# Introduction to Galois Representations

## Definitions and basic properties

**NATO ASI, Ohrid 2014**

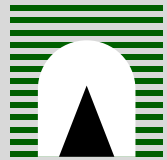
*Arithmetic of Hyperelliptic Curves*

August 25 - September 5, 2014

*Ohrid, the former Yugoslav Republic of Macedonia*,

Francesco Pappalardi
Dipartimento di Matematica e Fisica
Università Roma Tre





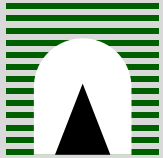
# The (general) Weierstraß Equation

An elliptic curve $E$ over a field $K$ is given by an equation

$$E : y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$$

where $a_1, a_3, a_2, a_4, a_6 \in K$





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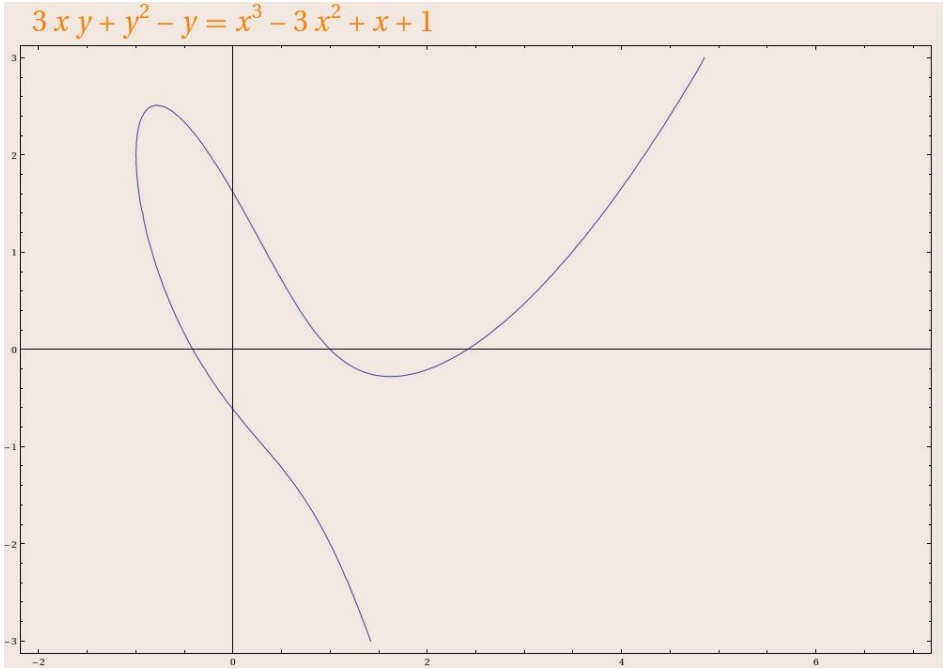


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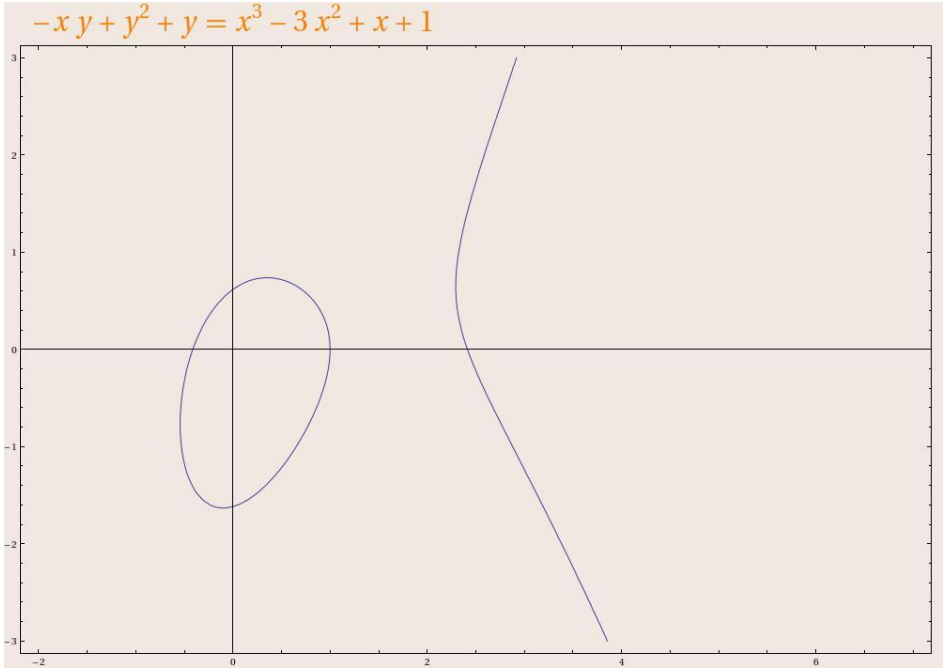




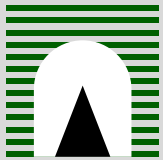

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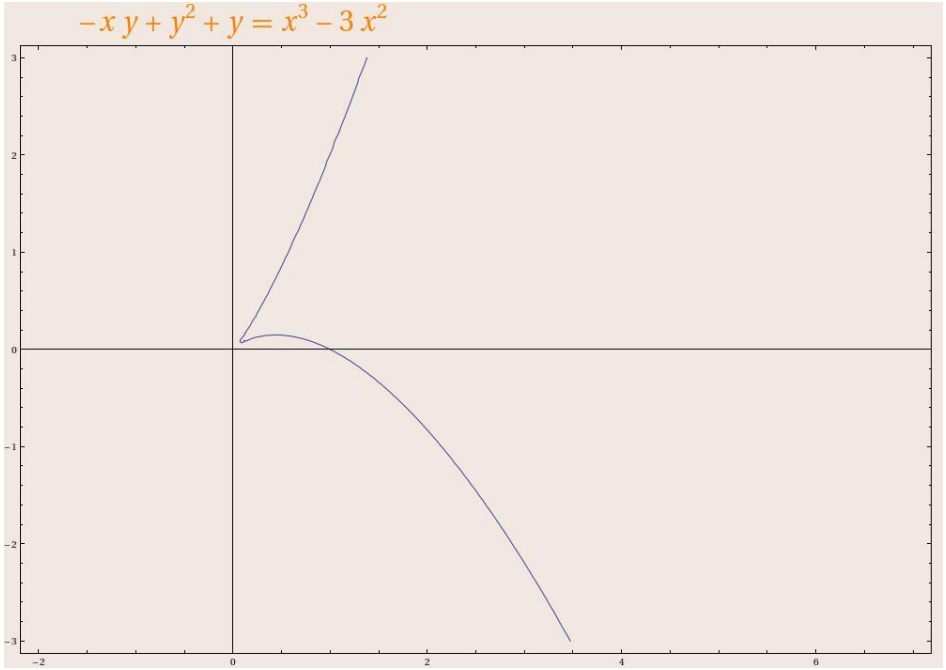




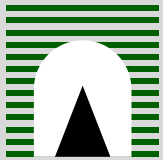

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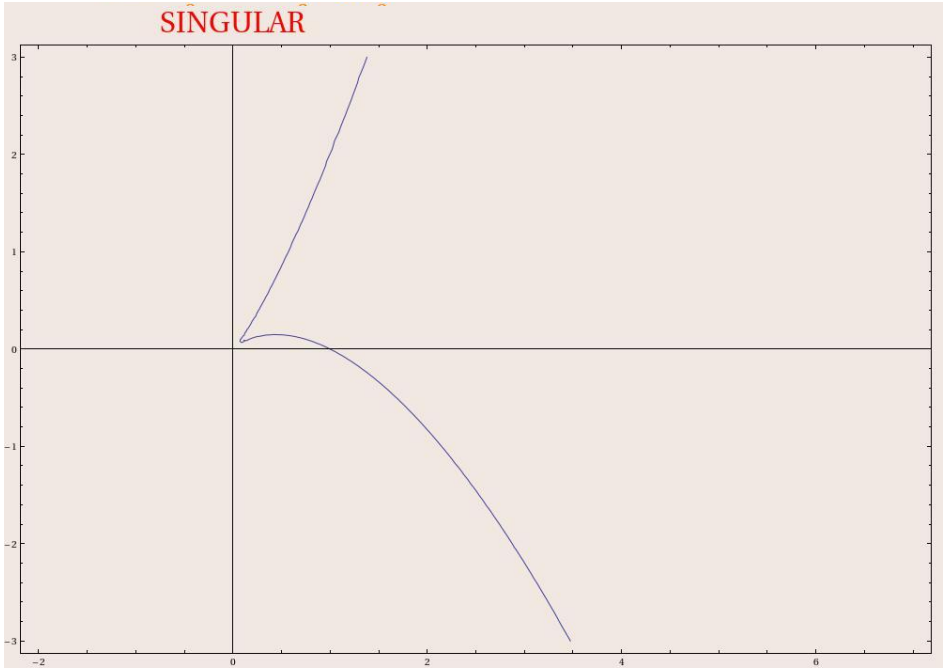




