# Lecture 3

## Elliptic curves over finite fields

### The group order

**Research School: Algebraic curves over finite fields**

*CIMPA-ICTP-UNESCO-MESR-MINECO-PHILIPPINES*

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# The division polynomials

**Definition (Division Polynomials of $E: y^2 = x^3 + Ax + B$ ($p > 3$))**

$$\begin{aligned}
\psi_0 =& 0, \psi_1 = 1, \psi_2 = 2y \\
\psi_3 =& 3x^4 + 6Ax^2 + 12Bx - A^2 \\
\psi_4 =& 4y(x^6 + 5Ax^4 + 20Bx^3 - 5A^2x^2 - 4ABx - 8B^2 - A^3) \\
& \vdots \\
\psi_{2m+1} =& \psi_{m+2}\psi_m^3 - \psi_{m-1}\psi_{m+1}^3 \qquad \text{for } m \geq 2 \\
\psi_{2m} =& \left(\frac{\psi_m}{2y}\right) \cdot (\psi_{m+2}\psi_{m-1}^2 - \psi_{m-2}\psi_{m+1}^2) \qquad \text{for } m \geq 3
\end{aligned}$$

The polynomial $\psi_m \in \mathbb{Z}[x, y]$ is the $m^{\text{th}}$ *division polynomial*

**Theorem ($E: Y^2 = X^3 + AX + B$ elliptic curve, $P = (x, y) \in E$)**

$$mP = m(x, y) = \left(\frac{\phi_m(x)}{\psi_m^2(x)}, \frac{\omega_m(x,y)}{\psi_m^3(x,y)}\right),$$

*where* $\phi_m = x\psi_m^2 - \psi_{m+1}\psi_{m-1}, \omega_m = \frac{\psi_{m+2}\psi_{m-1}^2 - \psi_{m-2}\psi_{m+1}^2}{4y}$

# Points of order $m$

## Definition ($m$–torsion point)

Let $E/K$ and let $\bar{K}$ an *algebraic closure of* $K$.

$$E[m] = \{P \in E(\bar{K}) : \ mP = \infty\}$$

## Theorem (Structure of Torsion Points)

*Let $E/K$ and $m \in \mathbb{N}$. If $p = \mathrm{char}(K) \nmid m$,*

$$E[m] \cong C_m \oplus C_m$$

*If $m = p^r m', p \nmid m'$,*

$$E[m] \cong C_m \oplus C_{m'} \qquad \text{or} \qquad E[m] \cong C_{m'} \oplus C_{m'}$$

## Idea of the proof:

Let $[m] : E \to E, P \mapsto mP$. Then

$$\#E[m] = \#\mathrm{Ker}[\mathrm{m}] \leq \partial\phi_m = m^2$$

equality holds iff $p \nmid m$.

## Remark.

- $E[2m+1] \setminus \{\infty\} = \{(x,y) \in E(\bar{K}) : \ \psi_{2m+1}(x) = 0\}$
- $E[2m] \setminus E[2] = \{(x,y) \in E(\bar{K}) : \ y^{-1}\psi_{2m}(x) = 0\}$

## Example

$$
\begin{aligned}
\psi_4(x) =& 2y(x^6 + 5Ax^4 + 20Bx^3 - 5A^2x^2 - 4BAx + \left(-A^3 - 8B^2\right)) \\
\psi_5(x) =& 5x^{12} + 62Ax^{10} + 380Bx^9 - 105A^2x^8 + 240BAx^7 \\
&+ \left(-300A^3 - 240B^2\right)x^6 - 696BA^2x^5 \\
&+ \left(-125A^4 - 1920B^2A\right)x^4 + \left(-80BA^3 - 1600B^3\right)x^3 \\
&+ \left(-50A^5 - 240B^2A^2\right)x^2 + \left(-100BA^4 - 640B^3A\right)x \\
&+ \left(A^6 - 32B^2A^3 - 256B^4\right) \\
\psi_6(x) =& 2y(6x^{16} + 144Ax^{14} + 1344Bx^{13} - 728A^2x^{12} + \left(-2576A^3 - 5376B^2\right)x^{10} \\
&- 9152BA^2x^9 + \left(-1884A^4 - 39744B^2A\right)x^8 + \left(1536BA^3 - 44544B^3\right)x^7 \\
&+ \left(-2576A^5 - 5376B^2A^2\right)x^6 + \left(-6720BA^4 - 32256B^3A\right)x^5 \\
&+ \left(-728A^6 - 8064B^2A^3 - 10752B^4\right)x^4 + \left(-3584BA^5 - 25088B^3A^2\right)x^3 \\
&+ \left(144A^7 - 3072B^2A^4 - 27648B^4A\right)x^2 \\
&+ \left(192BA^6 - 512B^3A^3 - 12288B^5\right)x + \left(6A^8 + 192B^2A^5 + 1024B^4A^2\right))
\end{aligned}
$$

