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ELLIPTIC CURVES CRYPTOGRAPHY

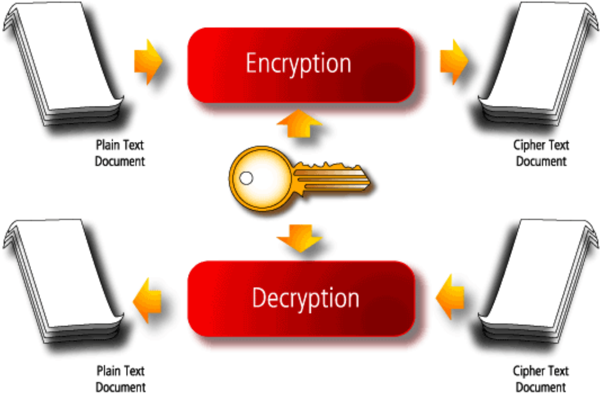
FRANCESCO PAPPALARDI

#2 - SECOND LECTURE.

SEPTEMBER 15TH 2015

National University of Mongolia
Ulan Baatar, Mongolia
September 15, 2015

Private key versus Public Key



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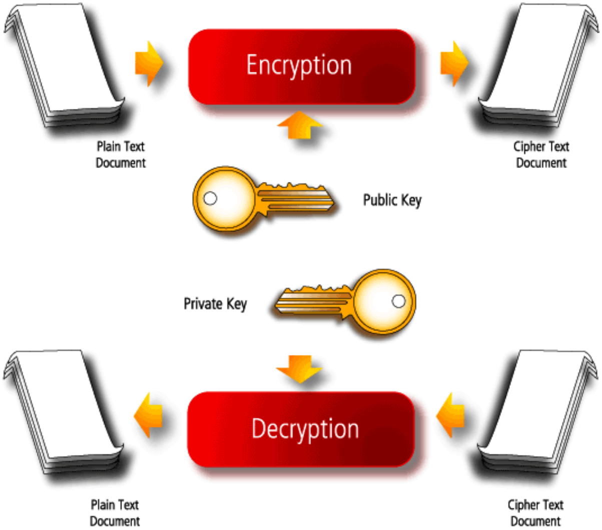

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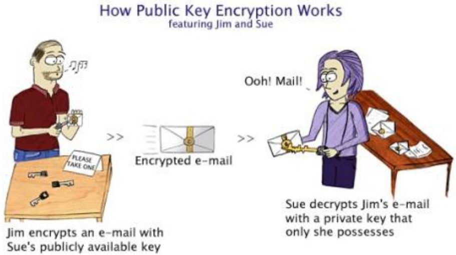
Classical General Examples of PKC

1. (1976) Diffie Hellmann Key exchange protocol *IEEE Trans. Information Theory IT-22 (1976)*
2. (1983) Massey Omura Cryptosystem *Proc. 4^{th} Benelux Symposium on Information Theory (1983)*
3. (1984) ElGamal Cryptosystem *IEEE Trans. Information Theory IT-31 (1985)*



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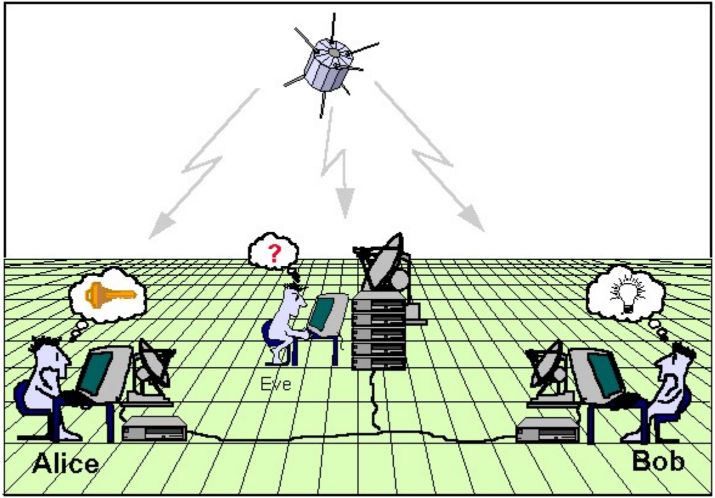
PKS



Elliptic curves over $\mathbb{F}_q$

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Diffie–Hellmann key exchange

