)	
Jan 8		4 [GEH] (Nielsen)	8 [GEH] (Nielsen)	Using a "gracefully degrading" lower bound for the numerat problem. Calculations confirmed here.
Jan 9			474,266? [EH] [m=4] (Sutherland)	
Jan 28			395,106? [m=2] (Sutherland)	
Jan 29		3 [GEH] (Nielsen)	6 [GEH] (Nielsen)	A new idea of Maynard exploits GEH to allow for cutoff func extends beyond the unit cube
Feb 9				Jan 29 results confirmed here
Feb 17		53?# (Nielsen)	264?# (Clark-Jarvis)	Managed to get the epsilon trick to be computationally feas
Feb 22		51?# (Nielsen)	252?# (Clark-Jarvis)	More efficient matrix computation allows for higher degrees
Mar 4				Jan 6 computations confirmed

Legend:

The race of the summer 2013 started on May 14,

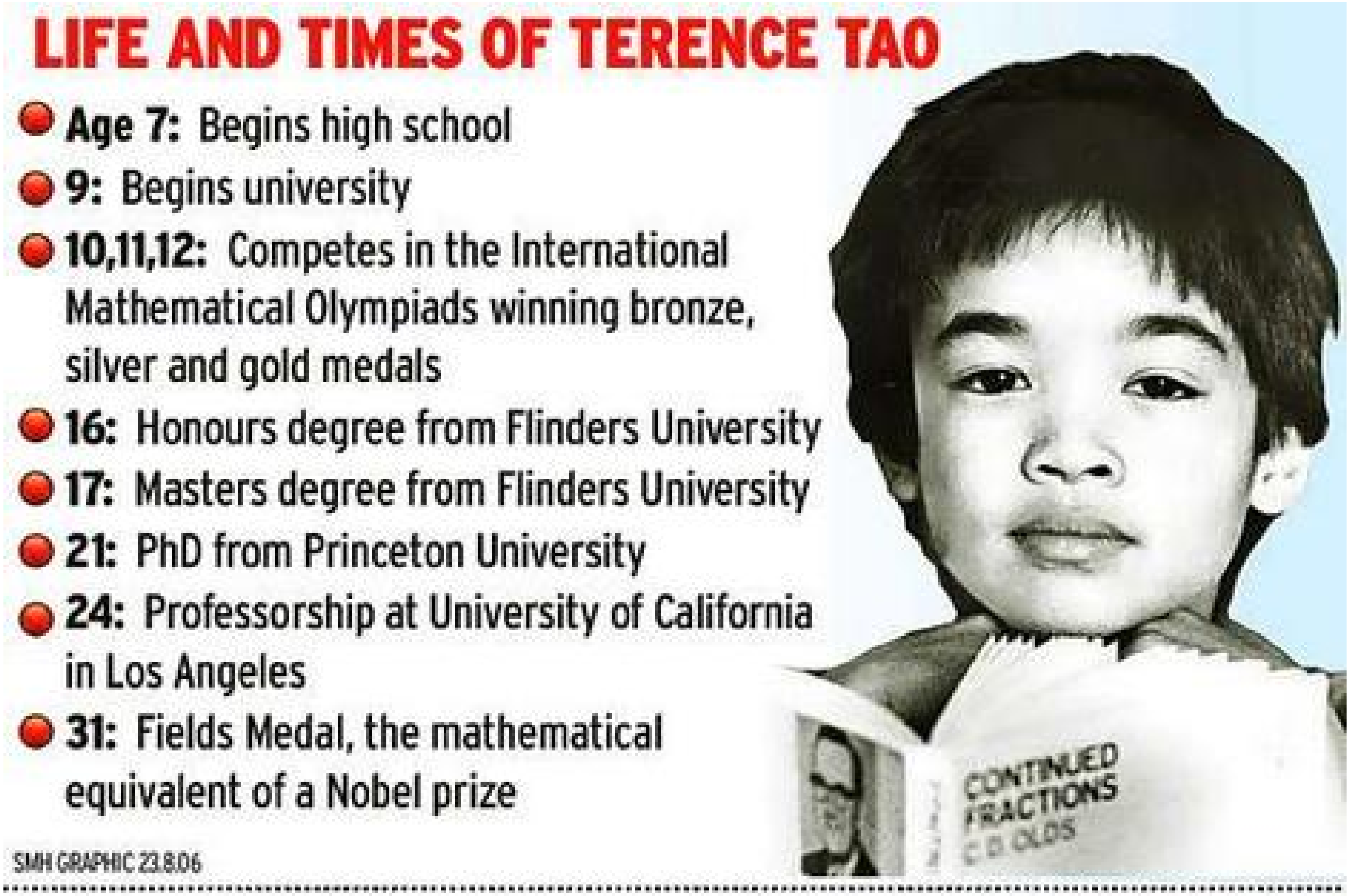
http://michaelnielsen.org/polymath1/index.php?title=Bounded_gaps_between_primes