# Group Structure of $E(\mathbb{F}_q)$

## Exercise

Use division polynomials in Sage to write a list of all curves $E$ over $\mathbb{F}_{103}$ such that $E(\mathbb{F}_{103}) \supset E[6]$. Do the same for curves over $\mathbb{F}_{5^4}$.

## Corollary (Corollary of the Theorem of Structure for torsion)

*Let $E/\mathbb{F}_q$. $\exists n, k \in \mathbb{N}$ are such that*

$$E(\mathbb{F}_q) \cong C_n \oplus C_{nk}$$

## Theorem

*Let $E/\mathbb{F}_q$ and $n, k \in \mathbb{N}$ such that $E(\mathbb{F}_q) \cong C_n \oplus C_{nk}$. Then $n \mid q - 1$.*

# Weil Pairing

Let $E/K$ and $m \in \mathbb{N}$ s.t. $p \nmid m$. Then

$$E[m] \cong C_m \oplus C_m$$

We set

$$\mu_m := \{x \in \bar{K} : x^m = 1\}$$

$\mu_m$ is a cyclic group with $m$ elements(since $p \nmid m$)

## Theorem (Existence of Weil Pairing)

*There exists a pairing* $e_m : E[m] \times E[m] \to \mu_m$ *called Weil Pairing, s.t.* $\forall P, Q \in E[m]$

1. $e_m(P +_E Q, R) = e_m(P, R) e_m(Q, R)$ *(bilinearity)*
2. $e_m(P, R) = 1 \forall R \in E[m] \Rightarrow P = \infty$ *(non degeneracy)*
3. $e_m(P, P) = 1$
4. $e_m(P, Q) = e_m(Q, P)^{-1}$
5. $e_m(\sigma P, \sigma Q) = \sigma e_m(P, Q) \; \forall \sigma \in \mathrm{Gal}(\bar{K}/K)$
6. $e_m(\alpha(P), \alpha(Q)) = e_m(P, Q)^{\deg \alpha} \; \forall \alpha$ *separable endomorphism*

The last one needs to be discussed further!!!

# Properties of Weil pairing

❶ $E[m] \cong C_m \oplus C_m \;\Rightarrow\; E[m]$ has a $\mathbb{Z}/m\mathbb{Z}$–*basis*

i.e. $\exists P, Q \in E[m] : \forall R \in E[m], \exists! \alpha, \beta \in \mathbb{Z}/m\mathbb{Z}, R = \alpha P + \beta Q$

❷ If $(P, Q)$ is a $\mathbb{Z}/m\mathbb{Z}$–basis, then $\zeta = e_m(P, Q) \in \mu_m$ is *primitive* (i.e. $\operatorname{ord} \zeta = m$)

**Proof.** Let $d = \operatorname{ord} \zeta$. Then $1 = e_m(P, Q)^d = e_m(P, dQ)$.
$\forall R \in E[m]$, $e_m(R, dQ) = e_m(P, dQ)^\alpha e_m(Q, Q)^{d\beta} = 1$.
So $dQ = \infty \;\Rightarrow\; m \mid d$.