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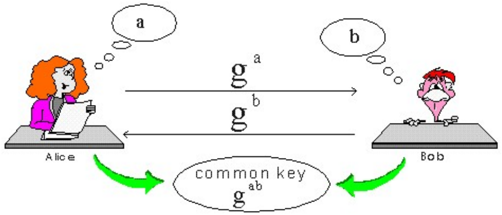
DHKEP

1. **Alice** and **Bob** agree on a cyclic group G and on a *generator* g in G
2. **Alice** picks a secret a, $0 \leq a \leq |G|$
3. **Bob** picks a secret b, $0 \leq b \leq |G|$
4. They compute and publish g^a (**Alice**) and g^b (**Bob**)
5. The common secret key is g^{ab}



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ElGamal Cryptosystem

Alice wants to sent a message $x \in G$ (cyclic group) to **Bob**

ElGamal SETUP:

1. **Alice** and **Bob** agree on a *generator* g in G
2. **Bob** picks a secret b, $0 < b \leq |G|$, he computes $\beta = g^b \in G$ and publishes β

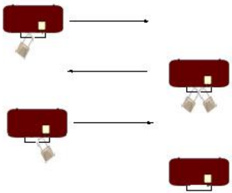
ElGamal ENCRYPTION: (Alice)

1. **Alice** picks a secret k, $0 < k \leq |G|$
2. She computes $\alpha = g^k \in G$ and $\gamma = x \cdot \beta^k \in G$
3. The encrypted message is $E(x) = (\alpha, \gamma) \in G \times G$

ElGamal DECRYPTION: (Bob)

1. **Bob** computes $D(\alpha, \gamma) = \gamma \cdot \alpha^{|G|-b}$
2. It works since $D(E(x)) = D(\alpha, \gamma) = x \cdot g^{bk} \cdot g^{k(|G|-b)} = x$ since $g^{k|G|} = 1$

Massey Omura on any finite Group G



Alice **Bob**

SETUP:

① **Alice** and **Bob** each
- pick a secret key $k_A, k_B \in U(\mathbb{Z}/|G|\mathbb{Z})$
- compute $\ell_A, \ell_B \in U(\mathbb{Z}/|G|\mathbb{Z})$ such that $k_A\ell_A \equiv 1(\mathrm{mod}\,|G|)$ and $k_B\ell_B \equiv 1(\mathrm{mod}\,|G|)$

④ **Alice** key is (k_A, ℓ_A) (k_A to lock and ℓ_A to unlock)

⑤ **Bob** key is (k_B, ℓ_B) (k_B to lock and ℓ_B to unlock)

WORKING: To send the message P

① **Alice** computes and sends $M = P^{k_A} \in G$

② **Bob** computes and sends back $N = M^{k_B} \in G$

③ **Alice** computes $L = N^{\ell_A} \in G$ and sends it back to **Bob**

④ **Bob** decrypt the message computing $P = L^{\ell_B} \in G$

It works: $P = L^{\ell_B} = N^{\ell_A \ell_B} = M^{k_B \ell_A l_B} = P^{k_A k_B \ell_A \ell_B} \in G$

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The generic Discrete Logarithms problem

- $G = \langle g \rangle$ cyclic group
- g a generator
- $x \in G$

Discrete Logarithm Problem:

Find $n \in \mathbb{Z}/|G|\mathbb{Z}$ such that $x = g^n$

- Need to specify how to make the operations in G
- If $G = (\mathbb{Z}/n\mathbb{Z}, +)$ then discrete logs are very easy.
- If $G = ((\mathbb{Z}/n\mathbb{Z})^*, \times)$ then G is cyclic iff $n = 2, 4, p^\alpha, 2 \cdot p^\alpha$ where p is an odd prime: famous theorem of Gauß.
- In $G = (\mathbb{Z}/p\mathbb{Z})^* =: \mathbb{F}_p^*$ there is no efficient algorithm to compute DL.
- We are interested in the case when $G = E(\mathbb{F}_q)$ where $E/\mathbb{F}_q$ is an elliptic curve
- Primordial public key cryptography is based on the difficulty of the Discrete Log problem