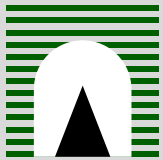

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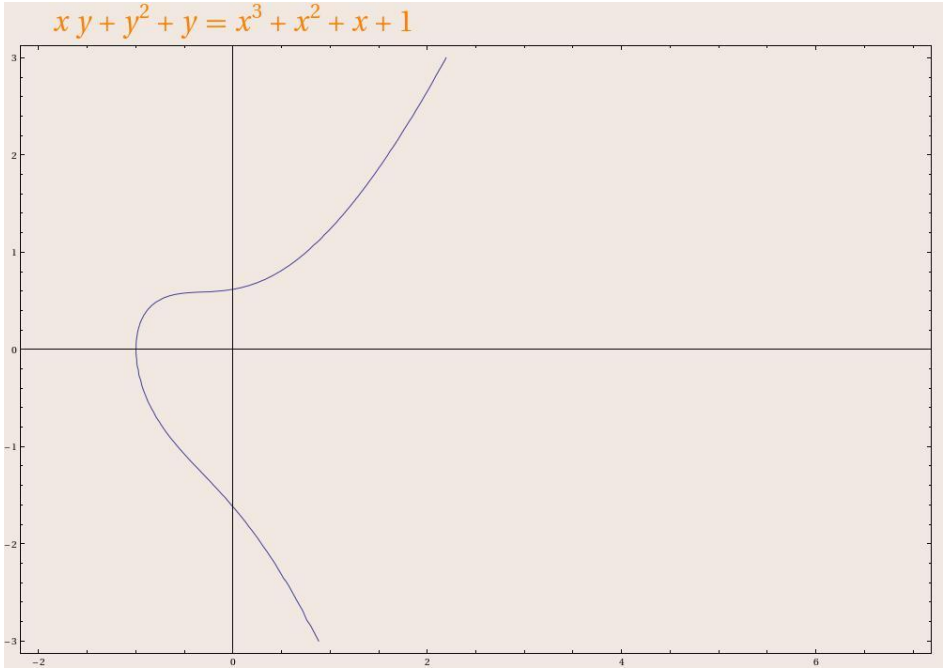




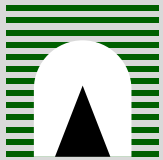

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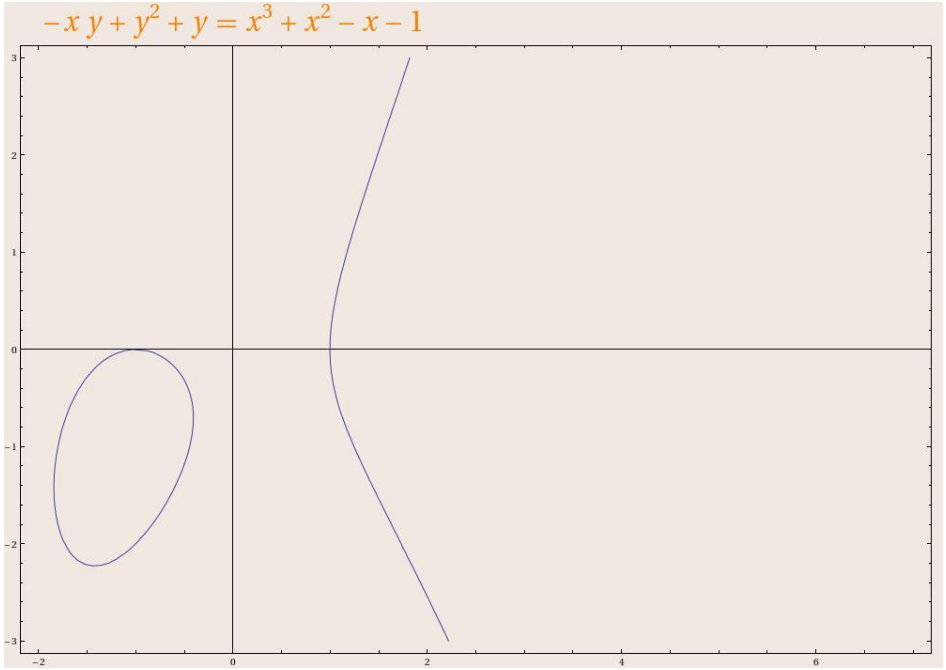




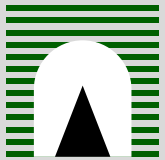

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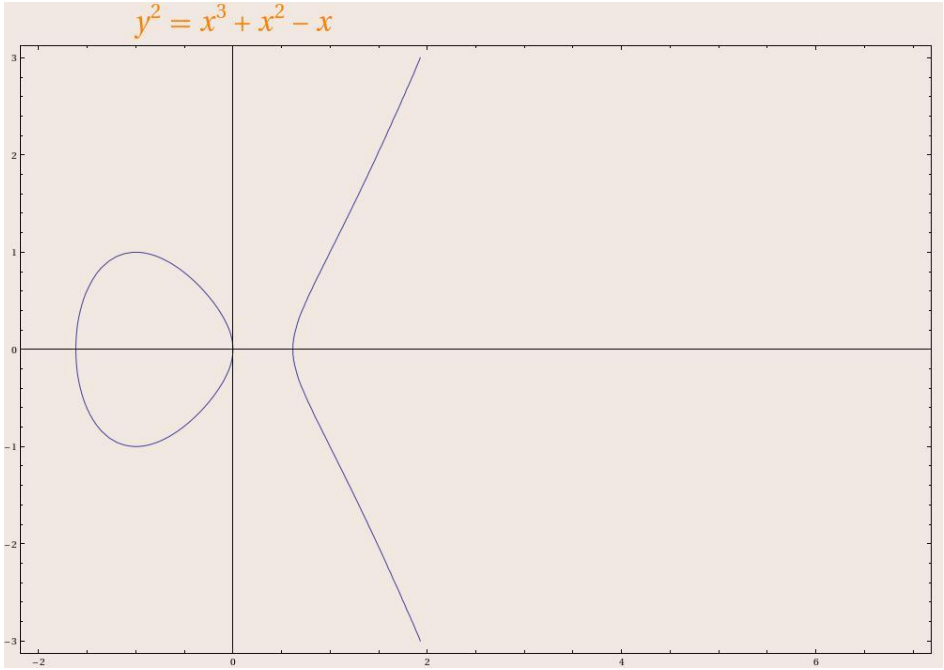


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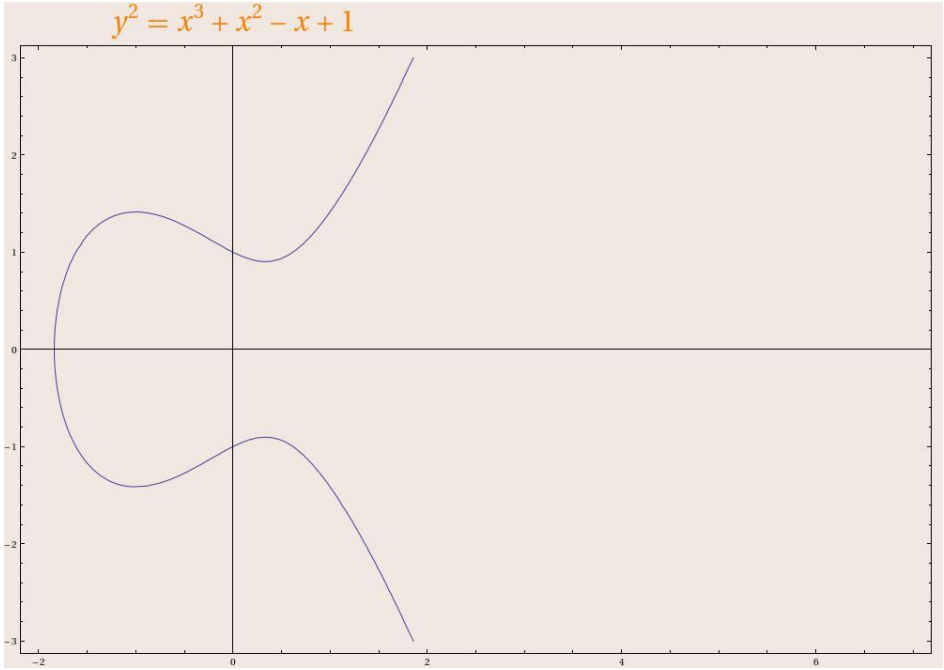




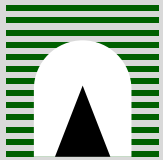

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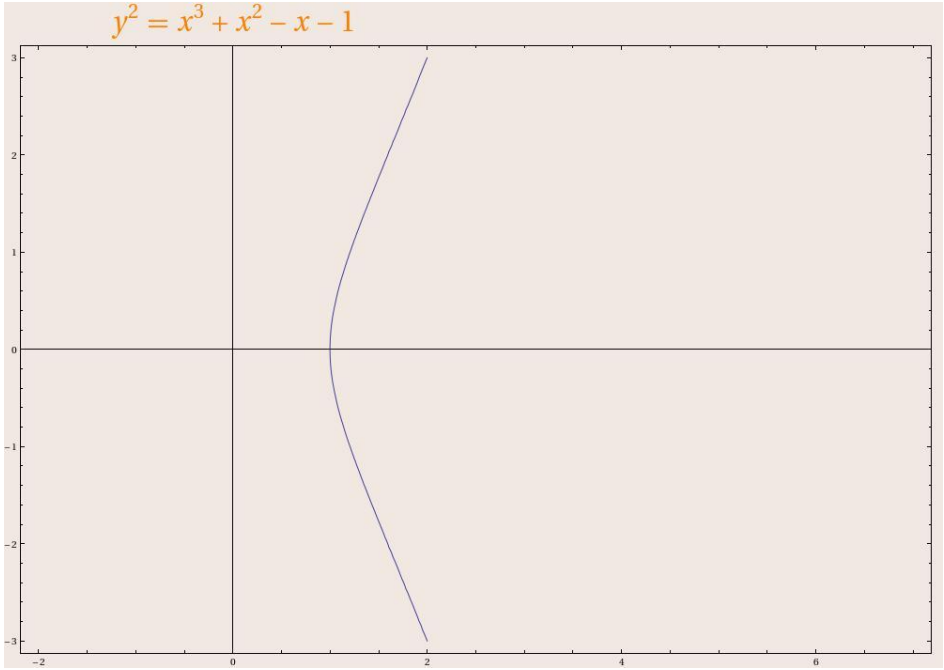


