ELLIPTIC CURVES OVER FINITE FIELDS

FRANCESCO PAPPALARDI

#3 - FIRST STEPS.

SEPTEMBER 4TH 2015

SEAMS School 2015
Number Theory and Applications in Cryptography and Coding Theory
University of Science, Ho Chi Minh, Vietnam
August 31 - September 08, 2015

Proto–History (from WIKIPEDIA)

Giulio Carlo, Count Fagnano, and Marquis de Toschi (December 6, 1682 – September 26, 1766) was an Italian mathematician. He was probably the first to direct attention to the theory of *elliptic integrals*. Fagnano was born in Senigallia.

He made his higher studies at the *Collegio Clementino* in Rome and there won great distinction, except in mathematics, to which his aversion was extreme. Only after his college course he took up the study of mathematics.

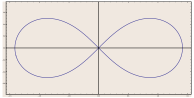
Later, without help from any teacher, he mastered mathematics from its foundations.

Some of His Achievements:

- $\pi = 2i \log \frac{1-i}{1+1}$
- Length of *Lemniscate*

Carlo Fagnano

Collegio Clementino

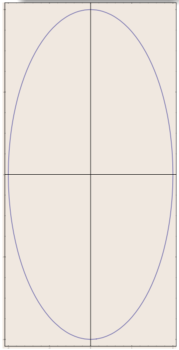


Lemniscate $(x^2 + y^2)^2 = 2a^2(x^2 - y^2)$

$$\ell = 4 \int_0^a \frac{a^2\,dr}{\sqrt{a^4 - r^4}} = \frac{a\sqrt{\pi}\,\Gamma(\frac{5}{4})}{\Gamma(\frac{3}{4})}$$

Length of Ellipses

$$\mathcal{E} : \frac{x^2}{4} + \frac{y^2}{16} = 1$$



The length of the arc of a plane curve $y = f(x), f : [a, b] \to \mathbb{R}$ is:

$$\ell = \int_a^b \sqrt{1 + (f'(t))^2}\, dt$$

Applying this formula to $\mathcal{E}$:

$$\ell(\mathcal{E}) = 4 \int_0^4 \sqrt{1 + \left(\frac{d\sqrt{16(1 - t^2/4)}}{dt} \right)^2}\, dt$$

$$= 4 \int_0^1 \sqrt{\frac{1 + 3x^2}{1 - x^2}}\, dx \qquad x = t/2$$

If y is the integrand, then we have the identity:

$$y^2(1 - x^2) = 1 + 3x^2$$

Apply the invertible change of variables:

$$\begin{cases} x = 1 - 2/t \\ y = \frac{u}{t-1} \end{cases}$$

Arrive to

$$u^2 = t^3 - 4t^2 + 6t - 3$$

What are Elliptic Curves?

Reasons to study them

Elliptic Curves

1. are curves and finite groups at the same time
2. are non singular projective curves of *genus* 1
3. have important applications in Algorithmic Number Theory and Cryptography
4. are the topic of the Birch and Swinnerton-Dyer conjecture (one of the seven Millennium Prize Problems)
5. have a group law that is a consequence of the fact that they intersect every line in exactly three points (in the projective plane over $\mathbb{C}$ and counted with multiplicity)
6. represent a mathematical world in itself ... Each of them does!!

Notations

Fields of characteristics 0

1. $\mathbb{Q}$ is the field of rational numbers
2. $\mathbb{R}$ and $\mathbb{C}$ are the fields of real and complex numbers
3. $K \subset \mathbb{C}$, $\dim_{\mathbb{Q}} K < \infty$ is a *number field*
   - $\mathbb{Q}[\sqrt{d}]$, $d \in \mathbb{Q}$
   - $\mathbb{Q}[\alpha]$, $f(\alpha) = 0$, $f \in \mathbb{Q}[X]$ irreducible

Finite fields

1. $\mathbb{F}_p = \{0, 1, \ldots, p-1\}$ is the prime field;
2. $\mathbb{F}_q$ is a finite field with $q = p^n$ elements
3. $\mathbb{F}_q = \mathbb{F}_p[\xi]$, $f(\xi) = 0$, $f \in \mathbb{F}_p[X]$ irreducible, $\partial f = n$
4. $\mathbb{F}_4 = \mathbb{F}_2[\xi]$, $\xi^2 = 1 + \xi$
5. $\mathbb{F}_8 = \mathbb{F}_2[\alpha]$, $\alpha^3 = \alpha + 1$ but also $\mathbb{F}_8 = \mathbb{F}_2[\beta]$, $\beta^3 = \beta^2 + 1$, $(\beta = \alpha^2 + 1)$
6. $\mathbb{F}_{101^{101}} = \mathbb{F}_{101}[\omega]$, $\omega^{101} = \omega + 1$