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Classical DL attacks

- Shanks baby-step, giant step (BSGS) *Proc. 2^{nd} Manitoba Conf. Numerical Mathematics (Winnipeg, 1972).*
- Pohlig–Hellmann Algorithm *IEEE Trans. Information Theory IT-24 (1978).*
- Index computation algorithm
- Sieving algorithms *La Macchia & Odlyzko, Designs Codes and Cryptography 1 (1991)*

NOTE: The last two are "very special" for $\mathbb{F}_q^*$

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DISCRETE LOGARITHMS: continues

Shanks Baby Step Giant Step algorithm

```
Input:   A group G = ⟨g⟩ and a ∈ G
Output:   k ∈ ℤ/|G|ℤ such that a = g^k
1.   M := ⌈√|G|⌉
2.   For j = 0,1,2,...,M.
           Compute g^j and store the pair (j, g^j) in a table
3.   A := g^{-M},  B := a
5.   For i = 0,1,2,...,M − 1.
         -1- Check if B is the second component (g^j) of any
              pair in the table
         -2- If so, return iM + j and halt.
         -3- If not B = B · A
```

- The BSGS algorithm is a generic algorithm. It works for every finite cyclic group.
- based on the fact that $\forall x \in \mathbb{Z}/n\mathbb{Z}$, $x = j + im$ with $m = \lceil\sqrt{n}\rceil$, $0 \le j < m$ and $0 \le i < m$
- Not necessary to know the order of the group G in advance. The algorithm still works if an upper bound on the group order is known.
- Usually the BSGS algorithm is used for groups whose order is prime.
- The running time of the algorithm and the space complexity is $O(\sqrt{|G|})$, much better than the $O(|G|)$ running time of the naive brute force
- The algorithm was originally developed by Daniel Shanks.

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DISCRETE LOGARITHMS: continues

The Pohlig–Hellman Algorithm

In some groups Discrete logs are easy. For example if G is a cyclic group and $\#G = 2^m$ then we know that there are subgroups:

$$\langle 1 \rangle = G_0 \subset G_1 \subset \cdots \subset G_m = G$$

such that G_i is cyclic and $\#G_i = 2^i$. Furthermore

$$G_i = \left\{ y \in G \text{ such that } y^{2^i} = 1 \right\}.$$

If $G = \langle g \rangle$, for any $a \in G$, either $a^{2^{m-1}} = 1$ or $a^{2^{m-1}} = g^{2^{m-1}}$. From this property we deduce the algorithm:

```
Input:   A group G = ⟨g⟩, |G| = 2^m and a ∈ G
Output:  k ∈ Z/|G|Z such that a = g^k
1.  A := a,  K = 0
2.  For j = 1,2,...,m.
          If A^(2^(m-j)) ≠ 1,  A := g^(-2^(j-1)) · A; K := K + 2^(j-1)
3.  Output K
```

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DISCRETE LOGARITHMS: continues

The Pohlig–Hellman Algorithm

- The above is a special case of the Pohlig-Hellman Algorithm which can be extended to the case when $|G|$ has only small prime divisors
- To avoid this situation one crucial requirement for a DL-resistent group in cryptography is that $\#G$ has a large prime divisor
- If $p = 2^k + 1$ is a Fermat prime, then DL in $(\mathbb{F}_p)^*$ are easy
- Classical algorithm for factoring have often analogues for computing discrete logs. A very important one is the *Pollard ρ–method*
- One of the strongest algorithms is the index calculus algorithm. NOT generic. It works only in $\mathbb{F}_q^*$

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DISCRETE LOGARITHMS: continues

Records

Discrete Logarithm Records:

- $G = \mathbb{F}_p^*$: $p \approx 10^{180}$ (596-bit)
  Cyril Bouvier, Pierrick Gaudry, Laurent Imbert, Hamza Jeljeli and Emmanuel Thomé (11 June 2014)
- $G = \mathbb{F}_{p^2}^*$: $p \approx 10^{80}$
  Razvan Barbulescu, Pierrick Gaudry, Aurore Guillevic, and François Morain (25 June 2014)
- $G = \mathbb{F}_{2^\alpha}^*$: $\alpha = 1279$
  Thorsten Kleinjung (17 October 2014)
- $G = E(\mathbb{F}_p)$: $p \approx 10^{35}$
  Joppe W. Bos, Marcelo E. Kaihara, T. Kleinjung, Arjen K. Lenstra and Peter L. Montgomery (July 2009)
  $p = 4451685225093714772084598273548427$
- $G = E(\mathbb{F}_{2^\alpha})$: $\alpha = 113$ Erich Wenger and Paul Wolfger (January 2015)

with *ECC* same security with $1/5$ of the size

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The problem of "Square Roots Modulo a prime"