❸ $E[m] \subset E(K) \;\Rightarrow\; \mu_m \subset K$

**Proof.** Let $\sigma \in \operatorname{Gal}(\bar{K}/K)$ since the basis $(P, Q) \subset E(K)$,
$\sigma(P) = P$, $\sigma(Q) = Q$. Hence
$\zeta = e_m(P, Q) = e_m(\sigma P, \sigma Q) = \sigma e_m(P, Q) = \sigma\zeta$
So $\zeta \in \bar{K}^{\operatorname{Gal}(\bar{K}/K)} = K \;\Rightarrow\; \mu_n = \langle\zeta\rangle \subset K^*$

❹ if $E(\mathbb{F}_q) \cong C_n \oplus C_{kn} \;\Rightarrow q \equiv 1 \bmod n$

**Proof.** $E[n] \subset E(\mathbb{F}_q) \Rightarrow \mu_n \subset \mathbb{F}_q^* \Rightarrow n \mid q - 1$

❺ If $E/\mathbb{Q} \;\Rightarrow\; E[m] \not\subseteq E(\mathbb{Q})$ for $m \geq 3$

# Endomorphisms

## Definition

A map $\alpha : E(\bar{K}) \to E(\bar{K})$ is called an endomorphism if

- $\alpha(P +_E Q) = \alpha(P) +_E \alpha(Q)$ ($\alpha$ is a group homomorphism)
- $\exists R_1, R_2 \in \bar{K}(x,y)$ s.t.
  $\alpha(x,y) = (R_1(x,y), R_2(x,y)) \qquad \forall (x,y) \notin \mathrm{Ker}(\alpha)$

($\bar{K}(x,y)$ is the field of *rational functions*, $\alpha(\infty) = \infty$ )

## Exercise (Show that we can always assume)

$$\alpha(x,y) = (r_1(x), y r_2(x)), \qquad \exists r_1, r_2 \in \bar{K}(x)$$

**Hint:** use $y^2 = x^3 + Ax + B$ and $\alpha(-(x,y)) = -\alpha(x,y)$,

## Remarks/Examples:

- if $r_1(x) = p(x)/q(x)$ with $\gcd(p,q) = 1$ and $(x_0, y_0) \in E(\bar{K})$ with $q(x_0) = 0 \Rightarrow \alpha(x_0, y_0) = \infty$
- $[m](x,y) = \left(\frac{\phi_m}{\psi_m^2}, \frac{\omega_m}{\psi_m^3}\right)$ is an endomorphism $\forall m \in \mathbb{Z}$
- $\Phi_q : E(\bar{\mathbb{F}}_q)) \to E(\bar{\mathbb{F}}_q)), (x,y) \mapsto (x^q, y^q)$ is called *Frobenius Endomorphism*

# Endomorphisms (continues)

**Theorem**

*If $\alpha \neq [0]$ is an endomorphism, then it is surjective.*

**Sketch of the proof.**

Assume $p > 3$, $\alpha(x, y) = (p(x)/q(x), yr_2(x))$ and $(a, b) \in E(\bar{K})$.

- If $p(x) - aq(x)$ is not constant, let $x_0$ be one of its roots. Choose $y_0$ a square root of $x_0^2 + AX_0 + B$.

  Then either $\alpha(x_0, y_0) = (a, b)$ or $\alpha(x_0, -y_0) = (a, b)$.

- If $p(x) - aq(x)$ is constant,

  this happens only for one value of $a$!

  Let $(a_1, b_1) \in E(\bar{K})$:
  $(a_1, b_1) \neq (a, \pm b)$ and $(a_1, b_1) +_E (a, b) \neq (a, \pm b)$.

  Then $(a_1, b_1) = \alpha(P_1)$ and $(a_1, b_1) +_E (a, b) = \alpha(P_2)$

  Finally $(a, b) = \alpha(P_2 - P_1)$

□

# Endomorphisms (continues)

**Definition**

Suppose $\alpha : E \to E, (x, y) = (r_1(x), yr_2(x))$ endomorphism.
Write $r_1(x) = p(x)/q(x)$ with $\gcd(p(x), q(x)) = 1$.