Notations

Algebraic Closure of $\mathbb{F}_q$

- $\mathbb{C} \supset \mathbb{Q}$ satisfies that *Fundamental Theorem of Algebra*! (i.e. $\forall f \in \mathbb{Q}[x], \partial f > 1, \exists \alpha \in \mathbb{C}, f(\alpha) = 0$)
- We need a field that plays the role, for $\mathbb{F}_q$, that $\mathbb{C}$ plays for $\mathbb{Q}$. It will be $\overline{\mathbb{F}}_q$, called *algebraic closure of* $\mathbb{F}_q$

1. $\forall n \in \mathbb{N}$, we fix an $\mathbb{F}_{q^n}$
2. We also require that $\mathbb{F}_{q^n} \subseteq \mathbb{F}_{q^m}$ if $n \mid m$
3. We let $\overline{\mathbb{F}}_q = \bigcup_{n \in \mathbb{N}} \mathbb{F}_{q^n}$

- **Fact:** $\overline{\mathbb{F}}_q$ is *algebraically closed*
  (i.e. $\forall f \in \mathbb{F}_q[x], \partial f > 1, \exists \alpha \in \overline{\mathbb{F}}_q, f(\alpha) = 0$)

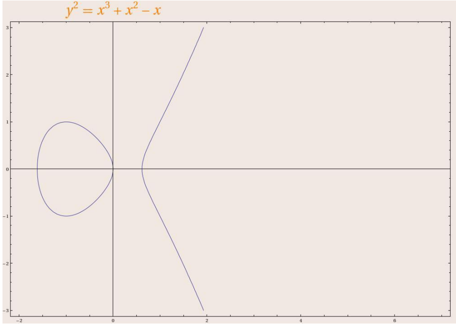
If $F(x, y) \in \mathbb{Q}[x, y]$ a *point of the curve* $F = 0$, means $(x_0, y_0) \in \mathbb{C}^2$ s.t. $F(x_0, y_0) = 0$.
If $F(x, y) \in \mathbb{F}_q[x, y]$ a *point of the curve* $F = 0$, means $(x_0, y_0) \in \overline{\mathbb{F}}_q^2$ s.t. $F(x_0, y_0) = 0$.

The (general) Weierstraß Equation

An elliptic curve E over a $\mathbb{F}_q$ (finite field) is given by an equation

$$E : y^2 + a_1 xy + a_3 y = x^3 + a_2 x^2 + a_4 x + a_6$$

where $a_1, a_3, a_2, a_4, a_6 \in \mathbb{F}_q$



The equation should not be *singular*

Tangent line to a plane curve

Given $f(x, y) \in \mathbb{F}_q[x, y]$ and a point (x_0, y_0) such that $f(x_0, y_0) = 0$, the *tangent line* is:

$$\frac{\partial f}{\partial x}(x_0, y_0)(x - x_0) + \frac{\partial f}{\partial y}(x_0, y_0)(y - y_0) = 0$$

If

$$\frac{\partial f}{\partial x}(x_0, y_0) = \frac{\partial f}{\partial y}(x_0, y_0) = 0,$$

such a tangent line cannot be computed and we say that (x_0, y_0) is *singular*

Definition

A non singular curve is a curve without any singular point

Example

The tangent line to $x^2 + y^2 = 1$ over $\mathbb{F}_7$ at $(2, 2)$ is

$$x + y = 4$$

Singular points

The classical definition

Definition

A *singular* point (x_0, y_0) on a curve $f(x, y) = 0$ is a point such that

$$\begin{cases} \frac{\partial f}{\partial x}(x_0, y_0) = 0 \\ \frac{\partial f}{\partial y}(x_0, y_0) = 0 \end{cases}$$

So, at a singular point there is no (unique) tangent line!! In the special case of Weierstraß equations:

$$E : y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$$

we have

$$\begin{cases} \partial_x = 0 \\ \partial_y = 0 \end{cases} \longrightarrow \begin{cases} a_1y = 3x^2 + 2a_2x + a_4 \\ 2y + a_1x + a_3 = 0 \end{cases}$$

We can express this condition in terms of the coefficients a_1, a_2, a_3, a_4, a_5.