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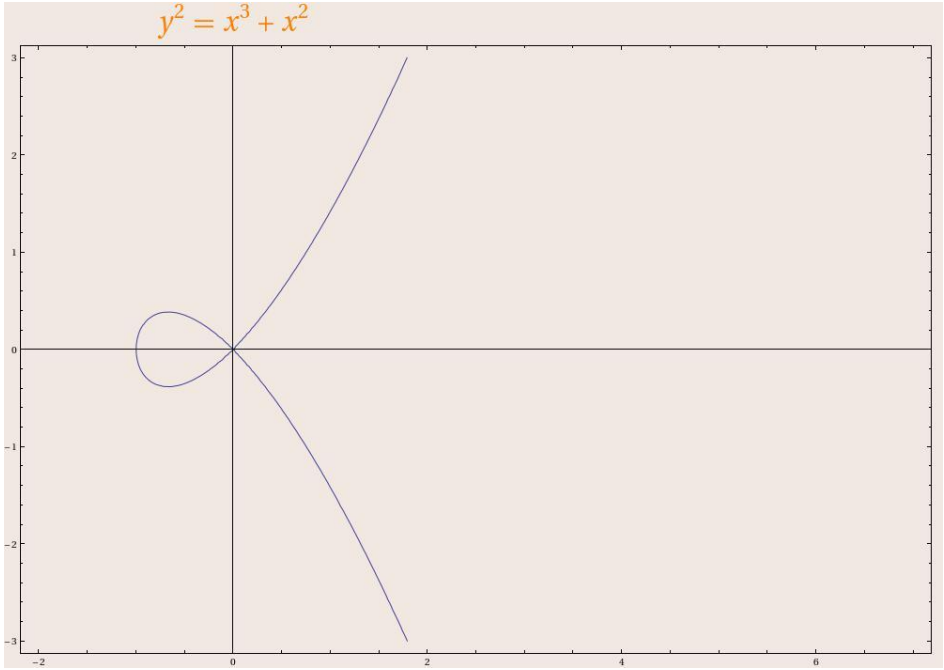




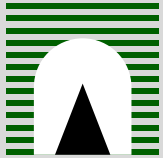

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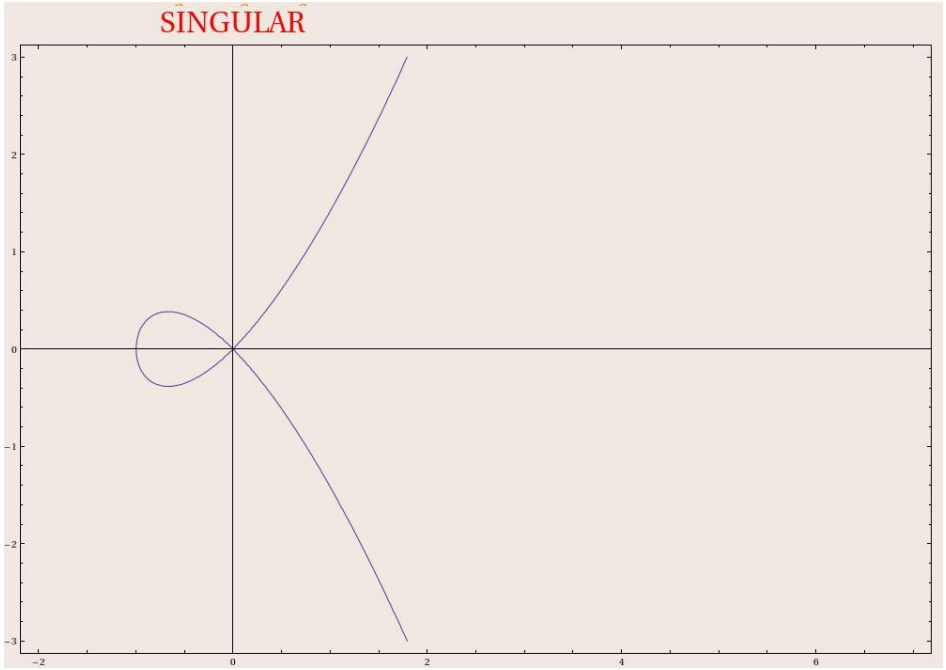


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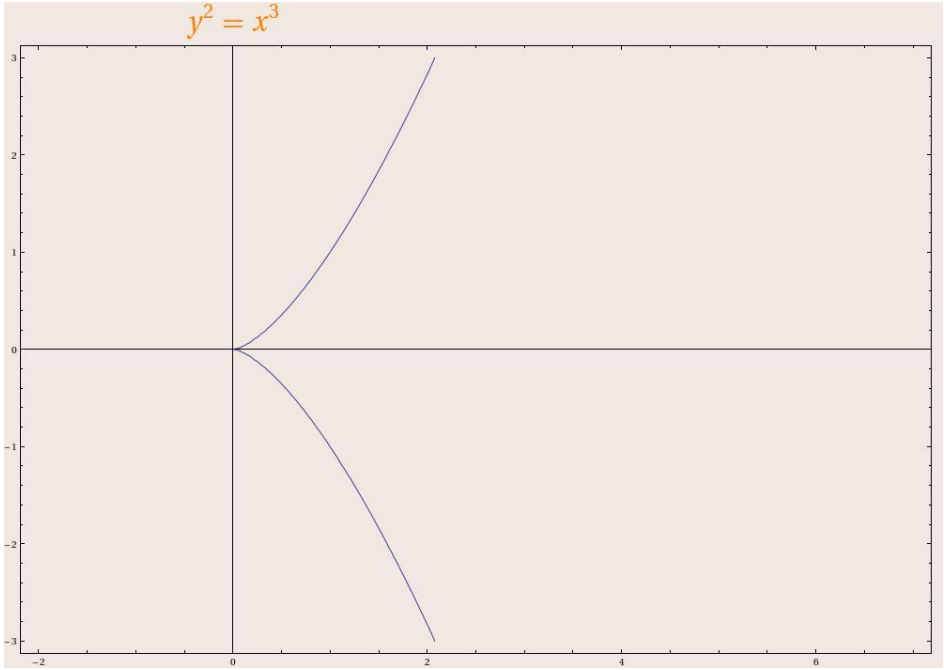




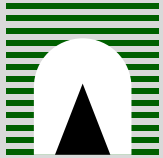

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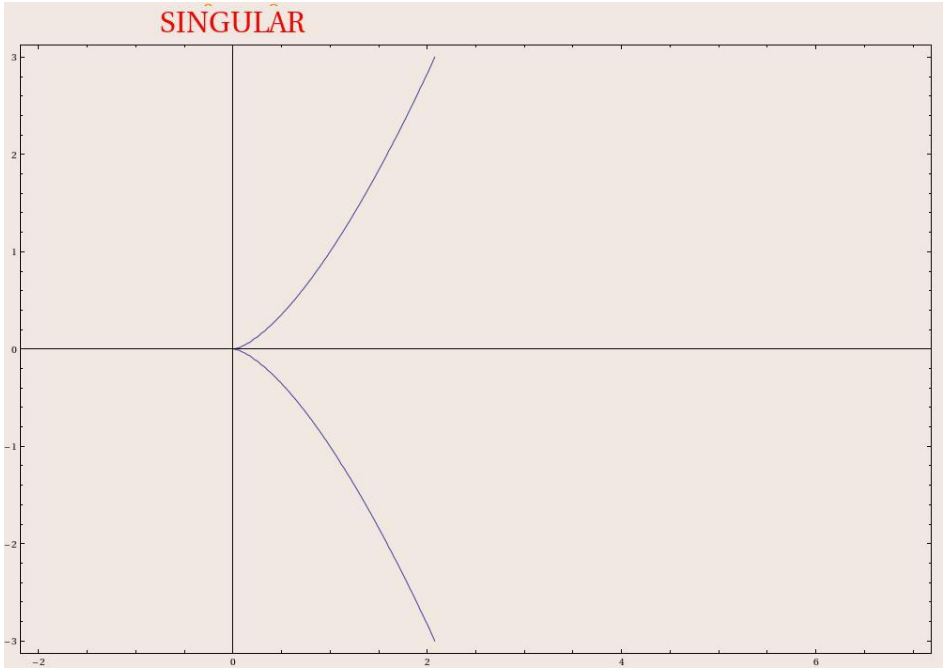