- The **degree** of $\alpha$ is $\deg \alpha := \max\{\deg p, \deg q\}$
- $\alpha$ is said **separable** if $(p'(x), q'(x)) \neq (0, 0)$ (identically)

**Lemma**

- *$\Phi_q(x, y) = (x^q, y^q)$ is a non separable endomorphism of degree $q$*
- *$[m](x, y) = \left(\frac{\phi_m}{\psi_m^2}, \frac{\omega_m}{\psi_m^3}\right)$ has degree $m^2$*
- *$[m]$ separable iff $p \nmid m$.*

**Proof.**

*First:* Use the fact that $x \mapsto x^q$ is the identity on $\mathbb{F}_q$ hence it fixes the coefficients of the Weierstraß equation. *Second:* already done. *Third* See [8, Proposition 2.28] □

# Endomorphisms (continues)

**Theorem**

*Let $\alpha \neq 0$ be an endomorphism. Then*

$$\#\,\mathrm{Ker}(\alpha) \begin{cases} = \deg\alpha & \text{if } \alpha \text{ is separable} \\ < \deg\alpha & \text{otherwise} \end{cases}$$

**Proof.**

It is same proof as $\#E[m] = \#\,\mathrm{Ker}[m] \leq \partial\phi_m = m^2$ (equality for $p \nmid m$) □

**Definition**

Let $E/K$. The *ring of endomorphisms*

$$\mathrm{End}(E) := \{\alpha : E \to E, \alpha \text{ is an endomorphism}\}.$$

where for all $\alpha_1, \alpha_2 \in \mathrm{End}(E)$,

- $(\alpha_1 + \alpha_2)P := \alpha_1(P) +_E \alpha_2(P)$
- $(\alpha_1\alpha_2)P = \alpha_1(\alpha_2(P))$

# Endomorphisms (continues)

**Properties of End$(E)$:**

- $[0]: P \mapsto \infty$ is the zero element
- $[1]: P \mapsto P$ is the identity element
- $\mathbb{Z} \hookrightarrow \mathrm{End}(E)$, $m \mapsto [m]$
- $\mathrm{End}(E)$ is not necessarily commutative
- if $K = \mathbb{F}_q$, $\Phi_q \in \mathrm{End}(E)$. So $\mathbb{Z}[\Phi_q] \subset \mathrm{End}(E)$

Recall that $\alpha \in \mathrm{End}(E)$ is said **separable** if $(p'(x), q'(x)) \neq (0,0)$ where $\alpha(x,y) = (p(x)/q(x), yr(x))$.

**Lemma**

*Let $\Phi_q : (x,y) \mapsto (x^q, y^q)$ be the Frobenius endomorphism and let $r, s \in \mathbb{Z}$. Then*

$$r\Phi_q + s \in \mathrm{End}(E) \text{ is separable} \Leftrightarrow p \nmid s$$

**Proof.**

See [8, Proposition 2.29] □

Recall that the **degree** if $\alpha$ is $\deg \alpha := \max\{\deg p, \deg q\}$ where $\alpha(x, y) = (p(x)/q(x), yr(x))$.

**Lemma**

$\forall r, s \in \mathbb{Z}$ *and* $\forall \alpha, \beta \in \mathrm{End}(E)$,

$$\deg(r\alpha + s\beta) = r^2 \deg \alpha + s^2 \deg \beta + rs(\deg(\alpha + \beta) - \deg \alpha - \deg \beta)$$

**Proof.**

Let $m \in \mathbb{N}$ with $p \nmid m$ and fix a basis $P, Q$ of $E[m] \cong C_m \oplus C_m$. Then $\alpha(P) = aP + bQ$ and $\alpha(Q) = cP + dQ$ with

$$\alpha_m = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ with entries in } \mathbb{Z}/m\mathbb{Z}.$$

We claim that $\deg(\alpha) \equiv \det \alpha_m \bmod m$. In fact if $\zeta = e_m(P, Q)$ is the Weil pairing (primitive root).