The Discriminant of an Equation

The condition of absence of singular points in terms of a_1, a_2, a_3, a_4, a_6

With a bit of `Mathematica`

```
Ell:=-a_6-a_4x-a_2x^2-x^3+a_3y+a_1xy+y^2;
SS := Solve[{D[Ell,x]==0,D[Ell,y]==0},{y,x}];
Simplify[ReplaceAll[Ell,SS[[1]]]*ReplaceAll[Ell,SS[[2]]]]
```

we obtain

$$\Delta'_E := \frac{1}{2^4 3^3}\Big(-a_1^5 a_3 a_4 - 8a_1^3 a_2 a_3 a_4 - 16 a_1 a_2^2 a_3 a_4 + 36 a_1^2 a_3^2 a_4 - a_1^4 a_4^2 - 8 a_1^2 a_2 a_4^2 - 16 a_2^2 a_4^2 + 96 a_1 a_3 a_4^2 + 64 a_4^3 + a_1^6 a_6 + 12 a_1^4 a_2 a_6 + 48 a_1^2 a_2^2 a_6 + 64 a_2^3 a_6 - 36 a_1^3 a_3 a_6 - 144 a_1 a_2 a_3 a_6 - 72 a_1^2 a_4 a_6 - 288 a_2 a_4 a_6 + 432 a_6^2\Big)$$

Definition

The *discriminant* of a Weierstraß equation over $\mathbb{F}_q$, $q = p^n$, $p \geq 5$ is

$$\Delta_E := 3^3 \Delta'_E$$

The discriminant of $E/\mathbb{F}_{2^\alpha}$

$$E: y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6,\ a_i \in \mathbb{F}_{2^\alpha}$$

If $p = 2$, the singularity condition becomes:

$$\begin{cases} \partial_x = 0 \\ \partial_y = 0 \end{cases} \longrightarrow \begin{cases} a_1y = x^2 + a_4 \\ a_1x + a_3 = 0 \end{cases}$$

Classification of Weierstraß equations over $\mathbb{F}_{2^\alpha}$

```
El:=a6+a4x+a2x^2+x^3+a3y+a1xy+y^2;
Simplify[ReplaceAll[El,{x→a3/a1,y→((a3/a1)^2+a4)/a1}]]
```

- Case $a_1 \neq 0$:

  we obtain

$$\Delta_E := (a_1^6a_6 + a_1^5a_3a_4 + a_1^4a_2a_3^2 + a_1^4a_4^2 + a_1^3a_3^3 + a_3^4)/a_1^6$$

- Case $a_1 = 0$ and $a_3 \neq 0$: curve non singular ($\Delta_E := a_3$)
- Case $a_1 = 0$ and $a_3 = 0$: *curve singular* (x_0, y_0), $(x_0^2 = a_4, y_0^2 = a_2a_4 + a_6)$ singular point!

Special Weierstraß equation of $E/\mathbb{F}_{p^\alpha}$, $p \neq 2$

$$E : y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6 \quad a_i \in \mathbb{F}_{p^\alpha}$$

If we “complete the squares“ by applying the transformation:

$$\begin{cases} x \leftarrow x \\ y \leftarrow y - \frac{a_1x+a_3}{2} \end{cases}$$

the Weierstraß equation becomes:

$$E' : y^2 = x^3 + a_2'x^2 + a_4'x + a_6'$$

where $a_2' = a_2 + \frac{a_1^2}{4}$, $a_4' = a_4 + \frac{a_1a_3}{2}$, $a_6' = a_6 + \frac{a_3^2}{4}$
If $p \geq 5$, we can also apply the transformation

$$\begin{cases} x \leftarrow x - \frac{a_2'}{3} \\ y \leftarrow y \end{cases}$$

obtaining the equations:

$$E'' : y^2 = x^3 + a_4''x + a_6''$$

where $a_4'' = a_4' - \frac{{a_2'}^2}{3}$, $a_6'' = a_6' + \frac{2{a_2'}^3}{27} - \frac{a_2'a_4'}{3}$

Special Weierstraß equation for $E/\mathbb{F}_{2^\alpha}$

Case $a_1 \neq 0$

$$E : y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6 \qquad a_i \in \mathbb{F}_{2^\alpha}$$

$$\Delta_E := \frac{a_1^6a_6+a_1^5a_3a_4+a_1^4a_2a_3^2+a_1^4a_4^2+a_1^3a_3^3+a_3^4}{a_1^6}$$

If we apply the affine transformation:

$$\begin{cases} x \longleftarrow a_1^2x + a_3/a_1 \\ y \longleftarrow a_1^3y + (a_1^2a_4 + a_3^2)/a_1^2 \end{cases}$$

we obtain

$$E' : y^2 + xy = x^3 + \left(\frac{a_2}{a_1^2} + \frac{a_3}{a_1^3}\right)x^2 + \frac{\Delta_E}{a_1^6}$$

Surprisingly $\Delta_{E'} = \Delta_E/a_1^6$

With Mathematica

```
El:=a6+a4x+a2x^2+x^3+a3y+a1xy+y^2;
Simplify[PolynomialMod[ReplaceAll[El,
             {x->a1^2 x+a3/a1, y->a1^3y+(a1^2a4+a3^2)/a1^3}],2]]
```