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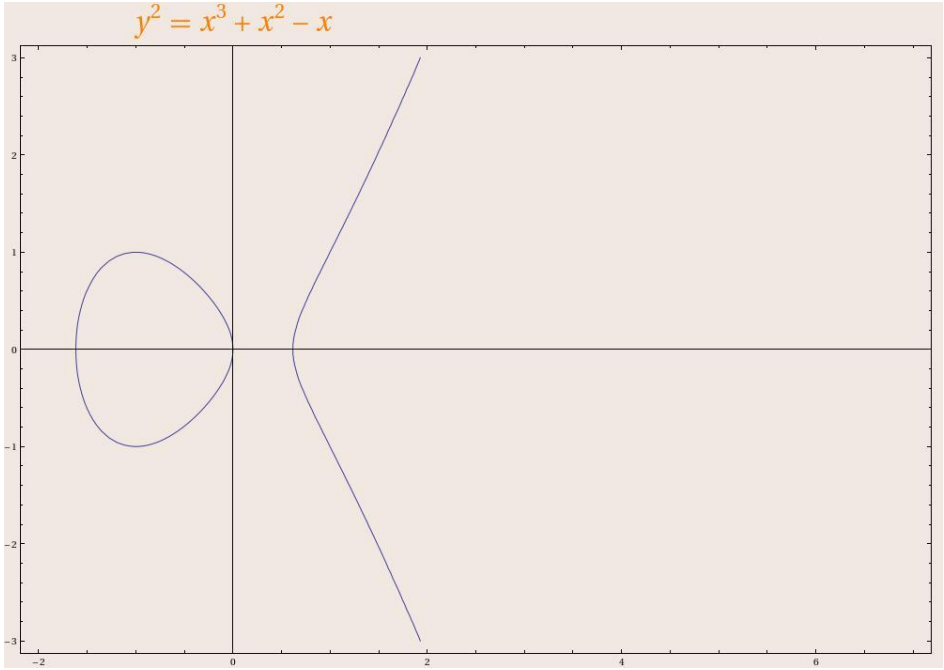


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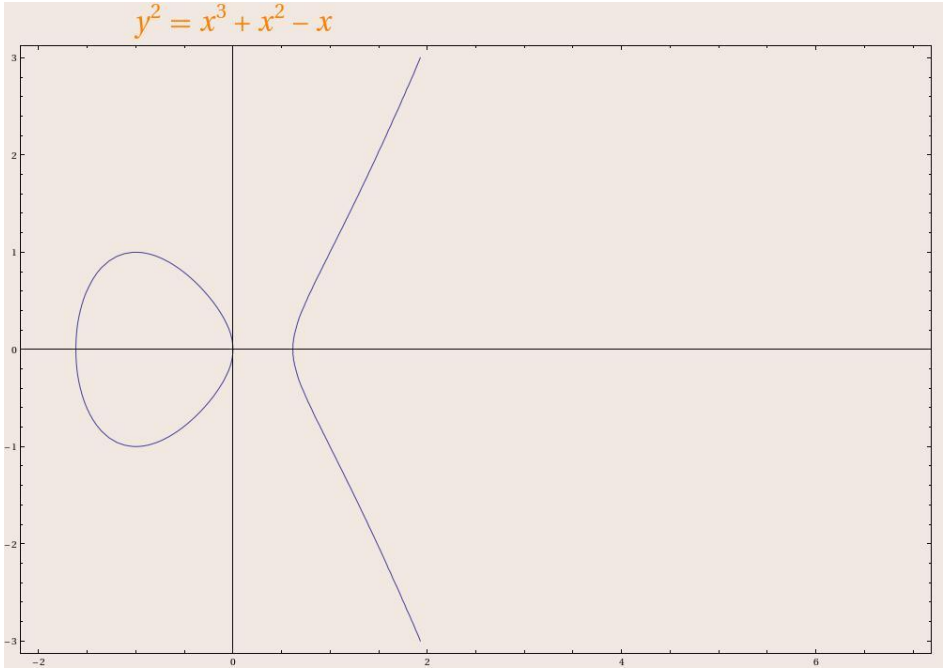


The equation should not be *singular*

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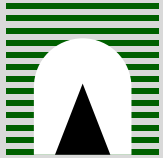
# The Discriminant of an Equation

## The condition of absence of singular points in terms of $a_1, a_2, a_3, a_4, a_6$





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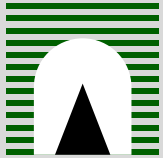
**Definition (The discriminant of a Weierstraß equation)**

$$\begin{aligned}\Delta_E := &-a_1^5a_3a_4 - 8a_1^3a_2a_3a_4 - 16a_1a_2^2a_3a_4 + 36a_1^2a_3^2a_4 \\ &- a_1^4a_4^2 - 8a_1^2a_2a_4^2 - 16a_2^2a_4^2 + 96a_1a_3a_4^2 + 64a_4^3 + \\ &a_1^6a_6 + 12a_1^4a_2a_6 + 48a_1^2a_2^2a_6 + 64a_2^3a_6 - 36a_1^3a_3a_6 \\ &- 144a_1a_2a_3a_6 - 72a_1^2a_4a_6 - 288a_2a_4a_6 + 432a_6^2\end{aligned}$$





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$E$ is non singular if and only if $\Delta_E \neq 0$

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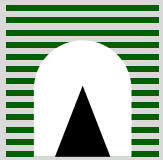

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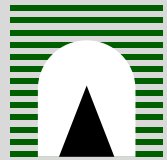
### Definition

Two Weierstraß equations over $K$ are said *(affinely) equivalent* if there exists a (affine) transformation of the following form

$$\begin{cases} x \longleftarrow u^2x + r \\ y \longleftarrow u^3y + u^2sx + t \end{cases} \qquad r, s, t, u \in K$$

that "takes" one equation into the other



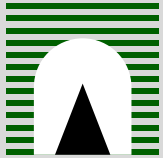


# The Weierstraß equation

## Classification of simplified forms

After applying a suitable affine transformation we can always assume that $E/K(p = \mathrm{char}(K))$ has a Weierstraß equation of the following form





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**Example (Classification)**

| $E$ | $p$ | $\Delta_E$ |
|---|---|---|
| $y^2 = x^3 + Ax + B$ | $\geq 5$ | $4A^3 + 27B^2$ |
| $y^2 + xy = x^3 + a_2x^2 + a_6$ | 2 | $a_6^2$ |
| $y^2 + a_3y = x^3 + a_4x + a_6$ | 2 | $a_3^4$ |
| $y^2 = x^3 + Ax^2 + Bx + C$ | 3 | $4A^3C - A^2B^2 - 18ABC$ $+4B^3 + 27C^2$ |

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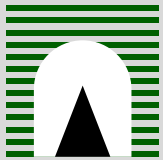
### Definition (Elliptic curve)

An elliptic curve is a non singular Weierstraß equation (i.e. $\Delta_E \neq 0$)





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**Note:** If $p = 0$ or $p \geq 3$, $\Delta_E = 0 \Leftrightarrow x^3 + Ax^2 + Bx + C$ has double roots

# The definition of $E(K)$

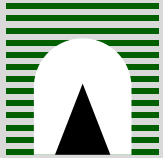
Let $E/K$ elliptic curve, $\infty := [0,1,0]$. Set

$$E(K) = \{[X,Y,Z] \in \mathbb{P}_2(K) : \\ Y^2Z + a_1XYZ + a_3YZ^2 = X^3 + a_2X^2Z + a_4XZ^2 + a_6Z^3\}$$

or equivalently

$$E(K) = \\ \{(x,y) \in K^2 : \ y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6\} \cup \{\infty\}$$





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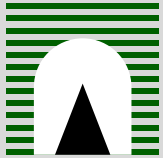

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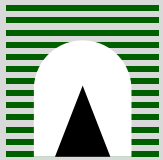

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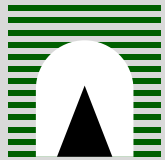

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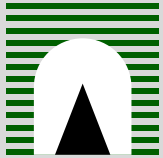

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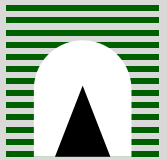
**Definition (line through points $P, Q \in E(K)$)**

$$r_{P,Q} : \begin{cases} \text{line through } P \text{ and } Q & \text{if } P \neq Q \\ \text{tangent line to } E \text{ at } P & \text{if } P = Q \end{cases}$$

projective or affine





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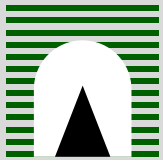
$$r_{P,\infty} : \text{vertical line through } P$$