Given an odd prime p and a quadratic residue a

Find x such that $x^2 \equiv a \bmod p$

It can be solved efficiently if we are given a quadratic nonresidue $g \in (\mathbb{Z}/p\mathbb{Z})^*$

1. Write $p-1 = 2^k \cdot q$ and we know that $(\mathbb{Z}/p\mathbb{Z})^*$ has a (cyclic) subgroup G with 2^k elements.
2. Note that $b = g^q$ is a generator of G (in fact if it was $b^{2^j} \equiv 1 \bmod p$ for $j < k$, then $g^{(p-1)/2} \equiv 1 \bmod p$) and that $a^q \in G$
3. Use the last algorithm to compute t such that $a^q = b^t$. Note that t is even since $a^{(p-1)/2} \equiv 1 \bmod p$.
4. Finally set $x = a^{(p-q)/2} b^{t/2}$ and observe that $x^2 = a^{(p-q)} b^t = a^p \equiv a \bmod p$.

REMARKS:

- The above is not deterministic. However Schoof in 1985 discovered a polynomial time algorithm which is however not efficient.
- To find a random point in an elliptic curve $E/\mathbb{F}_p$ one needs to compute square roots modulo p

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The problem of "Modular Square Roots"

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Given $n, a \in \mathbb{N}$

Find x (if it exists) such that $x^2 \equiv a \bmod n$

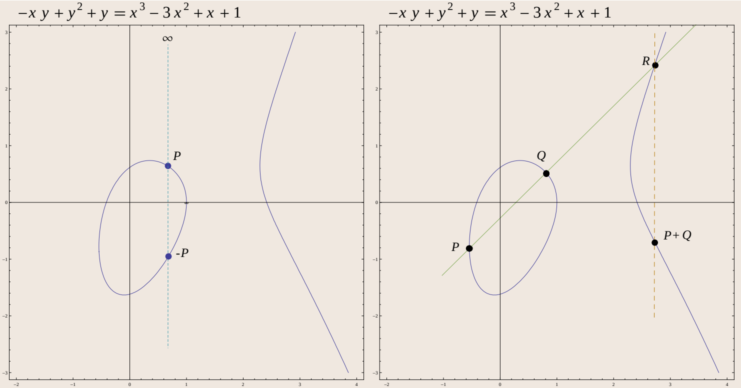
If the factorization of n is known, then this problem (efficiently) can be solved in 3 steps:

1. For each prime divisor p of n find x_p such that $x_p^2 \equiv a \bmod p$
2. Use the Hensel's Lemma to lift x_p to y_p where $y_p^2 \equiv a \bmod p^{v_p(n)}$
3. Use the Chinese remainder Theorem to find $x \in \mathbb{Z}/n\mathbb{Z}$ such that $x \equiv y_p \bmod p^{v_p(n)} \; \forall p \mid n$.
4. Finally $x^2 \equiv a \bmod n$.

Reminder from Yesterday

If $P, Q \in E(\mathbb{F}_q)$, $r_{P,Q}$: $\begin{cases} \text{line through } P \text{ and } Q & \text{if } P \neq Q \\ \text{tangent line to } E \text{ at } P & \text{if } P = Q, \end{cases}$

$r_{P,\infty}$: vertical line through P



$r_{P,\infty} \cap E(\mathbb{F}_q) = \{P, \infty, P'\}$ $\rightsquigarrow$ $\boxed{-P := P'}$

$r_{P,Q} \cap E(\mathbb{F}_q) = \{P, Q, R\}$ $\rightsquigarrow$ $\boxed{P +_E Q := -R}$

Formulas for Addition on E (Summary)

$$E : y^2 + a_1 xy + a_3 y = x^3 + a_2 x^2 + a_4 x + a_6$$

$P_1 = (x_1, y_1), P_2 = (x_2, y_2) \in E(\mathbb{F}_q) \setminus \{\infty\}$,

Addition Laws for the sum of affine points

- If $P_1 \neq P_2$
  - $x_1 = x_2$ $\Rightarrow$ $P_1 +_E P_2 = \infty$
  - $x_1 \neq x_2$

$$\lambda = \frac{y_2 - y_1}{x_2 - x_1} \qquad \nu = \frac{y_1 x_2 - y_2 x_1}{x_2 - x_1}$$