Special Weierstraß equation for $E/\mathbb{F}_{2^\alpha}$

Case $a_1 = 0$ and $\Delta_E := a_3 \neq 0$

$$E : y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6 \qquad a_i \in \mathbb{F}_{2^\alpha}$$

If we apply the affine transformation:

$$\begin{cases} x \longleftarrow x + a_2 \\ y \longleftarrow y \end{cases}$$

we obtain

$$E : y^2 + a_3y = x^3 + (a_4 + a_2^2)x + (a_6 + a_2a_4)$$

With `Mathematica`

```
E1:=a6+a4x+a2x^2+x^3+a3y+y^2; Simplify[PolynomialMod[ReplaceAll[E1,{x->x+a2,y->y}],2]]
```

Definition

Two Weierstraß equations over $\mathbb{F}_q$ are said (affinely) equivalent if there exists a (affine) change of variables that takes one into the other

Exercise

Prove that necessarily the change of variables has form

$$\begin{cases} x \longleftarrow u^2x + r \\ y \longleftarrow u^3y + u^2sx + t \end{cases} \qquad r, s, t, u \in \mathbb{F}_q$$

The Weierstraß equation

Classification of simplified forms

After applying a suitable affine transformation we can always assume that $E/\mathbb{F}_q(q = p^n)$ has a Weierstraß equation of the following form

Example (Classification)

E	p	Δ_E
$y^2 = x^3 + Ax + B$	≥ 5	$4A^3 + 27B^2$
$y^2 + xy = x^3 + a_2x^2 + a_6$	2	a_6^2
$y^2 + a_3y = x^3 + a_4x + a_6$	2	a_3^4
$y^2 = x^3 + Ax^2 + Bx + C$	3	$4A^3C - A^2B^2 - 18ABC + 4B^3 + 27C^2$

Definition (Elliptic curve)

An elliptic curve is the data of a non singular Weierstraß equation (i.e. $\Delta_E \neq 0$)

Note: If $p \geq 3$, $\Delta_E \neq 0 \Leftrightarrow x^3 + Ax^2 + Bx + C$ has no double root

Elliptic curves over $\mathbb{F}_2$

All possible Weierstraß equations over $\mathbb{F}_2$ are:

Weierstraß equations over $\mathbb{F}_2$

1. $y^2 + xy = x^3 + x^2 + 1$
2. $y^2 + xy = x^3 + 1$
3. $y^2 + y = x^3 + x$
4. $y^2 + y = x^3 + x + 1$
5. $y^2 + y = x^3$
6. $y^2 + y = x^3 + 1$

However the change of variables $\begin{cases} x \leftarrow x + 1 \\ y \leftarrow y + x \end{cases}$ takes the sixth curve into the fifth. Hence we can remove the sixth from the list.

Fact:

There are 5 affinely inequivalent elliptic curves over $\mathbb{F}_2$

Elliptic curves in characteristic 3

Via a suitable transformation ($x \to u^2x + r$, $y \to u^3y + u^2sx + t$) over $\mathbb{F}_3$, 8 inequivalent elliptic curves over $\mathbb{F}_3$ are found:

Weierstraß equations over $\mathbb{F}_3$

1. $y^2 = x^3 + x$
2. $y^2 = x^3 - x$
3. $y^2 = x^3 - x + 1$
4. $y^2 = x^3 - x - 1$
5. $y^2 = x^3 + x^2 + 1$
6. $y^2 = x^3 + x^2 - 1$
7. $y^2 = x^3 - x^2 + 1$
8. $y^2 = x^3 - x^2 - 1$

Exercise: Prove that

1. Over $\mathbb{F}_5$ there are 12 elliptic curves
2. Compute all of them
3. How many are there over $\mathbb{F}_4$, over $\mathbb{F}_7$ and over $\mathbb{F}_8$?

The definition of $E(\mathbb{F}_q)$

Let $E/\mathbb{F}_q$ elliptic curve, $\infty := [0,1,0]$. Set

$$E(\mathbb{F}_q) = \{[X,Y,Z] \in \mathbb{P}_2(\mathbb{F}_q) : Y^2Z + a_1XYZ + a_3YZ^2 = X^3 + a_2X^2Z + a_4XZ^2 + a_6Z^3\}$$

or equivalently

$$E(\mathbb{F}_q) = \{(x,y) \in \mathbb{F}_q^2 : y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6\} \cup \{\infty\}$$

We can think either

- $E(\mathbb{F}_q) \subset \mathbb{P}_2(\mathbb{F}_q)$ $\dashrightarrow$ geometric advantages
- $E(\mathbb{F}_q) \subset \mathbb{F}_q^2 \cup \{\infty\}$ $\dashrightarrow$ algebraic advantages

∞ might be though as the "vertical direction"

