Egyptian fractions: from Rhind Mathematical Papyrus to Erdős and Tao

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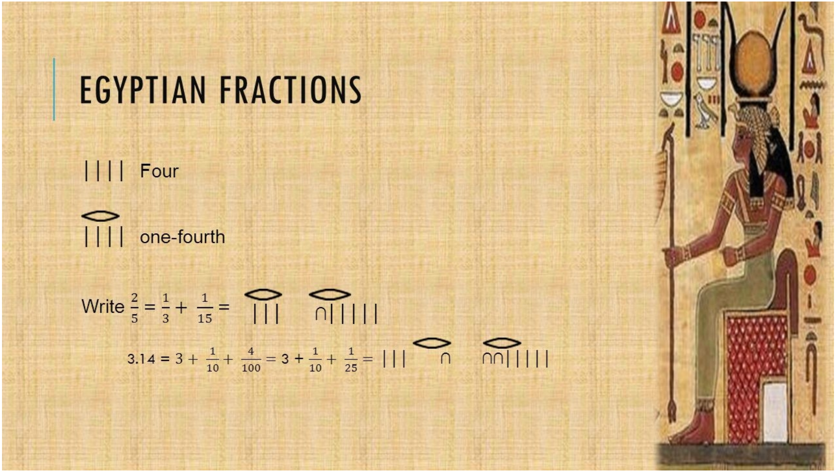
The Rhind Mathematical Papyrus

British Museum

Fractions in Egypt

Fractions in Egypt



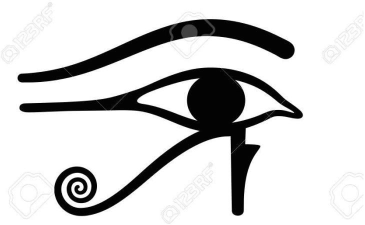
Egyptian fractions

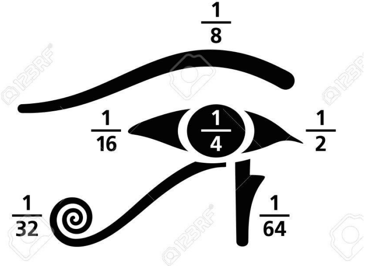
the Ancient Egyptians only used unit numerator fractions

they turned other fractions into sums of two or more fractions, all with a numerator of **1**

(apart from ⅔)

they used fractions with different denominators





1/8
1/16
1/4
1/2
1/32
1/64

Egyptian Fraction Expansion (EFE)

EFE

Given $a/b \in \mathbb{Q}^{>}$, an *Egyptian Fraction Expansion* of a/b with length k is the expression

$$\frac{a}{b} = \frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_k}$$

where $x_1, \ldots, x_k \in \mathbb{N}$

Every $a/b \in \mathbb{Q}^{>}$ has an EFE with distinct $x_1, \ldots, x_n$!!

The Greedy Algorithm

Fibonacci (1200's)

- Given $0 < a/b < 1$, the identity:

$$\frac{a}{b} = \frac{1}{b_1} + \frac{a_1}{bb_1}$$

can be found with

1. $b_1, a_1 \in \mathbb{N}$
2. $1 \leq a_1 < a$
3. $b_1 > 1$,

- Hence we can iterate the process to obtain EFE for a/b
- $\frac{a}{b} = \frac{1}{b_1} + \frac{1}{b_2} + \frac{a_2}{bb_1b_2} =$
  $= \frac{1}{b_1} + \frac{1}{b_2} + \frac{1}{b_3} + \frac{a_3}{bb_1b_2b_3} = \cdots$
- it takes at most a steps

Euclid ($\approx$ 300 BC)

- Given $a, b \in \mathbb{N}$, $\exists q, r \in \mathbb{N}$ s.t.

$$b = aq + r, \qquad 0 \leq r < a$$

- a quick computation shows

$$\frac{a}{b} = \frac{1}{q+1} + \frac{a-r}{b(q+1)}$$

- Hence
  1. $b_1 = q + 1 > 1$;
  2. $0 < a_1 = a - r < a$ since $\gcd(a, b) = 1$