- If $P_1 = P_2$
  - $2y_1 + a_1 x + a_3 = 0$ $\Rightarrow$ $P_1 +_E P_2 = 2P_1 = \infty$
  - $2y_1 + a_1 x + a_3 \neq 0$

$$\lambda = \frac{3x_1^2 + 2a_2 x_1 + a_4 - a_1 y_1}{2y_1 + a_1 x + a_3}, \nu = -\frac{a_3 y_1 + x_1^3 - a_4 x_1 - 2a_6}{2y_1 + a_1 x_1 + a_3}$$

Then

$$P_1 +_E P_2 = (\lambda^2 - a_1 \lambda - a_2 - x_1 - x_2, -\lambda^3 - a_1^2 \lambda + (\lambda + a_1)(a_2 + x_1 + x_2) - a_3 - \nu)$$

Formulas for Addition on E (Summary for special equation)

$$E : y^2 = x^3 + Ax + B$$

$P_1 = (x_1, y_1), P_2 = (x_2, y_2) \in E(\mathbb{F}_q) \setminus \{\infty\}$,

Addition Laws for the sum of affine points

- If $P_1 \neq P_2$
  - $x_1 = x_2$ $\Rightarrow$ $P_1 +_E P_2 = \infty$
  - $x_1 \neq x_2$

$$\lambda = \frac{y_2 - y_1}{x_2 - x_1} \qquad \nu = \frac{y_1 x_2 - y_2 x_1}{x_2 - x_1}$$

- If $P_1 = P_2$
  - $y_1 = 0$ $\Rightarrow$ $P_1 +_E P_2 = 2P_1 = \infty$
  - $y_1 \neq 0$

$$\lambda = \frac{3x_1^2 + A}{2y_1}, \nu = -\frac{x_1^3 - Ax_1 - 2B}{2y_1}$$

Then

$$P_1 +_E P_2 = (\lambda^2 - x_1 - x_2, -\lambda^3 + \lambda(x_1 + x_2) - \nu)$$

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The division polynomials

Definition (Division Polynomials of $E : y^2 = x^3 + Ax + B$ ($p > 3$))

$$
\begin{aligned}
\psi_0 &= 0,\ \psi_1 = 1,\ \psi_2 = 2y \\
\psi_3 &= 3x^4 + 6Ax^2 + 12Bx - A^2 \\
\psi_4 &= 4y(x^6 + 5Ax^4 + 20Bx^3 - 5A^2x^2 - 4ABx - 8B^2 - A^3) \\
&\vdots \\
\psi_{2m+1} &= \psi_{m+2}\psi_m^3 - \psi_{m-1}\psi_{m+1}^3 \qquad \text{for } m \geq 2 \\
\psi_{2m} &= \left(\frac{\psi_m}{2y}\right) \cdot (\psi_{m+2}\psi_{m-1}^2 - \psi_{m-2}\psi_{m+1}^2) \quad \text{for } m \geq 3
\end{aligned}
$$

The polynomial $\psi_m \in \mathbb{Z}[x, y]$ is the m^{th} *division polynomial*

Theorem ($E : Y^2 = X^3 + AX + B$ elliptic curve, $P = (x, y) \in E$)

$$mP = m(x, y) = \left(\frac{\phi_m(x)}{\psi_m^2(x)}, \frac{\omega_m(x, y)}{\psi_m^3(x, y)}\right),$$

where $\phi_m = x\psi_m^2 - \psi_{m+1}\psi_{m-1}$, $\omega_m = \dfrac{\psi_{m+2}\psi_{m-1}^2 - \psi_{m-2}\psi_{m+1}^2}{4y}$

Points of order m

Definition (m–torsion point)

Let E/K and let $\bar{K}$ an *algebraic closure of* K.

$$E[m] = \{P \in E(\bar{K}) : \ mP = \infty\}$$

Theorem (Structure of Torsion Points)

Let E/K and $m \in \mathbb{N}$.