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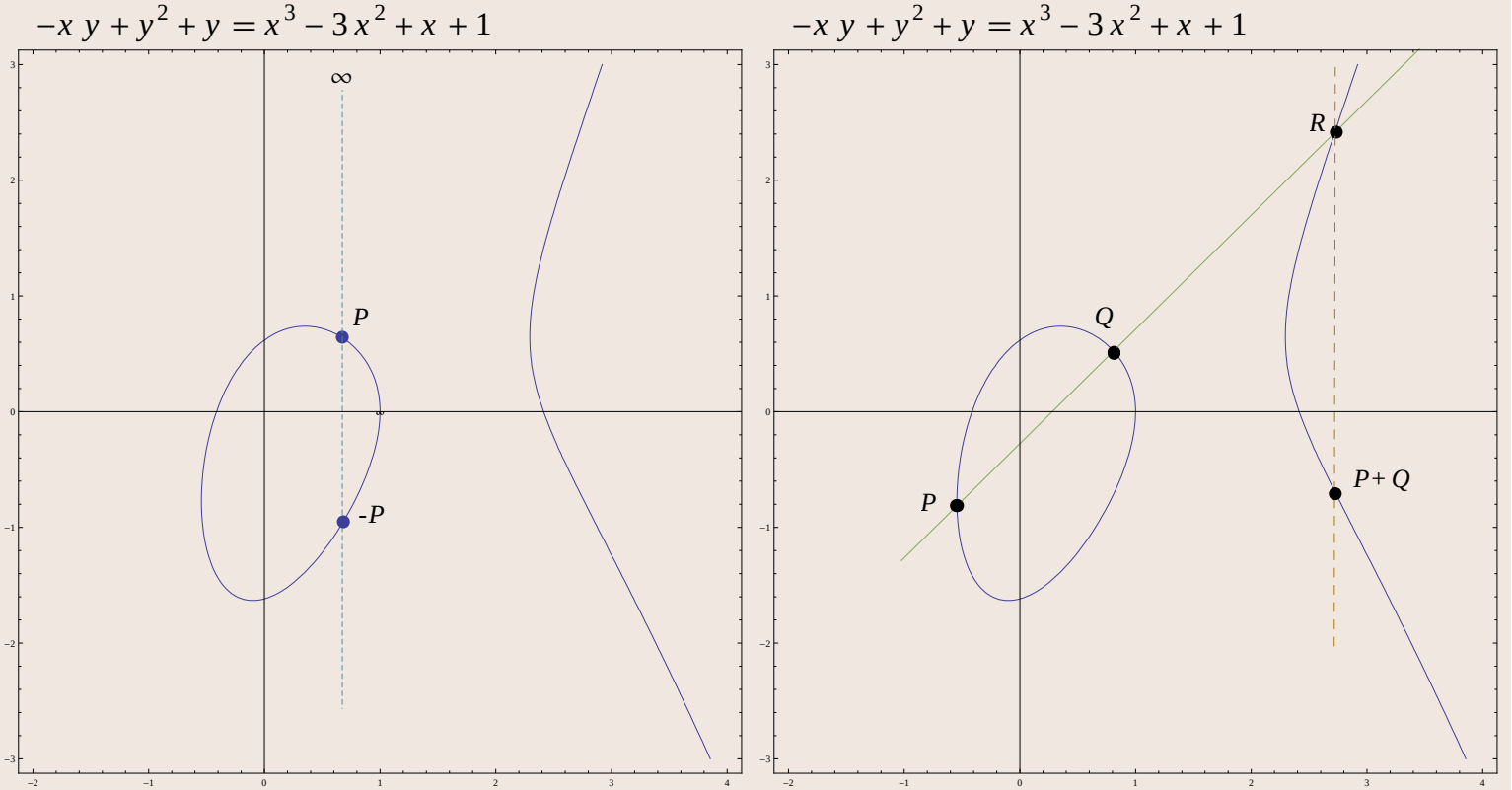




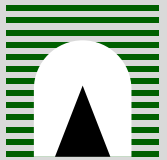

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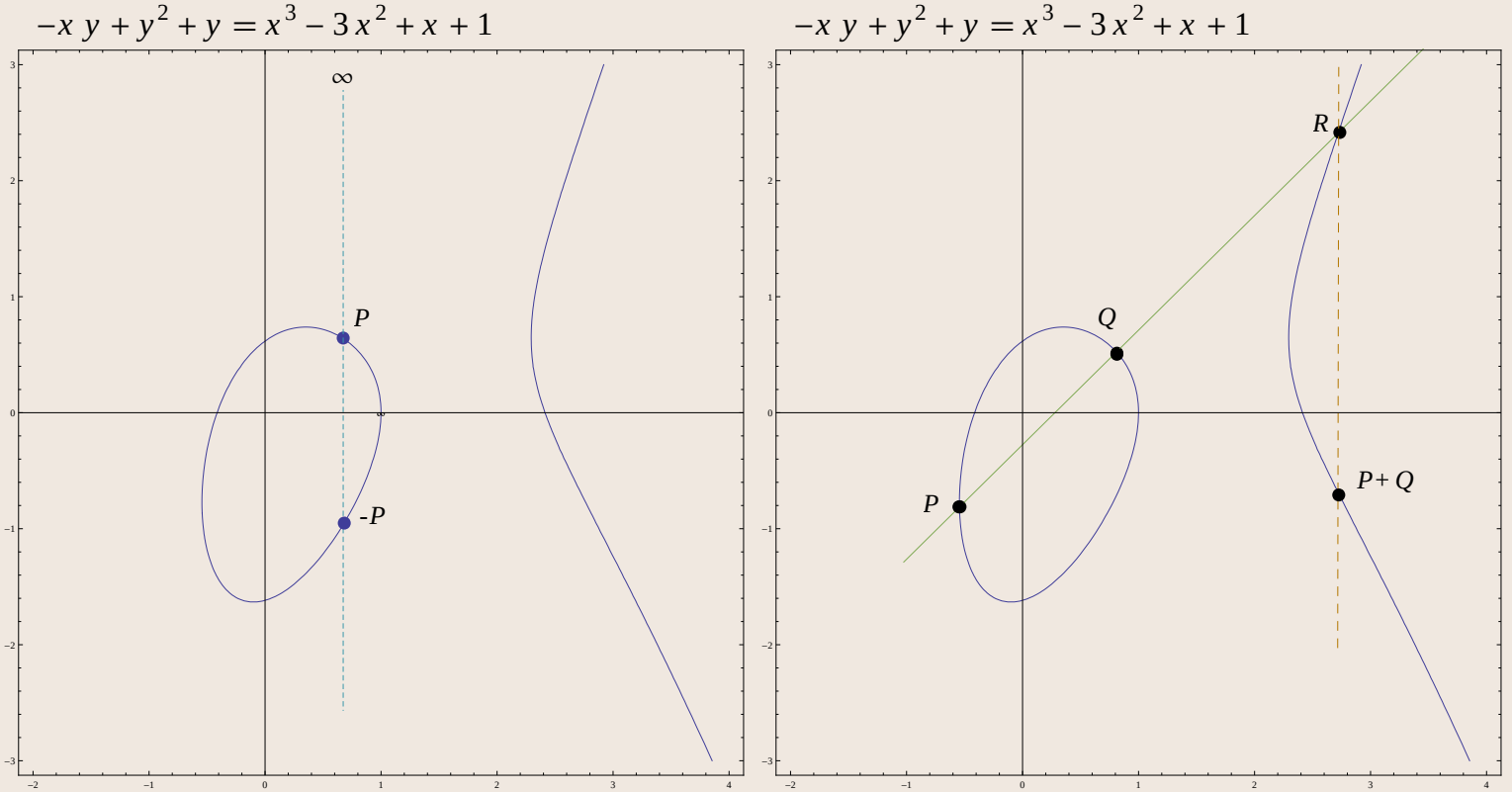



$r_{P,\infty} \cap E(K) = \{P, \infty, P'\}$ $\rightsquigarrow$ $-P := P'$

$r_{P,Q} \cap E(K) = \{P, Q, R\}$ $\rightsquigarrow$ $P +_E Q := -R$

## Theorem

*The addition law on $E/K$ ($K$ field) has the following properties:*

(a) $P +_E Q \in E$ $\quad \forall P, Q \in E$

(b) $P +_E \infty = \infty +_E P = P$ $\quad \forall P \in E$

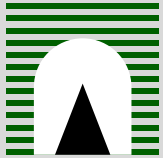
(c) $P +_E (-P) = \infty$ $\quad \forall P \in E$

(d) $P +_E (Q +_E R) = (P +_E Q) +_E R$ $\quad \forall P, Q, R \in E$

(e) $P +_E Q = Q +_E P$ $\quad \forall P, Q \in E$

*So $(E(\bar{K}), +_E)$ is an abelian group.*





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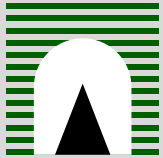
If $E/K \Rightarrow \forall L, K \subseteq L \subseteq \bar{K}, E(L)$ is an abelian group.





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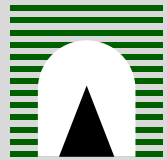
$$-P = -(x_1, y_1) = (x_1, -a_1 x_1 - a_3 - y_1)$$

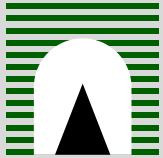




# Proof of the associativity

$$P +_E (Q +_E R) = (P +_E Q) +_E R \quad \forall P, Q, R \in E$$





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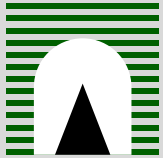

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Here we deal with the *generic case*. i.e. All the points $\pm P, \pm R, \pm Q, \pm(Q +_E R), \pm(P +_E Q), \infty$ all different