Example: The Greedy Algorithm at work

$$\begin{aligned}\frac{5}{121} &= \frac{1}{25} + \frac{4}{3025}\\ &= \frac{1}{25} + \frac{1}{757} + \frac{3}{2289925}\\ &= \cdots\\ &= \frac{1}{25} + \frac{1}{757} + \frac{1}{763309} + \frac{1}{873960180913} + \\ &\qquad + \frac{1}{1527612795642093418846225}\end{aligned}$$

However,

$$\frac{5}{121} = \frac{1}{33} + \frac{1}{121} + \frac{1}{363}$$

The Takenouchi Algorithm (1921)

how Takenouchi Algorithm works

1. based on the identity:

$$\frac{1}{b}+\frac{1}{b}=\begin{cases}\frac{1}{b/2} & \text{if } 2 \mid b\\ \frac{1}{\frac{b+1}{2}}+\frac{1}{\frac{b(b+1)}{2}} & \text{otherwise}\end{cases}$$

2. Write $\frac{a}{b}=\overbrace{\frac{1}{b}+\cdots+\frac{1}{b}}^{a-\text{times}}$
3. Apply the above identity $[a/2]$ times

$$\frac{a}{b}=\overbrace{\frac{1}{\frac{b+1}{2}}+\cdots+\frac{1}{\frac{b+1}{2}}}^{a/2-\text{times}}+\overbrace{\frac{1}{\frac{b(b+1)}{2}}+\cdots+\frac{1}{\frac{b(b+1)}{2}}}^{a/2-\text{times}}$$

4. reiterate using the first identity

The Takenouchi Algorithm (1921)

Example:

$$\frac{5}{121} = \frac{1}{121} + \frac{1}{121} + \frac{1}{121} + \frac{1}{121} + \frac{1}{121}$$
$$= \frac{1}{121} + \frac{1}{61} + \frac{1}{61} + \frac{1}{61 \times 121} + \frac{1}{61 \times 121}$$
$$= \frac{1}{121} + \frac{1}{31} + \frac{1}{1891} + \frac{1}{3691} + \frac{1}{27243271}$$

However it is still worse than,

$$\frac{5}{121} = \frac{1}{33} + \frac{1}{121} + \frac{1}{363}$$

Minimizing length & Denominators' sizes

Theorem (Tenenbaum – Yokota (1990))

Given $a/b \in \mathbb{Q} \cap (0,1)$, $\exists$ *EFE s.t.*

- *it has length* $O(\sqrt{\log b})$;
- *each denominator is* $O\left(b \log b (\log\log b)^4 (\log\log\log b)^2\right)$

thinking at ESE-expansion as a Waring problem with negative exponent...

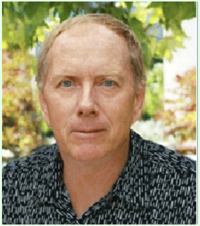
Theorem (Graham (1964))

Given $a/b \in \mathbb{Q}^{>}$,

$$\frac{a}{b} = \frac{1}{y_1^2} + \cdots + \frac{1}{y_k^2}$$

admits a solution in distinct integers $y_1, \ldots, y_k$

$$\iff \quad a/b \in (0, \pi^2/6 - 1) \cup [1, \pi^2/6)$$



Note: Graham result is quite general ... for example

$\frac{a}{b} = \frac{1}{y_1^2} + \cdots + \frac{1}{y_k^2}$ *with* $y_j^2 \equiv 4 \bmod 5$ *distinct* $\Leftrightarrow 5 \nmid b$ *and*

$$a/b \in (0, \alpha - \tfrac{13}{36}) \cap [\tfrac{1}{9}, \alpha - \tfrac{1}{4}) \cap [\tfrac{1}{4}, \alpha - \tfrac{1}{9}) \cap [\alpha, \tfrac{13}{36})$$

where $\alpha = 2(5 - \sqrt{5})\pi^2/125$

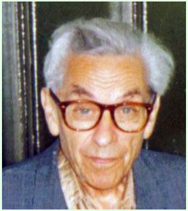
The Erdős-Strauß Conjecture

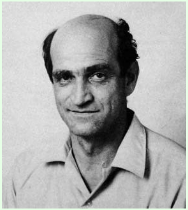
Erdős-Strauß Conjecture (ESC) (1950):

$\forall n > 2$,

$$\frac{4}{n} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z}$$

admits a solution in positive distinct integers x, y, z





Note:

- enough to consider (for prime $p \geq 3$), $\frac{4}{p} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z}$
- many computations. Record (2012) (Bello–Hernández, Benito and Fernández): ESC holds for $n \leq 2 \times 10^{14}$

The Schinzel Conjecture

Schinzel Conjecture:

given $a \in \mathbb{N}$, $\exists N_a$ s.t. if $n > N_a$,

$$\frac{a}{n} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z}$$

admits a solution in distinct integers x, y, z

Theorem (**Vaughan (1970):**)

$$\#\left\{ n \leq T : \begin{array}{c} \frac{a}{n} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z} \\ \textit{has no solution} \end{array} \right\} \ll \frac{T}{e^{c \log^{2/3} T}}$$

Elsholtz – Tao (2013): new results about ESC ... later

Fixing the denominator

Definition (Enumerating functions for fixed denominator)

Fix $n \in \mathbb{N}$ and set

1. $\mathcal{A}_k(n) = \left\{ a \in \mathbb{N} : \frac{a}{n} = \frac{1}{x_1} + \cdots + \frac{1}{x_k}, \exists x_1, \ldots, x_k \in \mathbb{N} \right\}$
2. $\mathcal{A}_k^*(n) = \{ a \in \mathcal{A}_k(n) : \gcd(a, n) = 1 \}$
3. $A_k(n) = \#\mathcal{A}_k(n)$
4. $A_k^*(n) = \#\mathcal{A}_k^*(n)$

Note that:

$$A_k(n) = \sum_{d|n} A_k^*(d)$$

Fixing the denominator

Numerics:

n	$A_2(n)$	$A_3(n)$	n	$A_2(n)$	$A_3(n)$	n	$A_2(n)$	$A_3(n)$	n	$A_2(n)$	$A_3(n)$
2	4	6	27	18	41	52	27	68	77	25	75
3	5	8	28	23	49	53	10	36	78	39	101
4	7	11	29	10	26	54	35	82	79	12	45
5	6	11	30	29	58	55	24	65	80	49	118
6	10	16	31	8	27	56	36	85	81	28	81
7	6	13	32	23	51	57	21	62	82	18	59
8	11	19	33	18	44	58	18	53	83	14	50
9	10	19	34	17	42	59	14	41	84	60	139
10	12	22	35	20	49	60	51	109	85	22	78
11	8	16	36	34	69	61	6	28	86	19	62
12	17	29	37	6	27	62	18	56	87	25	77
13	6	18	38	17	45	63	33	86	88	39	105
14	13	26	39	20	51	64	32	81	89	14	48
15	14	29	40	33	71	65	22	69	90	58	138
16	16	31	41	10	29	66	36	89	91	20	79
17	8	21	42	34	74	67	8	39	92	29	86
18	20	38	43	8	30	68	30	79	93	21	75
19	8	22	44	25	61	69	25	70	94	21	69
20	21	41	45	28	69	70	39	98	95	24	82
21	17	37	46	17	47	71	14	42	96	59	143
22	14	32	47	12	36	72	54	121	97	8	47
23	10	25	48	41	87	73	6	36	98	32	94
24	27	51	49	14	46	74	17	57	99	36	107
25	12	33	50	27	67	75	33	91	100	48	126

Fixing the denominator - the binary case

Croot, Dobbs, Friedlander, Hetzel, FP (2000):

1. $\forall \varepsilon > 0$,

$$A_2(n) \ll n^{\epsilon}$$

2. $$T \log^3 T \ll \sum_{n \le T} A_2(n) \ll T \log^3 T$$

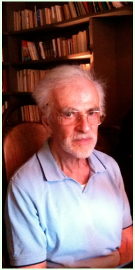
Fixing the denominator - the binary case

Lemma (Rav Criterion (1966))

Let $a,\ n \in \mathbb{N}$ *s.t.* $(a, n) = 1$.

$$\frac{a}{n} = \frac{1}{x} + \frac{1}{y}$$

has solution $x, y \in \mathbb{N} \quad \Leftrightarrow \quad \exists (u_1, u_2) \in \mathbf{N}^2$ *with* $(u_1, u_2) = 1$, $u_1 u_2 | n$ *and* $a | u_1 + u_2$