Definition (line through points $P, Q \in E(\mathbb{F}_q)$)

$$r_{P,Q} : \begin{cases} \text{line through } P \text{ and } Q & \text{if } P \neq Q \\ \text{tangent line to } E \text{ at } P & \text{if } P = Q \end{cases}$$

projective or affine

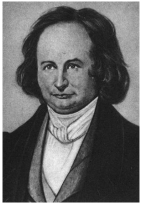
- if $\#(r_{P,Q} \cap E(\mathbb{F}_q)) \geq 2 \Rightarrow \#(r_{P,Q} \cap E(\mathbb{F}_q)) = 3$

if tangent line, contact point is counted with multiplicity

- $r_{\infty,\infty} \cap E(\mathbb{F}_q) = \{\infty, \infty, \infty\}$
- $r_{P,Q} : aX + bZ = 0$ (vertical) $\Rightarrow \infty = [0,1,0] \in r_{P,Q}$

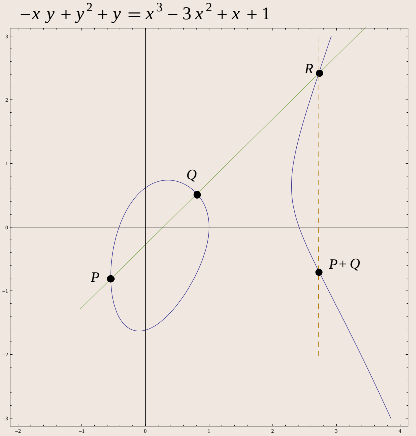
History (from Wikipedia)

Carl Gustav Jacob Jacobi (10/12/1804 – 18/02/1851) was a German mathematician, who made fundamental contributions to elliptic functions, dynamics, differential equations, and number theory.



Some of His Achievements:

- Theta and elliptic function
- Hamilton Jacobi Theory
- Inventor of determinants
- Jacobi Identity
  $[A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0$



$$r_{P,Q} \cap E(\mathbb{F}_q) = \{P, Q, R\}$$
$$r_{R,\infty} \cap E(\mathbb{F}_q) = \{\infty, R, R'\}$$

$$P +_E Q := R'$$

$$r_{P,\infty} \cap E(\mathbb{F}_q) = \{P, \infty, P'\}$$

$$-P := P'$$

Properties of the operation "$+_E$"

Theorem

The addition law on $E(\mathbb{F}_q)$ has the following properties:

(a) $P +_E Q \in E(\mathbb{F}_q)$ $\quad \forall P, Q \in E(\mathbb{F}_q)$

(b) $P +_E \infty = \infty +_E P = P$ $\quad \forall P \in E(\mathbb{F}_q)$

(c) $P +_E (-P) = \infty$ $\quad \forall P \in E(\mathbb{F}_q)$

(d) $P +_E (Q +_E R) = (P +_E Q) +_E R$ $\quad \forall P, Q, R \in E(\mathbb{F}_q)$

(e) $P +_E Q = Q +_E P$ $\quad \forall P, Q \in E(\mathbb{F}_q)$

- $(E(\mathbb{F}_q), +_E)$ commutative group
- All group properties are easy except associative law (d)
- Geometric proof of associativity uses *Pappo's Theorem*
- We shall comment on how to do it by explicit computation
- can substitute $\mathbb{F}_q$ with any field K; Theorem holds for $(E(K), +_E)$
- In particular, if $E/\mathbb{F}_q$, can consider the groups $E(\overline{\mathbb{F}}_q)$ or $E(\mathbb{F}_{q^n})$

Computing the inverse $-P$

$$E : y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$$

If $P = (x_1, y_1) \in E(\mathbb{F}_q)$

Definition: $-P := P'$ where $r_{P,\infty} \cap E(\mathbb{F}_q) = \{P, \infty, P'\}$

Write $P' = (x_1', y_1')$. Since $r_{P,\infty} : x = x_1 \Rightarrow x_1' = x_1$ and y_1 satisfies

$$y^2 + a_1x_1y + a_3y - (x_1^3 + a_2x_1^2 + a_4x_1 + a_6) = (y - y_1)(y - y_1')$$

So $y_1 + y_1' = -a_1x_1 - a_3$ (both coefficients of y) and

$$-P = -(x_1, y_1) = (x_1, -a_1x_1 - a_3 - y_1)$$

So, if $P_1 = (x_1, y_1), P_2 = (x_2, y_2) \in E(\mathbb{F}_q)$,

Definition: $P_1 +_E P_2 = -P_3$ where $r_{P_1,P_2} \cap E(\mathbb{F}_q) = \{P_1, P_2, P_3\}$