$$\zeta^{\deg(\alpha)} = e_m(\alpha(P), \alpha(Q)) = e_m(aP + bQ, cP + dQ) = \zeta^{ad-bc}$$

So $\deg(\alpha) \equiv ad - bc = \det \alpha_m (\bmod m)$. A calculation shows

$$\det(r\alpha_m + s\beta_m) = r^2 \det \alpha_m + s^2 \det \beta_m + rs \det(\alpha_m + \beta_m) - \det \alpha_m - \det \beta_m)$$

So $\quad \deg(r\alpha + s\beta) \equiv r^2 \deg \alpha + s^2 \deg \beta + rs \deg(\alpha + \beta) - \deg \alpha - \deg \beta \bmod m$

Since it holds for $\infty$–many $m$'s the above is an equality. □

## Theorem (Hasse)

*Let $E$ be an elliptic curve over the finite field $\mathbb{F}_q$. Then the order of $E(\mathbb{F}_q)$ satisfies*

$$|q + 1 - \#E(\mathbb{F}_q)| \leq 2\sqrt{q}.$$

So $\#E(\mathbb{F}_q) \in [(\sqrt{q} - 1)^2, (\sqrt{q} + 1)^2]$ the *Hasse interval* $\mathcal{I}_q$

## Example (Hasse Intervals)

| $q$ | $\mathcal{I}_q$ |
|---|---|
| 2 | $\{1, 2, 3, 4, 5\}$ |
| 3 | $\{1, 2, 3, 4, 5, 6, 7\}$ |
| 4 | $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ |
| 5 | $\{2, 3, 4, 5, 6, 7, 8, 9, 10\}$ |
| 7 | $\{3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13\}$ |
| 8 | $\{4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14\}$ |
| 9 | $\{4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16\}$ |
| 11 | $\{6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18\}$ |
| 13 | $\{7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21\}$ |
| 16 | $\{9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 25\}$ |
| 17 | $\{10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26\}$ |
| 19 | $\{12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28\}$ |
| 23 | $\{15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33\}$ |
| 25 | $\{16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36\}$ |
| 27 | $\{18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38\}$ |
| 29 | $\{20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40\}$ |
| 31 | $\{21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43\}$ |
| 32 | $\{22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44\}$ |

## The Frobenius endomorphism $\Phi_q$

$$\Phi_q : \bar{\mathbb{F}}_q \to \bar{\mathbb{F}}_q, x \mapsto x^q \text{ is a field automorphism}$$

Given $\alpha \in \bar{\mathbb{F}}_q$,

$$\alpha \in \mathbb{F}_{q^n} \Leftrightarrow \Phi_q^n(\alpha) = \alpha^{q^n} = \alpha$$

Fixed points of powers of $\Phi_q$ are exactly elements of $\mathbb{F}_{q^n}$

$$\Phi_q : E(\bar{\mathbb{F}}_q) \to E(\bar{\mathbb{F}}_q), (x, y) \mapsto (x^q, y^q), \infty \mapsto \infty$$

### Properties of $\Phi_q$

- $\Phi_q \in \text{End}(E)$, it is not separable and has degree $q$
- $\Phi_q(x, y) = (x, y) \iff (x, y) \in E(\mathbb{F}_q)$
- $\text{Ker}(\Phi_q - 1) = E(\mathbb{F}_q)$
- $\# \text{Ker}(\Phi_q - 1) = \deg(\Phi_q - 1)$ (since $\Phi_q - 1$ is separable)
- if we can compute $\deg(\Phi_q - 1)$, we can compute $\# E(\mathbb{F}_q)$
- $\Phi_q^n(x, y) = (x^{q^n}, y^{q^n})$ so $\Phi_q^n(x, y) = (x, y) \Leftrightarrow (x, y) \in \mathbb{F}_{q^n}$
- $\text{Ker}(\Phi_q^n - 1) = E(\mathbb{F}_{q^n})$