$$E[m] \cong \begin{cases} C_m \oplus C_m & \text{if } p = \mathrm{char}(K) \nmid m \\ C_m \oplus C_{m'} \quad \text{or} \quad E[m] \cong C_{m'} \oplus C_{m'} & \text{if } m = p^r m', p \nmid m' \end{cases}$$

FACTS:

- $E[2m+1] \setminus \{\infty\} = \{(x,y) \in E(\bar{K}) : \ \psi_{2m+1}(x) = 0\}$
- $E[2m] \setminus E[2] = \{(x,y) \in E(\bar{K}) : \ y^{-1}\psi_{2m}(x) = 0\}$
- Corollary of the Theorem of Structure for torsion $\exists n, k \in \mathbb{N}$ such that $E(\mathbb{F}_q) \cong C_n \oplus C_{nk}$
- Property of Weil pairing $n \mid q-1$.

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Theorem (Hasse)

Let E be an elliptic curve over the finite field $\mathbb{F}_q$. Then the order of $E(\mathbb{F}_q)$ satisfies

$$|q+1-\#E(\mathbb{F}_q)| \le 2\sqrt{q}.$$

So $\#E(\mathbb{F}_q) \in [(\sqrt{q}-1)^2, (\sqrt{q}+1)^2]$ the *Hasse interval* $\mathcal{I}_q$

Example (Hasse Intervals)

q	$\mathcal{I}_q$
2	$\{1,2,3,4,5\}$
3	$\{1,2,3,4,5,6,7\}$
4	$\{1,2,3,4,5,6,7,8,9\}$
5	$\{2,3,4,5,6,7,8,9,10\}$
7	$\{3,4,5,6,7,8,9,10,11,12,13\}$
8	$\{4,5,6,7,8,9,10,11,12,13,14\}$
9	$\{4,5,6,7,8,9,10,11,12,13,14,15,16\}$
11	$\{6,7,8,9,10,11,12,13,14,15,16,17,18\}$
13	$\{7,8,9,10,11,12,13,14,15,16,17,18,19,20,21\}$
16	$\{9,10,11,12,13,14,15,16,17,18,19,20,21,22,23,25\}$
17	$\{10,11,12,13,14,15,16,17,18,19,20,21,22,23,24,25,26\}$
19	$\{12,13,14,15,16,17,18,19,20,21,22,23,24,25,26,27,28\}$
23	$\{15,16,17,18,19,20,21,22,23,24,25,26,27,28,29,30,31,32,33\}$
25	$\{16,17,18,19,20,21,22,23,24,25,26,27,28,29,30,31,32,33,34,35,36\}$
27	$\{18,19,20,21,22,23,24,25,26,27,28,29,30,31,32,33,34,35,36,37,38\}$
29	$\{20,21,22,23,24,25,26,27,28,29,30,31,32,33,34,35,36,37,38,39,40\}$
31	$\{21,22,23,24,25,26,27,28,29,30,31,32,33,34,35,36,37,38,39,40,41,42,43\}$
32	$\{22,23,24,25,26,27,28,29,30,31,32,33,34,35,36,37,38,39,40,41,42,43,44\}$

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Theorem (Waterhouse)

Let $q = p^n$ and let $N = q + 1 - a$.

$$\exists E/\mathbb{F}_q \text{ s.t.} \#E(\mathbb{F}_q) = N \Leftrightarrow |a| \leq 2\sqrt{q} \text{ and}$$

one of the following is satisfied:

(i) $\gcd(a, p) = 1$;

(ii) *n even and one of the following is satisfied:*
1. $a = \pm 2\sqrt{q}$;
2. $p \not\equiv 1 \pmod 3$, *and* $a = \pm\sqrt{q}$;
3. $p \not\equiv 1 \pmod 4$, *and* $a = 0$;

(iii) *n is odd, and one of the following is satisfied:*
1. $p = 2$ *or* 3, *and* $a = \pm p^{(n+1)/2}$;
2. $a = 0$.