Finally, if $P_3 = (x_3, y_3)$, then

$$P_1 +_E P_2 = -P_3 = (x_3, -a_1x_3 - a_3 - y_3)$$

Lines through points of E

$$E : y^2 + a_1 xy + a_3 y = x^3 + a_2 x^2 + a_4 x + a_6$$

where $a_1, a_3, a_2, a_4, a_6 \in \mathbb{F}_q$,

$P_1 = (x_1, y_1), P_2 = (x_2, y_2) \in E(\mathbb{F}_q)$

$$\lambda = \frac{y_2 - y_1}{x_2 - x_1}, \qquad \nu = \frac{y_1 x_2 - x_1 y_2}{x_2 - x_1}$$

1. $P_1 \neq P_2$ and $x_1 \neq x_2 \implies r_{P_1,P_2} : y = \lambda x + \nu$
2. $P_1 \neq P_2$ and $x_1 = x_2 \implies r_{P_1,P_2} : x = x_1$
3. $P_1 = P_2$ and $2y_1 + a_1 x_1 + a_3 \neq 0 \implies r_{P_1,P_2} : y = \lambda x + \nu$

$$\lambda = \frac{3x_1^2 + 2a_2 x_1 + a_4 - a_1 y_1}{2y_1 + a_1 x_1 + a_3}, \nu = -\frac{a_3 y_1 + x_1^3 - a_4 x_1 - 2a_6}{2y_1 + a_1 x_1 + a_3}$$

4. $P_1 = P_2$ and $2y_1 + a_1 x_1 + a_3 = 0 \implies r_{P_1,P_2} : x = x_1$
5. $r_{P_1,\infty} : x = x_1$ $\qquad r_{\infty,\infty} : Z = 0$

Intersection between a line and E

We want to compute $P_3 = (x_3, y_3)$ where $r_{P_1,P_2} : y = \lambda x + \nu$,

$$r_{P_1,P_2} \cap E(\mathbb{F}_q) = \{P_1, P_2, P_3\}$$

We find the intersection:

$$r_{P_1,P_2} \cap E(\mathbb{F}_q) = \begin{cases} E : y^2 + a_1 xy + a_3 y = x^3 + a_2 x^2 + a_4 x + a_6 \\ r_{P_1,P_2} : y = \lambda x + \nu \end{cases}$$

Substituting

$$(\lambda x + \nu)^2 + a_1 x(\lambda x + \nu) + a_3(\lambda x + \nu) = x^3 + a_2 x^2 + a_4 x + a_6$$

Since x_1 and x_2 are solutions, we can find x_3 by comparing

$$\begin{aligned} x^3 + a_2 x^2 + a_4 x + a_6 - ((\lambda x + \nu)^2 + a_1 x(\lambda x + \nu) + a_3(\lambda x + \nu)) &= \\ x^3 + (a_2 - \lambda^2 - a_1 \lambda) x^2 + \cdots &= \\ (x - x_1)(x - x_2)(x - x_3) = x^3 - (x_1 + x_2 + x_3) x^2 + \cdots & \end{aligned}$$

Equating coeffcients of x^2,

$$x_3 = \lambda^2 - a_1 \lambda - a_2 - x_1 - x_2, \qquad y_3 = \lambda x_3 + \nu$$

Finally

$$P_3 = (\lambda^2 - a_1 \lambda - a_2 - x_1 - x_2, \lambda^3 - a_1 \lambda^2 - \lambda(a_2 + x_1 + x_2) + \nu)$$

Formulas for Addition on E (Summary)

$$E : y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$$

$P_1 = (x_1, y_1), P_2 = (x_2, y_2) \in E(\mathbb{F}_q) \setminus \{\infty\}$,

Addition Laws for the sum of affine points

- If $P_1 \neq P_2$
  - $x_1 = x_2$ $\Rightarrow$ $P_1 +_E P_2 = \infty$
  - $x_1 \neq x_2$

$$\lambda = \frac{y_2 - y_1}{x_2 - x_1} \qquad \nu = \frac{y_1x_2 - y_2x_1}{x_2 - x_1}$$

- If $P_1 = P_2$
  - $2y_1 + a_1x + a_3 = 0$ $\Rightarrow$ $P_1 +_E P_2 = 2P_1 = \infty$
  - $2y_1 + a_1x + a_3 \neq 0$

$$\lambda = \frac{3x_1^2 + 2a_2x_1 + a_4 - a_1y_1}{2y_1 + a_1x + a_3}, \nu = -\frac{a_3y_1 + x_1^3 - a_4x_1 - 2a_6}{2y_1 + a_1x_1 + a_3}$$

Then

$$P_1 +_E P_2 = (\lambda^2 - a_1\lambda - a_2 - x_1 - x_2, -\lambda^3 - a_1^2\lambda + (\lambda + a_1)(a_2 + x_1 + x_2) - a_3 - \nu)$$