# Proof of Hasse's Theorem

**Lemma**

*Let $E/\mathbb{F}_q$ and write $a = q+1-\#E(\mathbb{F}_q) = q+1-\deg(\Phi_q - 1)$. Then $\forall r, s \in \mathbb{Z}$, $\gcd(q,s) = 1$,*

$$\deg(r\phi + s) = r^2 q + s^2 - rsa$$

**Proof.**

Proof of the Lemma From a previous proposition, we know that

$\deg(r\Phi_q + s) = r^2\deg(\Phi_q) + s^2\deg([-1]) - rs(\deg(\Phi_q - 1) - \deg(\Phi_q) - \deg([-1]))$

But

$$\deg(\Phi_q) = q, \deg([-1]) = 1 \text{ and } \deg(\Phi_q - 1) - q - 1 = -a$$

□

**Proof of Hasse's Theorem.**

$$q\left(\frac{r}{s}\right)^2 - a\left(\frac{r}{s}\right) + 1 = \frac{\deg(r\Phi_q + s)}{s^2} \geq 0$$

on a dense set of rational numbers.

This implies $\forall X \in \mathbb{R}$, $X^2 - aX + q \geq 0$.So

$$a^2 - 4q \leq 0 \iff |a| \leq 2\sqrt{q}!!$$

□

# Proof of Hasse's Theorem (continues)

**Ingredients for the proof:**

1. $E(\mathbb{F}_q) = \mathrm{Ker}(\Phi_q - 1)$
2. $\Phi_q - 1$ is separable
3. $\#\mathrm{Ker}(\Phi_q - 1) = \deg(\Phi_q - 1)$

**Corollary**

*Let $a = q + 1 - \#E(\mathbb{F}_q)$. Then*

1. $$\Phi_q^2 - a\Phi_q + q = 0$$ *is an identity of endomorphisms.*
2. *$a \in \mathbb{Z}$ is the unique integer $k$ such that $\Phi_q^2 - k\Phi_q + q = 0$*
3. $$a \equiv \mathrm{Tr}((\Phi_q)_m) \bmod m \; \forall m \text{ s.t. } \gcd(m, q) = 1$$

## Sketch of the Proof of Corollary.

Let $m \in \mathbb{N}$ s.t. $\gcd(m, q) = 1$. Choose a basis for $E[m]$ and write

$$(\Phi_q)_m = \begin{pmatrix} s & t \\ u & v \end{pmatrix}$$

$\Phi_q - 1$ separable implies

$$\begin{aligned} \# \mathrm{Ker}(\Phi_q - 1) = \deg(\Phi_q - 1) &\equiv \det((\Phi_q)_m - I)) \\ &= \det((\Phi_q)_m) - \mathrm{Tr}((\Phi_q)_m) + 1 (\bmod m). \end{aligned}$$

Hence

$$\mathrm{Tr}((\Phi_q)_m) \equiv a (\bmod m)$$

By Cayley–Hamilton

$$(\Phi_q)_m^2 - a(\Phi_q)_m + qI \equiv 0 (\bmod m)$$

Since this happens for infinitely many $m$'s,

$$\Phi_q^2 - a\Phi_q + q = 0$$

as endomorphism. □

# Subfield curves (continues)

### Definition

Let $E/\mathbb{F}_q$ and write $E(\mathbb{F}_q) = q + 1 - a$, $(|a| \leq 2\sqrt{q})$. The *characteristic* polynomial of $E$ is

$$P_E(T) = T^2 - aT + q \in \mathbb{Z}[T].$$

and its roots:

$$\alpha = \frac{1}{2}\left(a + \sqrt{a^2 - 4q}\right) \qquad \beta = \frac{1}{2}\left(a - \sqrt{a^2 - 4q}\right)$$

are called *characteristic roots of Frobenius* ($P_E(\Phi_q) = 0$).