Mathematica code

```
L[x_,y_,r_,s_]:=(s-y)/(r-x);
M[x_,y_,r_,s_]:=(yr-sx)/(r-x);
A[{x_,y_},{r_,s_}]:={(L[x,y,r,s])^2-(x+r),
              -(L[x,y,r,s])^3+L[x,y,r,s](x+r)-M[x,y,r,s]}
Together[A[A[{x,y},{u,v}],{h,k}]-A[{x,y},A[{u,v},{h,k}]]]
det = Det[({{1,x1,x1^3-y1^2},{1,x2,x2^3-y2^2},{1,x3,x3^3-y3^2}})]
PolynomialQ[Together[Numerator[Factor[res[[1]]]]/det],
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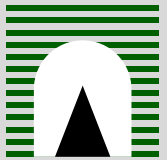

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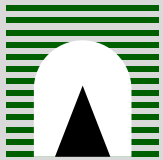
- runs in 2 seconds on a PC





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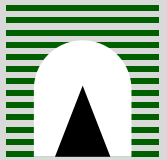
- More cases to check. e.g $P +_E 2Q = (P +_E Q) +_E Q$





# Formulas for Addition on $E$ (Summary)

$$E : y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$$

$P_1 = (x_1, y_1), P_2 = (x_2, y_2) \in E(K) \setminus \{\infty\},$

## Addition Laws for the sum of affine points

- If $P_1 \neq P_2$
  - $x_1 = x_2$ $\Rightarrow$ $P_1 +_E P_2 = \infty$
  - $x_1 \neq x_2$

$$\lambda = \frac{y_2 - y_1}{x_2 - x_1} \qquad \nu = \frac{y_1x_2 - y_2x_1}{x_2 - x_1}$$

- If $P_1 = P_2$
  - $2y_1 + a_1x + a_3 = 0$ $\Rightarrow$ $P_1 +_E P_2 = 2P_1 = \infty$
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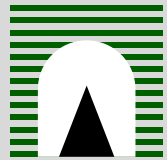

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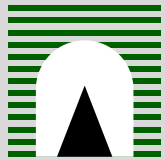
Then

$$P_1 +_E P_2 = (\lambda^2 - a_1\lambda - a_2 - x_1 - x_2, -\lambda^3 - a_1^2\lambda + (\lambda + a_1)(a_2 + x_1 + x_2) - a_3 - \nu)$$





# Formulas for Addition on $E$ (Summary for special equation)

$$E : y^2 = x^3 + Ax + B$$

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- If $P_1 = P_2$
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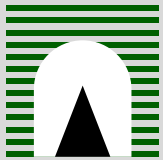
  - $y_1 \neq 0$

$$\lambda = \frac{3x_1^2 + A}{2y_1}, \nu = -\frac{x_1^3 - Ax_1 - 2B}{2y_1}$$

Then

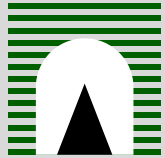
$$P_1 +_E P_2 = (\lambda^2 - x_1 - x_2, -\lambda^3 + \lambda(x_1 + x_2) - \nu)$$





# Points of order (dividing) $m$





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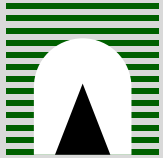

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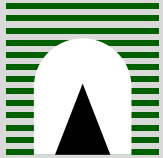

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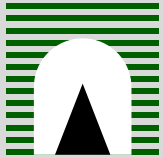

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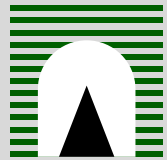
$$E/\mathbb{F}_p \text{ is called } \begin{cases} \textit{ordinary} & \text{if } E[p] \cong C_p \\ \textit{supersingular} & \text{if } E[p] = \{\infty\} \end{cases}$$





# Group Structure of $E(\mathbb{F}_q)$

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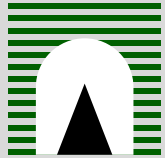

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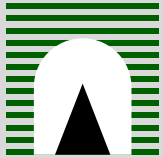

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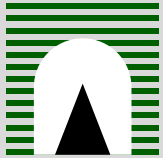
with $n_i | n_{i+1}$ for $i \geq 1$.





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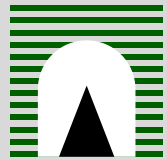

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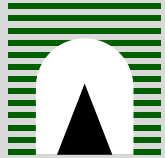
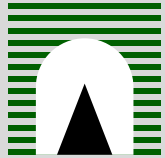
From *Structure of Torsion Theorem*, $\#E[n_1] \leq n_1^2$. So $r \leq 2$ □





# The division polynomials





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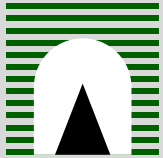

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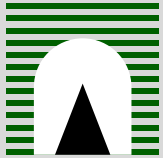
The polynomial $\psi_m \in \mathbb{Z}[x, y]$ is called the $m^{\text{th}}$ *division polynomial*





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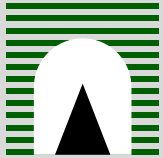
There are more more complicated formulas for general Weierstraß equations.





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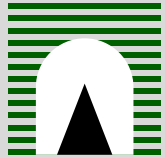

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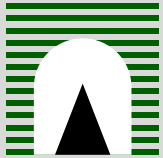

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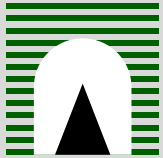

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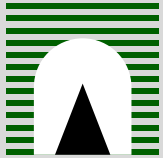

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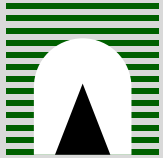

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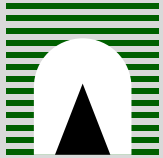

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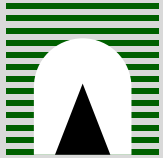

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- The structure theorem of $E[m]$ follows form these properties

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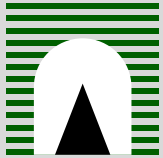

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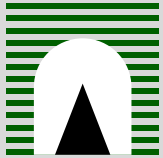

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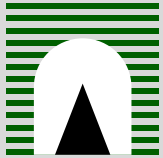

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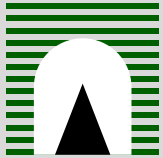

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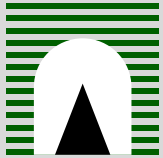

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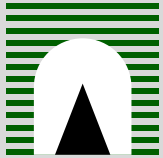

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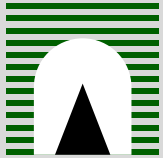

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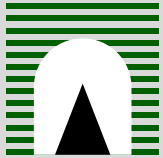

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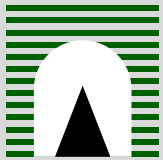

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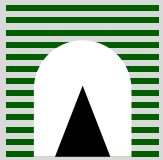

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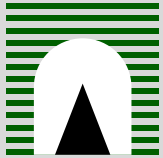
- $\Phi_q(x, y) = (x^q, y^q)$ is endomorphism





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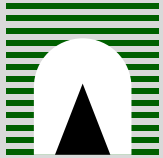

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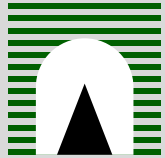

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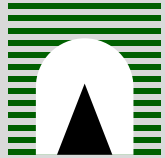

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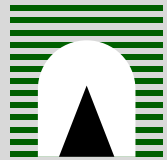
- Let $\alpha \neq 0$ be an endomorphism. Then

$$\#\operatorname{Ker}(\alpha) \begin{cases} = \deg \alpha & \text{if } \alpha \text{ is separable} \\ < \deg \alpha & \text{otherwise} \end{cases}$$





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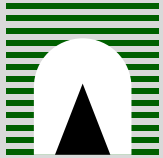

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Let $E/K$. The *ring of endomorphisms*

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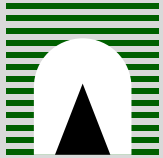

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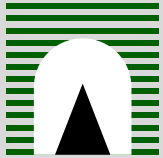

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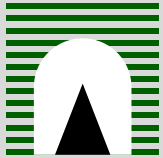

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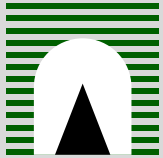

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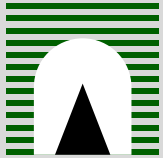

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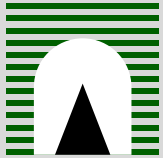

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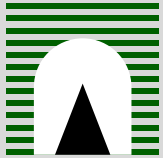

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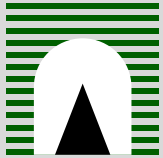

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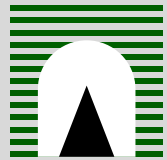

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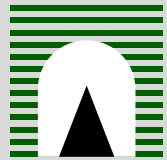
- if $K = \mathbb{F}_q$, $\Phi_q \in \mathrm{End}(E)$. So $\mathbb{Z}[\Phi_q] \subset \mathrm{End}(E)$
- $\Phi_q$ satisfied in $\mathrm{End}(E)$ the polynomial $X^2 - a_q X + q$ where $E(\mathbb{F}_q) = q + 1 - a_q$





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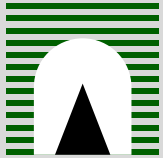

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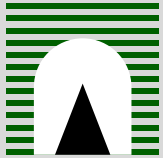

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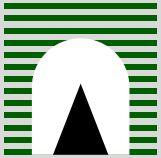

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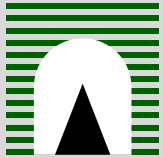

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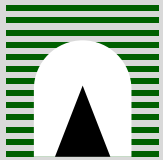

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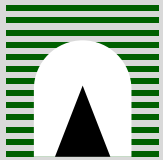

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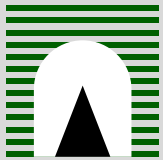