Formulas for Addition on E (Summary for special equation)

$$E : y^2 = x^3 + Ax + B$$

$P_1 = (x_1, y_1), P_2 = (x_2, y_2) \in E(\mathbb{F}_q) \setminus \{\infty\}$,

Addition Laws for the sum of affine points

- If $P_1 \neq P_2$
  - $x_1 = x_2$ $\Rightarrow$ $P_1 +_E P_2 = \infty$
  - $x_1 \neq x_2$

$$\lambda = \frac{y_2 - y_1}{x_2 - x_1} \qquad \nu = \frac{y_1 x_2 - y_2 x_1}{x_2 - x_1}$$

- If $P_1 = P_2$
  - $y_1 = 0$ $\Rightarrow$ $P_1 +_E P_2 = 2P_1 = \infty$
  - $y_1 \neq 0$

$$\lambda = \frac{3x_1^2 + A}{2y_1}, \nu = -\frac{x_1^3 - Ax_1 - 2B}{2y_1}$$

Then

$$P_1 +_E P_2 = (\lambda^2 - x_1 - x_2, -\lambda^3 + \lambda(x_1 + x_2) - \nu)$$

A Finite Field Example

Over $\mathbb{F}_p$ geometric pictures don't make sense.

Example

Let $E : y^2 = x^3 - 5x + 8/\mathbb{F}_{37}$, $P = (6,3), Q = (9,10) \in E(\mathbb{F}_{37})$

$$r_{P,Q} : y = 27x + 26 \quad r_{P,P} : y = 11x + 11$$

$$r_{P,Q} \cap E(\mathbb{F}_{37}) = \begin{cases} y^2 = x^3 - 5x + 8 \\ y = 27x + 26 \end{cases} = \{(6,3),(9,10),(11,27)\}$$

$$r_{P,P} \cap E(\mathbb{F}_{37}) = \begin{cases} y^2 = x^3 - 5x + 8 \\ y = 11x + 11 \end{cases} = \{(6,3),(6,3),(35,26)\}$$

$$P +_E Q = (11,10) \qquad 2P = (35,11)$$

$3P = (34,25), 4P = (8,6), 5P = (16,19), \ldots 3P + 4Q = (31,28), \ldots$

Exercise

- Compute the order and the Group Structure of $E(\mathbb{F}_{37})$
- Show that if $E_1/\mathbb{F}_q$ is equivalent to $E_2/\mathbb{F}_q$, then $E_1(\mathbb{F}_{q^n}) \cong E_2(\mathbb{F}_{q^n}) \forall n \in \mathbb{N}$.

Group Structure

Theorem (Classification of finite abelian groups)

If G is abelian and finite, $\exists n_1, \ldots, n_k \in \mathbb{N}^{>1}$ such that

1. $n_1 \mid n_2 \mid \cdots \mid n_k$
2. $G \cong C_{n_1} \oplus \cdots \oplus C_{n_k}$

Furthermore $n_1, \ldots, n_k$ (Group Structure) are unique

Example (One can verify that:)

$$C_{2400} \oplus C_{72} \oplus C_{1440} \cong C_{12} \oplus C_{60} \oplus C_{15200}$$

Shall show that

$$E(\mathbb{F}_q) \cong C_n \oplus C_{nk} \qquad \exists n, k \in \mathbb{N}^{>0}$$

(i.e. $E(\mathbb{F}_q)$ is either cyclic ($n = 1$) or the product of 2 cyclic groups)

Proof of the associativity

$$P +_E (Q +_E R) = (P +_E Q) +_E R \quad \forall P, Q, R \in E$$

We should verify the above in many different cases according to if $Q = R$, $P = Q$, $P = Q +_E R, \ldots$
Here we deal with the *generic case*. i.e. All the points $\pm P, \pm R, \pm Q, \pm(Q +_E R), \pm(P +_E Q), \infty$ all different

Mathematica code

```
L[x_,y_,r_,s_]:=(s-y)/(r-x);
M[x_,y_,r_,s_]:=(yr-sx)/(r-x);
A[{x_,y_},{r_,s_}]:={(L[x,y,r,s])^2-(x+r),
              -(L[x,y,r,s])^3+L[x,y,r,s](x+r)-M[x,y,r,s]}
Together[A[A[{x,y},{u,v}],{h,k}]-A[{x,y},A[{u,v},{h,k}]]]
det = Det[({{1,x1,x1^3-y1^2},{1,x2,x2^3-y2^2},{1,x3,x3^3-y3^2}})]
PolynomialQ[Together[Numerator[Factor[res[[1]]]]/det],
                            {x1,x2,x3,y1,y2,y3}]
PolynomialQ[Together[Numerator[Factor[res[[2]]]]/det],
                            {x1,x2,x3,y1,y2,y3}]
```