### Theorem

$\forall n \in \mathbb{N}$

$$\#E(\mathbb{F}_{q^n}) = q^n + 1 - (\alpha^n + \beta^n).$$

## Subfield curves (continues)

**Theorem**

$\forall n \in \mathbb{N}\ \#E(\mathbb{F}_{q^n}) = q^n + 1 - (\alpha^n + \beta^n).$

**Proof.**

Note that

1. Result is true for $n = 1$, $\alpha + \beta = a$
2. $\alpha^n + \beta^n \in \mathbb{Z}$, $(\alpha\beta)^n = q^n$
3. $f(X) = (X^n - \alpha^n)(X^n - \beta^n) = X^{2n} - (\alpha^n + \beta^n)X^n + q^n \in \mathbb{Z}[X]$
4. $f(X)$ is divisible by $X^2 - aX + q = (X - \alpha)(X - \beta)$
5. $(\Phi_q)^n|_{\bar{\mathbb{F}}_{q^n}} = \Phi_{q^n} : (x, y) \mapsto (x^{q^n}, y^{q^n})$
6. $(\Phi_q^n)^2 - (\alpha^n + \beta^n)\Phi_q^n + q^n = Q(\Phi_q))(\Phi_q^2 - a\Phi_q + q) = 0$
   where $f(X) = Q(X)(X^2 - aX + q)$

Hence $\Phi_q^n$ satisfies

$$X^2 - ((\alpha^n + \beta^n))X + q.$$

So

$$\alpha^n + \beta^n = q^n + 1 - \#E(\mathbb{F}_{q^n}).$$

Characteristic polynomial of $\Phi_{q^n}$: $X^2 - (\alpha^n + \beta^n)X + q^n$ □

# Subfield curves (continues)

$$E(\mathbb{F}_q) = q + 1 - a \;\Rightarrow\; E(\mathbb{F}_{q^n}) = q^n + 1 - (\alpha^n + \beta^n)$$

where $P_E(T) = T^2 - aT + q = (T - \alpha)(T - \beta) \in \mathbb{Z}[T]$

## Curves $/\mathbb{F}_2$

| $E$ | $a$ | $P_E(T)$ | $(\alpha, \beta)$ |
|---|---|---|---|
| $y^2 + xy = x^3 + x^2 + 1$ | $1$ | $T^2 - T + 2$ | $\frac{1}{2}(1 \pm \sqrt{-7})$ |
| $y^2 + xy = x^3 + 1$ | $-1$ | $T^2 + T + 2$ | $\frac{1}{2}(-1 \pm \sqrt{-7})$ |
| $y^2 + y = x^3 + x$ | $-2$ | $T^2 + 2T + 2$ | $-1 \pm i$ |
| $y^2 + y = x^3 + x + 1$ | $2$ | $T^2 - 2T + 2$ | $1 \pm i$ |
| $y^2 + y = x^3$ | $0$ | $T^2 + 2$ | $\pm\sqrt{-2}$ |

$E : y^2 + xy = x^3 + x^2 + 1 \;\Rightarrow$

$$E(\mathbb{F}_{2^{100}}) = 2^{100} + 1 - \left(\frac{1+\sqrt{-7}}{2}\right)^{100} - \left(\frac{1-\sqrt{-7}}{2}\right)^{100} = 1267650600228229382588845215376$$

# Subfield curves

$$E(\mathbb{F}_q) = q + 1 - a \Rightarrow E(\mathbb{F}_{q^n}) = q^n + 1 - (\alpha^n + \beta^n)$$