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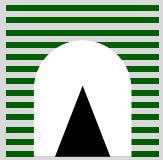

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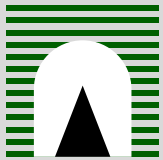

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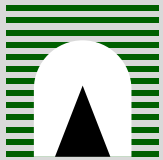

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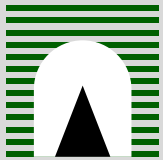

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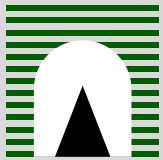

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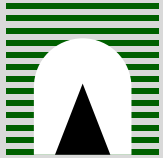

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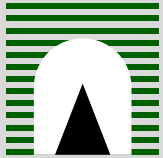

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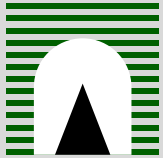

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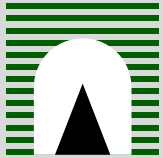

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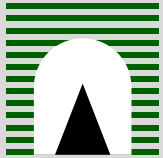

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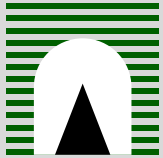
We shall focus on elliptic curves without CM.





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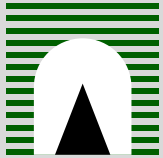

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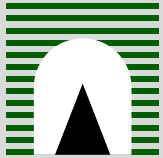

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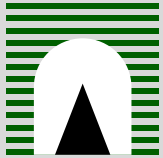

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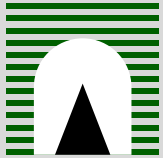

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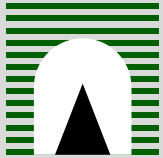

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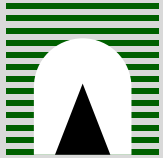

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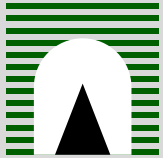

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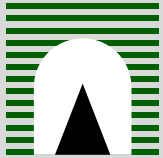

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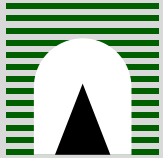

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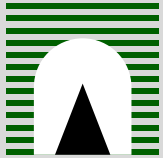

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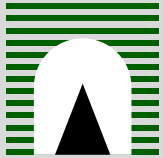

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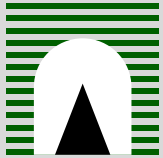

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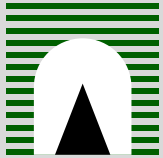
    apply $\sigma$ to the equations defining the group law





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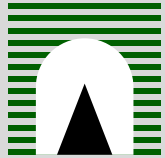

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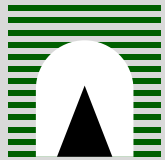

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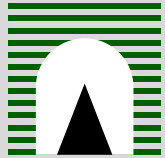

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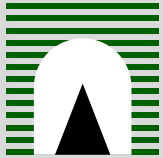
- By identifying $\mathrm{Aut}(E[m])$ with $\mathrm{Aut}(\mathbb{Z}/m\mathbb{Z} \otimes \mathbb{Z}/m\mathbb{Z})$, we can think at the image of $\rho_{E,m}$ as a subgroup of $\mathrm{GL}_2(\mathbb{Z}/m\mathbb{Z})$





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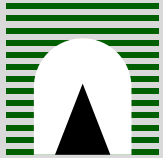
Assume that $E$ is without complex multiplication ($\mathrm{End}(E) \cong \mathbb{Z}$) then $\rho_{E,\ell}$, is usually surjective.





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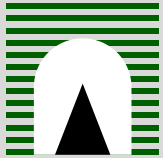

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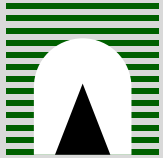

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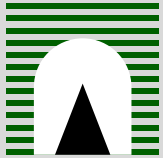

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For each number field $K$ there is a uniform bound $\ell_{\max}$ such that

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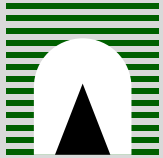

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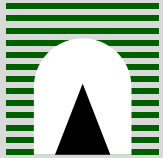
For $K = \mathbb{Q}$, it is generally believed that $\ell_{\max} = 37$.





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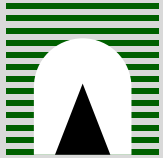

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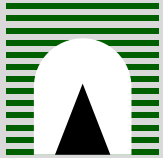

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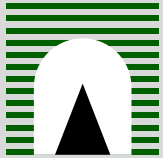

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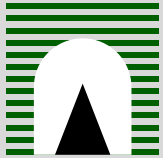

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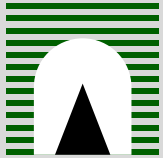

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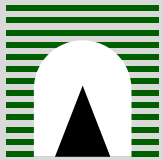

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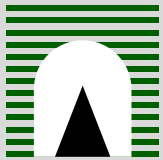

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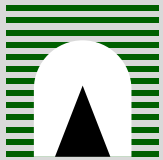
Even when $E$ has a rational $\ell$-torsion point, this does not determine the image of $\rho_{E,\ell}$.





# Absolute Galois Group

- The absolute Galois group
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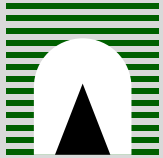

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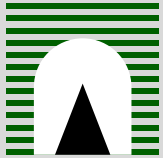

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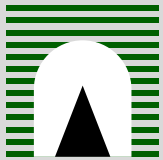
- So $G_{\mathbb{Q}}$ admits as quotient any possible Galois Group of Galois extensions of $\mathbb{Q}$ and it is the projective limit of its finite quotients





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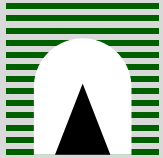
  $G_{\mathbb{Q}} := \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) = \{\sigma : \overline{\mathbb{Q}} \to \overline{\mathbb{Q}},\ \text{field automorphism}\}$
  is a profinite group
- If $K$ is any Galois extension of $\mathbb{Q}$, then

$$\mathrm{Gal}(K/\mathbb{Q}) \cong G_{\mathbb{Q}}/\{\sigma \in G_{\mathbb{Q}} : \sigma_{|K} = \mathrm{id}_K\}$$

- So $G_{\mathbb{Q}}$ admits as quotient any possible Galois Group of Galois extensions of $\mathbb{Q}$ and it is the projective limit of its finite quotients
- Recall *$n$–torsion field* $\mathbb{Q}(E[n])$ and $G_m = \mathrm{Gal}(\mathbb{Q}(E[n])/\mathbb{Q})$.





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- Recall *$n$–torsion field* $\mathbb{Q}(E[n])$ and $G_m = \mathrm{Gal}(\mathbb{Q}(E[n])/\mathbb{Q})$.
- The mod $m$–representation

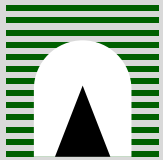
$$\rho_{E,n} : G_n \hookrightarrow \mathrm{Aut}(E[n]) \cong \mathrm{GL}_2(\mathbb{Z}/n\mathbb{Z})$$

can be extended to

$$\rho_E : G_{\mathbb{Q}} \longrightarrow \mathrm{Aut}(E[\infty])$$

where $E[\infty] = \cup_{m\in\mathbb{N}} E[m]$ is the *torsion subgroup* of $E(\overline{\mathbb{Q}})$.





# $\ell$–adic representations

Consider the decomposition:

$$\mathrm{Aut}(E[\infty]) = \prod_{\ell \text{ prime}} \mathrm{Aut}(E[\ell^\infty]) \cong \prod_{\ell \text{ prime}} \mathrm{GL}_2(\mathbb{Z}_\ell).$$

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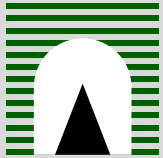

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For every fixed prime $\ell$, the projection

$$\rho_{E,\ell^\infty} : G_{\mathbb{Q}} \longrightarrow \mathrm{GL}_2(\mathbb{Z}_\ell)$$

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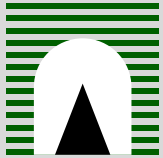

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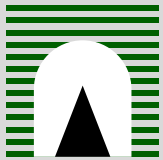

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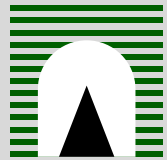

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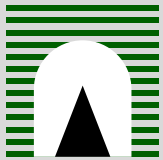

# Serre Uniformity Theorem

The statements:

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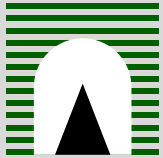

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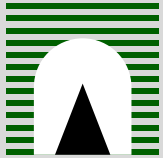

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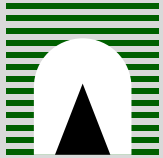

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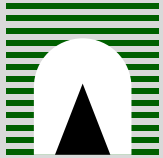

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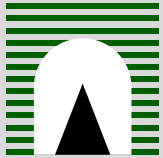

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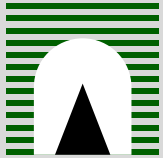

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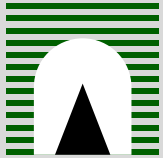

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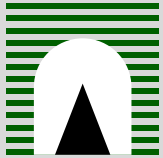

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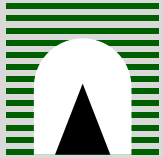