- runs in 2 seconds on a PC
- For an elementary proof: "An Elementary Proof of the Group Law for Elliptic Curves." Department of Mathematics: Rice University. Web. 20 Nov. 2009.

  http://math.rice.edu/~friedl/papers/AAELLIPTIC.PDF
- More cases to check. e.g $P +_E 2Q = (P +_E Q) +_E Q$

EXAMPLE: Elliptic curves over $\mathbb{F}_2$

From our previous list:

Groups of points

E	$E(\mathbb{F}_2)$	$\lvert E(\mathbb{F}_2)\rvert$
$y^2 + xy = x^3 + x^2 + 1$	$\{\infty, (0,1)\}$	2
$y^2 + xy = x^3 + 1$	$\{\infty, (0,1), (1,0), (1,1)\}$	4
$y^2 + y = x^3 + x$	$\{\infty, (0,0), (0,1), (1,0), (1,1)\}$	5
$y^2 + y = x^3 + x + 1$	$\{\infty\}$	1
$y^2 + y = x^3$	$\{\infty, (0,0), (0,1)\}$	3

So for each curve $E(\mathbb{F}_2)$ is cyclic except possibly for the second for which we need to distinguish between C_4 and $C_2 \oplus C_2$.

Note: each C_i, $i = 1, \ldots, 5$ is represented by a curve $/\mathbb{F}_2$

EXAMPLE: Elliptic curves over $\mathbb{F}_3$

From our previous list:

Groups of points

i	E_i	$E_i(\mathbb{F}_3)$	$E_i(\mathbb{F}_3)$
1	$y^2 = x^3 + x$	$\{\infty, (0,0), (2,1), (2,2)\}$	C_4
2	$y^2 = x^3 - x$	$\{\infty, (1,0), (2,0), (0,0)\}$	$C_2 \oplus C_2$
3	$y^2 = x^3 - x + 1$	$\{\infty, (0,1), (0,2), (1,1), (1,2), (2,1), (2,2)\}$	C_7
4	$y^2 = x^3 - x - 1$	$\{\infty\}$	$\{1\}$
5	$y^2 = x^3 + x^2 - 1$	$\{\infty, (1,1), (1,2)\}$	C_3
6	$y^2 = x^3 + x^2 + 1$	$\{\infty, (0,1), (0,2), (1,0), (2,1), (2,2)\}$	C_6
7	$y^2 = x^3 - x^2 + 1$	$\{\infty, (0,1), (0,2), (1,1), (1,2), \}$	C_5
8	$y^2 = x^3 - x^2 - 1$	$\{\infty, (2,0))\}$	C_2

Note: each $C_i, i = 1, \ldots, 7$ is represented by a curve $/\mathbb{F}_3$

Exercise: let $\left(\frac{a}{q}\right)$ be the kronecker symbol. Show that the number of non–isomorphic (i.e. inequivalent) classes of elliptic curves over $\mathbb{F}_q$ is

$$2q + 3 + \left(\frac{-4}{q}\right) + 2\left(\frac{-3}{q}\right)$$

EXAMPLE: Elliptic curves over $\mathbb{F}_5$ and $\mathbb{F}_4$

$\forall E/\mathbb{F}_5$ (12 elliptic curves), $\#E(\mathbb{F}_5) \in \{2,3,4,5,6,7,8,9,10\}$. $\forall n, 2 \leq n \leq 10 \exists! E/\mathbb{F}_5 : \#E(\mathbb{F}_5) = n$ with the exceptions:

Example (Elliptic curves over $\mathbb{F}_5$)

- $E_1 : y^2 = x^3 + 1$ and $E_2 : y^2 = x^3 + 2$ both order 6

$$\begin{cases} x \longleftarrow 2x \\ y \longleftarrow \sqrt{3}y \end{cases}$$

E_1 and E_2 affinely equivalent over $\mathbb{F}_5[\sqrt{3}] = \mathbb{F}_{25}$ (*twists*)

- $E_3 : y^2 = x^3 + x$ and $E_4 : y^2 = x^3 + x + 2$ order 4

$$E_3(\mathbb{F}_5) \cong C_2 \oplus C_2 \qquad E_4(\mathbb{F}_5) \cong C_4$$

- $E_5 : y^2 = x^3 + 4x$ and $E_6 : y^2 = x^3 + 4x + 1$ both order 8

$$E_5(\mathbb{F}_5) \cong C_2 \times \oplus C_4 \qquad E_6(\mathbb{F}_5) \cong C_8$$

- $E_7 : y^2 = x^3 + x + 1$ order 9 and $E_7(\mathbb{F}_5) \cong C_9$

Exercise: Classify all elliptic curves over $\mathbb{F}_4 = \mathbb{F}_2[\xi], \xi^2 = \xi + 1$

Further Reading...

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