Race to the solution of a more general problem

H_m = least integer s.t. among $n, n+1, \cdots, n+H_m$ there are m consecutive primes

m	Conjectural	Assuming EH	Without EH	Without EH or Deligne
1	2	6 (on GEH) 12 (on EH only)	**252**	252
2	6	270	395,106	474,266
3	8	52,116	24,462,654	32,313,878
4	12	474,266	1,497,901,734	2,186,561,568
5	16	4,137,854	82,575,303,678	131,161,149,090
m	$(1+o(1))m\log m$	$O(me^{2m})$	$O(m\exp((4-\frac{52}{283})m))$	$O(m\exp((4-\frac{4}{43})m))$

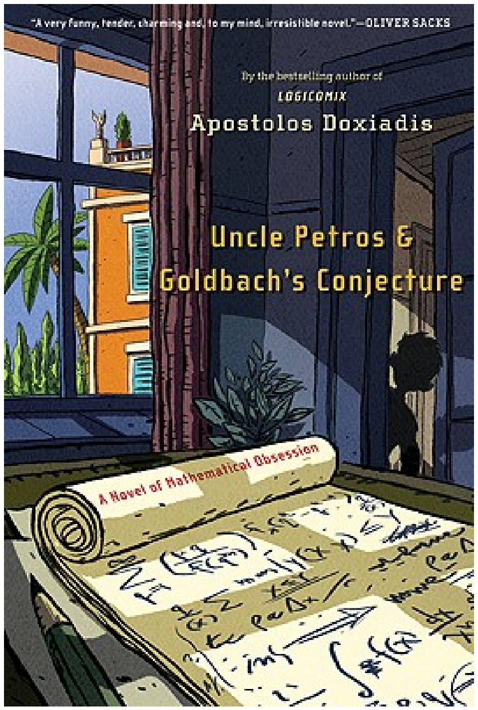
Polymath8 and Terry Tao



The five conjectures today – any News?

☞ **Goldbach conjecture**

Every even number (except for 2) can be written as the sum of two primes



Equivalent formulation:

every integer greater than 5
can be written as the sum of three primes

From Vinogradov to Helfgott

Harald Helfgott

- (Vinogradov – 1937) every odd integer greater than $3^{3^{15}}$ is the sum of three primes
- (Helfgott – 2013) every odd integer greater than 5 is the sum of three primes

Hooley's Contribution

Riemann Hypothesis implies Artin Conjecture.

The period of $1/p$ has length $p - 1$ for infinitely many primes p

for example:

$\frac{1}{7} = 0.\overline{142857},$

$\frac{1}{17} = 0,\overline{0588235294117647},$

$\frac{1}{19} = 0.\overline{052631578947368421},$

$\vdots$

$\frac{1}{47} = 0.\overline{0212765957446808510638297872340425531914893617}\cdots$

Primes with this property: $7, 17, 19, 23, 29, 47, 59, 61, 97, 109, 113, 131, 149, 167, 179, 181, 193, \ldots$