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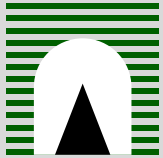
$$[\mathrm{Aut}(E[\infty]) : \mathcal{H}_E] = 2$$

$\mathcal{H}_E$ is the *Serre's Subgroup*





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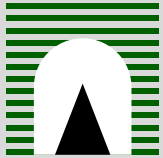

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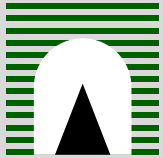

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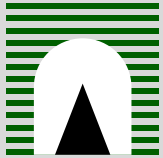

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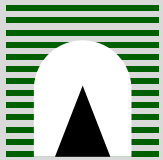

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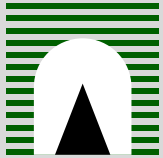

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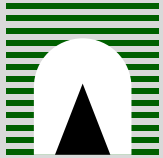
**Theorem (N. Jones (2010))**

*Almost all elliptic curves are Serre's curves*





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If $E$ admits a rational $\ell$–isogeny (a $\mathbb{Q}$–rational morphism of degree $\ell$, $E' \to E$), then $E$ it is NOT a Serre's curve

# The Frobenius Elements

## Definition

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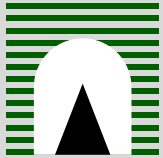

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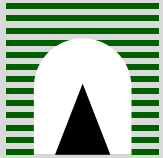

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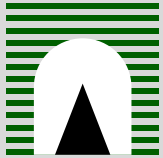

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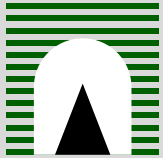

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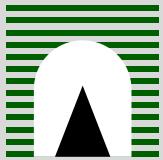

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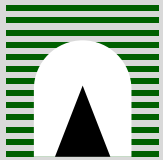

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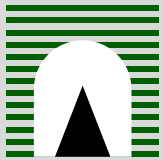

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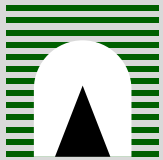

# Chebotarev Density Theorem

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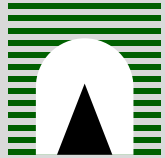

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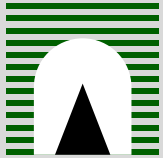

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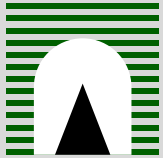

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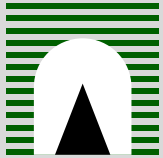

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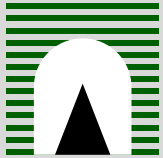

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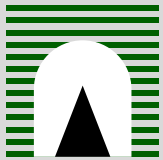

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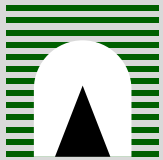
- The Generalized Riemann Hypothesis implies

$$\pi_{\mathcal{C}/\mathcal{G}}(x) = \frac{\#\mathcal{C}}{\#\mathcal{G}} \int_2^x \frac{dt}{\log x} + O\left(\sqrt{\#\mathcal{C}}\sqrt{x}\log(xM\#\mathcal{G})\right)$$

where $M$ is the product of primes numbers that ramify in $K/\mathbb{Q}$.





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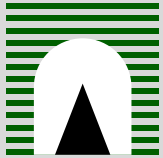

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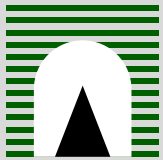

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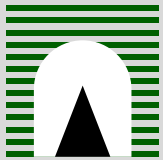
- In the case when $\mathcal{C} = \mathcal{G}_{\mathrm{tr}=r} = \{\sigma \in \mathcal{G} : \mathrm{tr}\,\sigma = t\}$, and $\ell$ is a sufficiently large prime so that $\mathrm{Gal}(\mathbb{Q}(E[\ell])/\mathbb{Q}) = \mathrm{GL}_2(\mathbb{F}_\ell)$, then

$$\#\,\mathrm{GL}_2(\mathbb{F}_\ell)_{\mathrm{tr}=r} = \begin{cases} \ell^2(\ell-1) & \text{if } r = 0 \\ \ell(\ell^2-\ell-1) & \text{otherwise.} \end{cases}$$





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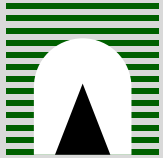

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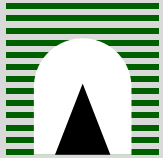

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The following asymptotic formula holds

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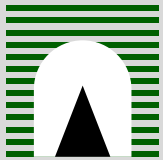

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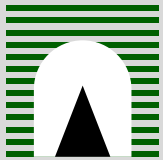

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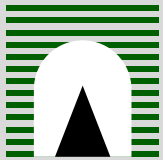

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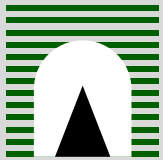
$$\prod_{\ell} \left(1 - \frac{1}{(\ell^2 - \ell)(\ell^2 - 1)}\right).$$





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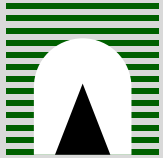

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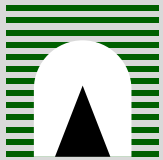

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Let $E/\mathbb{Q}$, $r \in \mathbb{Z}$ and set

$$\pi_E^r(x) = \#\{p \leq x : p \nmid \Delta_E \text{ and } \#\overline{E}(\mathbb{F}_p) = p + 1 - r\}$$

**Conjecture (Lang – Trotter (1970))**

If either $r \neq 0$ or if $E$ has no CM, then the following asymptotic formula holds

$$\pi_E^r(x) \sim C_{E,r} \frac{\sqrt{x}}{\log x} \qquad x \to \infty$$

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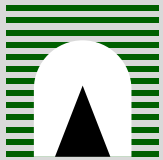

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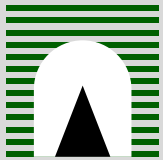

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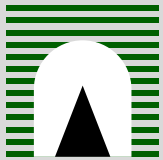
$$C_{E,r} = \frac{2}{\pi} \lim_{m\to\infty} \frac{k_m \# \operatorname{Gal}(\mathbb{Q}(E[k_m])/\mathbb{Q})_{\text{trace}=r}}{\# \operatorname{Gal}(\mathbb{Q}(E[K_m])/\mathbb{Q})}$$





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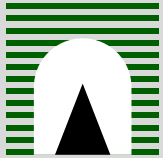

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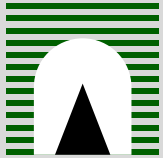

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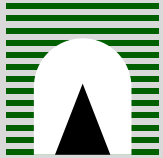

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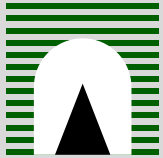

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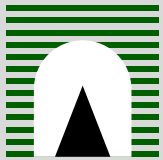
$$\hat{G}_m = r_m^{-1}(\mathrm{Gal}(\mathbb{Q}(E[m])/\mathbb{Q}))$$

where

$$r_m : \prod_{\ell|m} \mathrm{GL}_2(\mathbb{Z}_\ell) \to \mathrm{Gal}(\mathbb{Q}(E[m])/\mathbb{Q})$$

is the reduction map





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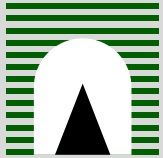
**Theorem (Serre)**

*Let $E/\mathbb{Q}$ be an elliptic curve with out CM. Then there exists $m \in \mathbb{N}$ that splits and stabilizes $\rho_E$*





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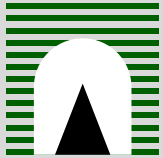

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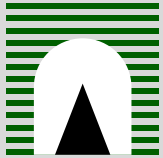

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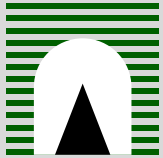
Although it is hard to compute in General, there is a simple formula to compute the Serre's conductor of Serre's curves. (more next lecture)

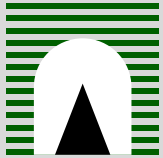




# Lang Trotter Conjecture for trace of Frobenius

## An application of $\ell$–adic representations and of the Chebotarev density Theorem





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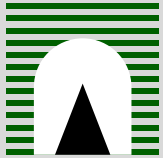

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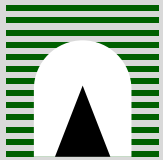
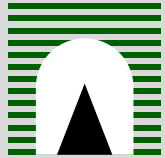


# Lang Trotter Conjecture for Primitive points





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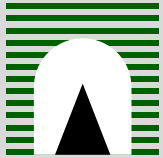
## Definition

Let $E/\mathbb{Q}$ and let $P \in E(\mathbb{Q})$ be of infinite order. $P$ is called *primitive* for a prime $p$ if the reduction $\overline{P}$ of $P \bmod p$ $\langle \overline{P} \rangle = \overline{E}(\mathbb{F}_p)$





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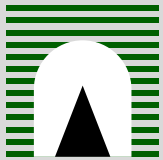

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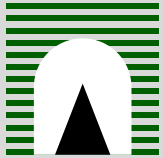

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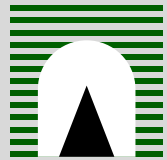
$$\delta_{E,P} = \sum_{n=1}^{\infty} \mu(n) \frac{\#\mathcal{C}_{P,n}}{\#\operatorname{Gal}(\mathbb{Q}(E[n], n^{-1}P)/\mathbb{Q})}$$

where $\mathbb{Q}(E[n], n^{-1}P)$ is the extension of $\mathbb{Q}(E[n])$ of the coordinates of the points $Q \in E(\bar{\mathbb{Q}})$ such that $nQ = P$ and $\mathcal{C}_{P,n}$ is a union of conjugacy classes in $\operatorname{Gal}(\mathbb{Q}(E[n], n^{-1}P)/\mathbb{Q})$. (more next lecture)





# Some reading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