where $P_E(T) = T^2 - aT + q = (T - \alpha)(T - \beta) \in \mathbb{Z}[T]$

## Curves $/\mathbb{F}_2$

| $i$ | $E_i$ | $a$ | $P_{E_i}(T)$ | $(\alpha, \beta)$ |
|---|---|---|---|---|
| 1 | $y^2 = x^3 + x$ | 0 | $T^2 + 3$ | $\pm\sqrt{-3}$ |
| 2 | $y^2 = x^3 - x$ | 0 | $T^2 + 3$ | $\pm\sqrt{-3}$ |
| 3 | $y^2 = x^3 - x + 1$ | $-3$ | $T^2 + 3T + 3$ | $\frac{1}{2}(-3 \pm \sqrt{-3})$ |
| 4 | $y^2 = x^3 - x - 1$ | 3 | $T^2 - 3T + 3$ | $\frac{1}{2}(3 \pm \sqrt{-3})$ |
| 5 | $y^2 = x^3 + x^2 - 1$ | 1 | $T^2 - T + 3$ | $\frac{1}{2}(1 \pm \sqrt{-11})$ |
| 6 | $y^2 = x^3 - x^2 + 1$ | $-1$ | $T^2 + T + 3$ | $\frac{1}{2}(-1 \pm \sqrt{-11})$ |
| 7 | $y^2 = x^3 + x^2 + 1$ | $-2$ | $T^2 + 2T + 3$ | $-1 \pm \sqrt{-2}$ |
| 8 | $y^2 = x^3 - x^2 - 1$ | 2 | $T^2 - 2T + 3$ | $1 \pm \sqrt{-2}$ |

## Lemma

*Let $s_n = \alpha^n + \beta^n$ where $\alpha\beta = q$ and $\alpha + \beta = a$. Then*

$$s_0 = 2, \quad , s_1 = a \quad \text{and} \quad s_{n+1} = as_n - qs_{n-1}$$

## Legendre Symbols

Recall the *Finite field Legendre symbols*: let $x \in \mathbb{F}_q$,

$$\left(\frac{x}{\mathbb{F}_q}\right) = \begin{cases} +1 & \text{if } t^2 = x \text{ has a solution } t \in \mathbb{F}_q^* \\ -1 & \text{if } t^2 = x \text{ has no solution } t \in \mathbb{F}_q \\ 0 & \text{if } x = 0 \end{cases}$$

**Theorem**

*Let* $E : y^2 = x^3 + Ax + B$ *over* $\mathbb{F}_q$. *Then*

$$\#E(\mathbb{F}_q) = q + 1 + \sum_{x \in \mathbb{F}_q} \left(\frac{x^3+Ax+B}{\mathbb{F}_q}\right)$$

**Proof.**

Note that

$$1 + \left(\frac{x_0^3+Ax_0+B}{\mathbb{F}_q}\right) = \begin{cases} 2 & \text{if } \exists y_0 \in \mathbb{F}_q^* \text{ s.t. } (x_0, \pm y_0) \in E(\mathbb{F}_q) \\ 1 & \text{if } (x_0, 0) \in E(\mathbb{F}_q) \\ 0 & \text{otherwise} \end{cases}$$

Hence

$$\#E(\mathbb{F}_q) = 1 + \sum_{x \in \mathbb{F}_q} \left(1 + \left(\frac{x^3+Ax+B}{\mathbb{F}_q}\right)\right)$$

□

# Last Slide

## Corollary

*Let* $E : y^2 = x^3 + Ax + B$ *over* $\mathbb{F}_q$ *and*
$E_\mu : y^2 = x^3 + \mu^2 Ax + \mu^3 B$, $\mu \in \mathbb{F}_q^* \setminus (\mathbb{F}_q^*)^2$ *its twist. Then*

$$\#E(\mathbb{F}_q) = q + 1 - a \;\Leftrightarrow\; \#E_\mu(\mathbb{F}_q) = q + 1 + a$$

*and*

$$\#E(\mathbb{F}_{q^2}) = \#E_\mu(\mathbb{F}_{q^2}).$$

## Proof.

$$\begin{aligned}\#E_\mu(\mathbb{F}_q) &= q + 1 + \sum_{x \in \mathbb{F}_q} \left( \frac{x^3 + \mu^2 Ax + \mu^3 B}{\mathbb{F}_q} \right) \\ &= q + 1 + \left( \frac{\mu}{\mathbb{F}_q} \right) \sum_{x \in \mathbb{F}_q} \left( \frac{x^3 + Ax + B}{\mathbb{F}_q} \right)\end{aligned}$$

and $\left( \frac{\mu}{\mathbb{F}_q} \right) = -1$ □

# Further Reading...


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