Example (q prime $\forall N \in I_q, \exists E/\mathbb{F}_q, \#E(\mathbb{F}_q) = N$. q not prime:)

q	$a \in$
$4 = 2^2$	$\{-4, -3, -2, -1, 0, 1, 2, 3, 4\}$
$8 = 2^3$	$\{-5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5\}$
$9 = 3^2$	$\{-6, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, 6\}$
$16 = 2^4$	$\{-8, -7, -6, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, 6, 7, 8\}$
$25 = 5^2$	$\{-10, -9, -8, -7, -6, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$
$27 = 3^3$	$\{-10, -9, -8, -7, -6, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$
$32 = 2^5$	$\{-11, -10, -9, -8, -7, -6, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$

Theorem (Rück)

Suppose N is a possible order of an elliptic curve $/\mathbb{F}_q$, $q = p^n$. Write

$$N = p^e n_1 n_2, \quad p \nmid n_1 n_2 \quad \text{and} \quad n_1 \mid n_2 \text{ (possibly } n_1 = 1).$$

There exists $E/\mathbb{F}_q$ s.t.

$$E(\mathbb{F}_q) \cong C_{n_1} \oplus C_{n_2 p^e}$$

if and only if

1. *$n_1 = n_2$ in the case (ii).1 of Waterhouse's Theorem;*
2. *$n_1 | q - 1$ in all other cases of Waterhouse's Theorem.*

Example

- If $q = p^{2n}$ and $\#E(\mathbb{F}_q) = q + 1 \pm 2\sqrt{q} = (p^n \pm 1)^2$, then
$$E(\mathbb{F}_q) \cong C_{p^n \pm 1} \oplus C_{p^n \pm 1}.$$
- Let $N = 100$ and $q = 101 \Rightarrow \exists E_1, E_2, E_3, E_4/\mathbb{F}_{101}$ s.t.
$$E_1(\mathbb{F}_{101}) \cong C_{10} \oplus C_{10} \qquad E_2(\mathbb{F}_{101}) \cong C_2 \oplus C_{50}$$
$$E_3(\mathbb{F}_{101}) \cong C_5 \oplus C_{20} \qquad E_4(\mathbb{F}_{101}) \cong C_{100}$$

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Subfield curves

Definition

Let $E/\mathbb{F}_q$ and write $E(\mathbb{F}_q) = q + 1 - a$, ($|a| \leq 2\sqrt{q}$). The *characteristic* polynomial of E is

$$P_E(T) = T^2 - aT + q \in \mathbb{Z}[T].$$

and its roots:

$$\alpha = \frac{1}{2}\left(a + \sqrt{a^2 - 4q}\right) \qquad \beta = \frac{1}{2}\left(a - \sqrt{a^2 - 4q}\right)$$

are called *characteristic roots of Frobenius* ($P_E(\Phi_q) = 0$).

Theorem

$\forall n \in \mathbb{N}$

$$\#E(\mathbb{F}_{q^n}) = q^n + 1 - (\alpha^n + \beta^n).$$

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Subfield curves (continues)

$$E(\mathbb{F}_q) = q + 1 - a \;\Rightarrow\; E(\mathbb{F}_{q^n}) = q^n + 1 - (\alpha^n + \beta^n)$$

where $P_E(T) = T^2 - aT + q = (T - \alpha)(T - \beta) \in \mathbb{Z}[T]$

Curves $/\mathbb{F}_2$

E	a	$P_E(T)$	(α, β)
$y^2 + xy = x^3 + x^2 + 1$	1	$T^2 - T + 2$	$\frac{1}{2}(1 \pm \sqrt{-7})$
$y^2 + xy = x^3 + 1$	-1	$T^2 + T + 2$	$\frac{1}{2}(-1 \pm \sqrt{-7})$
$y^2 + y = x^3 + x$	-2	$T^2 + 2T + 2$	$-1 \pm i$
$y^2 + y = x^3 + x + 1$	2	$T^2 - 2T + 2$	$1 \pm i$
$y^2 + y = x^3$	0	$T^2 + 2$	$\pm\sqrt{-2}$

$$E: y^2 + xy = x^3 + x^2 + 1 \;\Rightarrow\; E(\mathbb{F}_{2^{100}}) = 2^{100} + 1 - \left(\frac{1+\sqrt{-7}}{2}\right)^{100} - \left(\frac{1-\sqrt{-7}}{2}\right)^{100} = 1267650600228229382588845215376$$

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Subfield curves

$$E(\mathbb{F}_q) = q + 1 - a \ \Rightarrow\ E(\mathbb{F}_{q^n}) = q^n + 1 - (\alpha^n + \beta^n)$$
$$\text{where } P_E(T) = T^2 - aT + q = (T - \alpha)(T - \beta) \in \mathbb{Z}[T]$$