Consequence: let $\tau(n)$ be number of divisors of n and $[m, n]$ be the lowest common multiple of n and m

$$A_2^*(p^k) = \tau([p^k + 1, p^{k-1} + 1, \ldots, p + 1])$$

Fixing the denominator - the general case

Theorem (Croot, Dobbs, Friedlander, Hetzel, IP (2000))

- $\forall \varepsilon > 0, \qquad A_3(n) \ll_\epsilon n^{1/2+\epsilon}$
- *by an induction argument,* $\forall \varepsilon > 0,$

$$A_k(n) \ll_\epsilon n^{\alpha_k+\epsilon}$$

where $\alpha_k = 1 - 2/(3^{k-2}+1)$




Theorem (Banderier, Luca, IP (2018))

- $\forall \varepsilon > 0, \qquad A_3(n) \ll_\epsilon n^{1/3+\epsilon}$
- *by an induction argument,* $\forall \varepsilon > 0,$

$$A_k(n) \ll_\epsilon n^{\beta_k+\epsilon}$$

where $\beta_k = 1 - 2/(2 \cdot 3^{k-3}+1)$


Lemma

Let $a/n \in \mathbb{Q}^{>}$. $a/n = 1/x + 1/y + 1/z$ *for some* $x, y, z \in \mathbb{N}$
$\Leftrightarrow \exists$ *six positive integers* $D_1, D_2, D_3, v_1, v_2, v_3$ *with*

(i) $[D_1, D_2, D_3] \mid n$;

(ii) $v_1 v_2 v_3 \mid D_1 v_1 + D_2 v_2 + D_3 v_3$;

(iii) $a \mid (D_1 v_1 + D_2 v_2 + D_3 v_3)/(v_1 v_2 v_3)$

Conversely, if there are such integers, then by putting $E = [D_1, D_2, D_3]$, $f_1 := n/E$, $f_2 = (D_1 v_1 + D_2 v_2 + D_3 v_3)/(a v_1 v_2 v_3)$ *and* $f = f_1 f_2$, *a representation is*

$$\frac{a}{n} = \frac{1}{(E/D_1) v_2 v_3 f} + \frac{1}{(E/D_2) v_1 v_3 f} + \frac{1}{(E/D_3) v_1 v_2 f}$$

back to Erdős-Strauß Conjecture

the polynomial families of solution

Polynomial families of solutions

- $$\frac{4}{n} = \frac{1}{n} + \frac{1}{(n+1)/3} + \frac{1}{n(n+1)/3}$$
  $\Longrightarrow$ if $n \equiv 2 \bmod 3$, ESC holds for n
- $$\frac{4}{n} = \frac{1}{n/3} + \frac{1}{4n/3} + \frac{1}{4n}$$
  $\Longrightarrow$ if $n \equiv 0 \bmod 3$, ESC holds for n
- Need to solve ESC for $n \equiv 1 \bmod 3$
- **idea can be pushed:** $4/n$ requires four terms with the *greedy algorithm* if and only if $n \equiv 1$ or $17 (\bmod 24)$
- **example** if $n = 5 + 24t$

$$\frac{4}{n} = \frac{1}{6t+1} + \frac{1}{(2+8t)(6t+1)} + \frac{1}{(5+24t)(6t+1)(2+8t)}$$

(Another) example ($n \equiv 7 \bmod 24$)

$$\frac{4}{7+24t} = \frac{1}{6t+2} + \frac{1}{(8+24t)(6t+2)} + \frac{1}{(7+24t)(8+24t)(6t+2)}$$

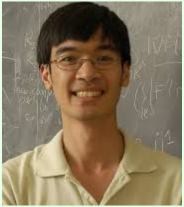
Definition (solvable congruences)

We say that $r(\bmod q) \in \mathbb{Z}/q\mathbb{Z}^*$ is *solvable by polynomials* if $\exists P_1, P_2, P_3 \in \mathbb{Q}[x]$ which take positive integer values for sufficiently large integer argument and such that for all $n \equiv r(\bmod q)$:

$$\frac{4}{n} = \frac{1}{P_1(n)} + \frac{1}{P_2(n)} + \frac{1}{P_3(n)}$$

Theorem (Elsholtz–Tao (2013))