Curves $/\mathbb{F}_3$

i	E_i	a	$P_{E_i}(T)$	(α, β)
1	$y^2 = x^3 + x$	0	$T^2 + 3$	$\pm\sqrt{-3}$
2	$y^2 = x^3 - x$	0	$T^2 + 3$	$\pm\sqrt{-3}$
3	$y^2 = x^3 - x + 1$	-3	$T^2 + 3T + 3$	$\frac{1}{2}(-3 \pm \sqrt{-3})$
4	$y^2 = x^3 - x - 1$	3	$T^2 - 3T + 3$	$\frac{1}{2}(3 \pm \sqrt{-3})$
5	$y^2 = x^3 + x^2 - 1$	1	$T^2 - T + 3$	$\frac{1}{2}(1 \pm \sqrt{-11})$
6	$y^2 = x^3 - x^2 + 1$	-1	$T^2 + T + 3$	$\frac{1}{2}(-1 \pm \sqrt{-11})$
7	$y^2 = x^3 + x^2 + 1$	-2	$T^2 + 2T + 3$	$-1 \pm \sqrt{-2}$
8	$y^2 = x^3 - x^2 - 1$	2	$T^2 - 2T + 3$	$1 \pm \sqrt{-2}$

Lemma

Let $s_n = \alpha^n + \beta^n$ where $\alpha\beta = q$ and $\alpha + \beta = a$. Then

$$s_0 = 2, \quad , s_1 = a \quad \textit{and} \quad s_{n+1} = as_n - qs_{n-1}$$

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Recall the *Finite field Legendre symbols*: let $x \in \mathbb{F}_q$,

$$\left(\frac{x}{\mathbb{F}_q}\right) = \begin{cases} +1 & \text{if } t^2 = x \text{ has a solution } t \in \mathbb{F}_q^* \\ -1 & \text{if } t^2 = x \text{ has no solution } t \in \mathbb{F}_q \\ 0 & \text{if } x = 0 \end{cases}$$

Theorem

Let $E : y^2 = x^3 + Ax + B$ *over* $\mathbb{F}_q$. *Then*

$$\#E(\mathbb{F}_q) = q + 1 + \sum_{x \in \mathbb{F}_q} \left(\frac{x^3+Ax+B}{\mathbb{F}_q}\right)$$

Proof.

Note that

$$1 + \left(\frac{x_0^3+Ax_0+B}{\mathbb{F}_q}\right) = \begin{cases} 2 & \text{if } \exists y_0 \in \mathbb{F}_q^* \text{ s.t. } (x_0, \pm y_0) \in E(\mathbb{F}_q) \\ 1 & \text{if } (x_0, 0) \in E(\mathbb{F}_q) \\ 0 & \text{otherwise} \end{cases}$$

Hence

$$\#E(\mathbb{F}_q) = 1 + \sum_{x \in \mathbb{F}_q} \left(1 + \left(\frac{x^3+Ax+B}{\mathbb{F}_q}\right)\right)$$

□

Last Slide

Corollary

Let $E: y^2 = x^3 + Ax + B$ *over* $\mathbb{F}_q$ *and* $E_\mu: y^2 = x^3 + \mu^2 Ax + \mu^3 B$, $\mu \in \mathbb{F}_q^* \setminus (\mathbb{F}_q^*)^2$ *its twist. Then*

$$\boxed{\#E(\mathbb{F}_q) = q + 1 - a \Leftrightarrow \#E_\mu(\mathbb{F}_q) = q + 1 + a}$$

and

$$\boxed{\#E(\mathbb{F}_{q^2}) = \#E_\mu(\mathbb{F}_{q^2}).}$$

Proof.

$$\begin{aligned}\#E_\mu(\mathbb{F}_q) &= q + 1 + \sum_{x \in \mathbb{F}_q} \left(\frac{x^3 + \mu^2 Ax + \mu^3 B}{\mathbb{F}_q}\right) \\ &= q + 1 + \left(\frac{\mu}{\mathbb{F}_q}\right) \sum_{x \in \mathbb{F}_q} \left(\frac{x^3 + Ax + B}{\mathbb{F}_q}\right)\end{aligned}$$

and $\left(\frac{\mu}{\mathbb{F}_q}\right) = -1$ □

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