There is a classification of solvable conguences by polynomials



Theorem (Mordell (1969))

All (primitive) congruence classes $r \pmod{840}$ are solvable by polynomials unless r is a perfect square

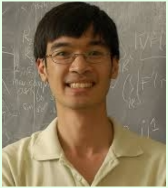
(i.e. $r = 1^2, 11^2, 13^2, 17^2, 19^2, 23^2$)

Remarks & Definitions:

- Up to reordering, solutions of $\frac{4}{p} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z}$ are of two types:
  I. $p \mid x$ & $p \nmid yz$
  II. $p \mid \gcd(x, y)$ & $p \nmid z$
- in analogy, we say that, up to reordering, a solutions of $\frac{4}{n} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z}$ is of type:
  I. if $n \mid x$ & $\gcd(n, yz) = 1$
  II. $n \mid \gcd(x, y)$ & $\gcd(n, z) = 1$
- $f(n)$ be the number of solutions of $4/n = 1/x + 1/y + 1/z$
- Set $f_I(n)$ (resp $f_{II}(n)$) be the number of solutions of type I (resp II) of $4/n = 1/x + 1/y + 1/z$
- $f(p) = 3f_I(p) + 3f_{II}(p)$
- $f(n) \geq 3f_I(n) + 3f_{II}(n)$

Theorem (some of Elsholtz – Tao's results)

- $f_I(n) \ll n^{3/5+\varepsilon}, \quad f_{II}(n) \ll n^{2/5+\varepsilon}$
- $N\log^3 N \ll \sum_{n\le N} f_I(n) \ll N\log^3 N$
- $N\log^3 N \ll \sum_{n\le N} f_{II}(n) \ll N\log^3 N$
- $N\log^2 N \ll \sum_{p\le N} f_I(p) \ll N\log^2 N \log\log N$
- $N\log^2 N \ll \sum_{p\le N} f_{II}(p) \ll N\log^2 N$
- $f(n) \gg e^{\left((\log 3+o(1))\frac{\log n}{\log\log n}\right)}$ *for* ∞ n
- $f(n) \gg (\log n)^{0.54}$ *for almost all* n
- $f(p) \gg (\log p)^{0.54}$ *for almost all* p



Let $S_{m,n} = \{(x, y, z) \in \mathbb{C}^3 : mxyz = nyz + nxy + nxz\} \subset \mathbb{C}^3$.
$A_3(n)$ equals the number of $m \in \mathbb{N}$ s.t. $S_{m,n} \cap \mathbb{N}^3 \neq \emptyset$. Set

$$\Sigma^{\mathrm{I}}_{m,n} = \left\{ (a, b, c, d, e, f) \in \mathbb{C}^6 : \begin{array}{l} mabd = ne + 1, ce = a + b \\ mabcd = n(a + b) + c \\ macde = ne + ma^2d + 1 \\ mbcde = ne + mb^2d + 1 \\ macd = n + f, ef = ma^2d + 1 \\ bf = na + c \\ n^2 + mc^2d = f(mbcd - n) \end{array} \right\}$$

which is a 3-dimensional algebraic variety. The map

$$\pi^{\mathrm{I}}_{m,n} : \Sigma^{\mathrm{I}}_{m,n} \longrightarrow S_{m,n}, (a, b, c, d, e) \mapsto (abdn, acd, bcd)$$

is well defined after quotienting by the dilation symmetry
$(a, b, c, d, e, f) \mapsto (\lambda a, \lambda b, \lambda c, \lambda^{-2}d, e, f)$ this map is bijective

back to $A_3(p)$

Adapting Elsholtz – Tao construction

Theorem (Banderier, Luca, FP (2018))

$$\sum_{p\le N} A_{II,3}(p) \ll N\log^2 N\log\log N$$

where $A_{II,3}(p)$ *is the number of* $a\in\mathbb{N}$ *s.t.*

$$\frac{a}{p}=\frac{1}{px}+\frac{1}{py}+\frac{1}{z}$$

admits a solution $x,y,z\in\mathbb{N}$

these are classical *elementary analytic number theory* proof:

- Dirichlet average divisor in special sparse sequences
- Prime in arithmetic progression
- Brun Titchmarsh estimates
- Bombieri–Vinogradov Theorem