Lecture 2

Elliptic curves over finite fields

The Group structure

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Elliptic curves over $\mathbb{F}_q$

Definition (Elliptic curve)

An elliptic curve over a field K is the data of a non singular Weierstraß equation
$E : y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6, a_i \in K$

If $p = \operatorname{char} K > 3$,

$$\Delta_E := \frac{1}{2^4}\left(-a_1^5a_3a_4 - 8a_1^3a_2a_3a_4 - 16a_1a_2^2a_3a_4 + 36a_1^2a_3^2a_4\right.$$
$$- a_1^4a_4^2 - 8a_1^2a_2a_4^2 - 16a_2^2a_4^2 + 96a_1a_3a_4^2 + 64a_4^3 +$$
$$a_1^6a_6 + 12a_1^4a_2a_6 + 48a_1^2a_2^2a_6 + 64a_2^3a_6 - 36a_1^3a_3a_6$$
$$\left.-144a_1a_2a_3a_6 - 72a_1^2a_4a_6 - 288a_2a_4a_6 + 432a_6^2\right)$$

Elliptic curves over K

After applying a suitable affine transformation we can always assume that E/K has a Weierstraß equation of the following form

Example (Classification ($p = \text{char } K$))

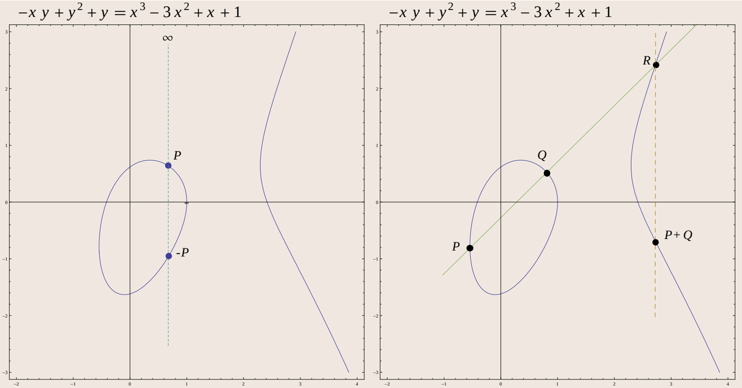
E	p	Δ_E
$y^2 = x^3 + Ax + B$	≥ 5	$4A^3 + 27B^2$
$y^2 + xy = x^3 + a_2x^2 + a_6$	2	a_6^2
$y^2 + a_3y = x^3 + a_4x + a_6$	2	a_3^4
$y^2 = x^3 + Ax^2 + Bx + C$	3	$4A^3C - A^2B^2 - 18ABC$ $+4B^3 + 27C^2$

Let $E/\mathbb{F}_q$ elliptic curve, $\infty := [0, 1, 0]$. Set

$$E(\mathbb{F}_q) = \{(x, y) \in \mathbb{F}_q^2 : \ y^2 = x^3 + Ax + B\} \cup \{\infty\}$$

If $P, Q \in E(\mathbb{F}_q)$, $r_{P,Q}$: $\begin{cases} \text{line through } P \text{ and } Q & \text{if } P \neq Q \\ \text{tangent line to } E \text{ at } P & \text{if } P = Q, \end{cases}$

$r_{P,\infty}$: vertical line through P



$r_{P,\infty} \cap E(\mathbb{F}_q) = \{P, \infty, P'\}$ $\rightsquigarrow$ $\boxed{-P := P'}$

$r_{P,Q} \cap E(\mathbb{F}_q) = \{P, Q, R\}$ $\rightsquigarrow$ $\boxed{P +_E Q := -R}$

Theorem

The addition law on E/K (K field) has the following properties:

(a) $P +_E Q \in E \qquad \forall P, Q \in E$

(b) $P +_E \infty = \infty +_E P = P \qquad \forall P \in E$

(c) $P +_E (-P) = \infty \qquad \forall P \in E$

(d) $P +_E (Q +_E R) = (P +_E Q) +_E R \qquad \forall P, Q, R \in E$

(e) $P +_E Q = Q +_E P \qquad \forall P, Q \in E$

So $(E(\bar{K}), +_E)$ is an abelian group.

Remark:

If $E/K \Rightarrow \forall L, K \subseteq L \subseteq \bar{K}, E(L)$ is an abelian group.

$$-P = -(x_1, y_1) = (x_1, -a_1 x_1 - a_3 - y_1)$$

Formulas for Addition on E (Summary)

$$E : y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$$

$$P_1 = (x_1, y_1), P_2 = (x_2, y_2) \in E(K) \setminus \{\infty\},$$

Addition Laws for the sum of affine points

- If $P_1 \neq P_2$
  - $x_1 = x_2 \quad \Rightarrow \quad P_1 +_E P_2 = \infty$
  - $x_1 \neq x_2$

    $$\lambda = \frac{y_2 - y_1}{x_2 - x_1} \qquad \nu = \frac{y_1x_2 - y_2x_1}{x_2 - x_1}$$
- If $P_1 = P_2$
  - $2y_1 + a_1x + a_3 = 0 \quad \Rightarrow \quad P_1 +_E P_2 = 2P_1 = \infty$
  - $2y_1 + a_1x + a_3 \neq 0$

    $$\lambda = \frac{3x_1^2 + 2a_2x_1 + a_4 - a_1y_1}{2y_1 + a_1x + a_3}, \nu = -\frac{a_3y_1 + x_1^3 - a_4x_1 - 2a_6}{2y_1 + a_1x_1 + a_3}$$

Then

$$P_1 +_E P_2 = (\lambda^2 - a_1\lambda - a_2 - x_1 - x_2, -\lambda^3 - a_1^2\lambda + (\lambda + a_1)(a_2 + x_1 + x_2) - a_3 - \nu)$$

Formulas for Addition on E (Summary for special equation)

$$E: y^2 = x^3 + Ax + B$$

$P_1 = (x_1, y_1), P_2 = (x_2, y_2) \in E(K) \setminus \{\infty\}$,

Addition Laws for the sum of affine points

- If $P_1 \neq P_2$
  - $x_1 = x_2$ $\Rightarrow$ $P_1 +_E P_2 = \infty$
  - $x_1 \neq x_2$

$$\lambda = \frac{y_2 - y_1}{x_2 - x_1} \qquad \nu = \frac{y_1 x_2 - y_2 x_1}{x_2 - x_1}$$

- If $P_1 = P_2$
  - $y_1 = 0$ $\Rightarrow$ $P_1 +_E P_2 = 2P_1 = \infty$
  - $y_1 \neq 0$

$$\lambda = \frac{3x_1^2 + A}{2y_1}, \nu = -\frac{x_1^3 - Ax_1 - 2B}{2y_1}$$

Then

$$P_1 +_E P_2 = (\lambda^2 - x_1 - x_2, -\lambda^3 + \lambda(x_1 + x_2) - \nu)$$

Notations

Finite fields

1. $\mathbb{F}_p = \{0, 1, \dots, p-1\}$ is the prime field;
2. $\mathbb{F}_q$ is a finite field with $q = p^n$ elements;
3. $\mathbb{F}_q = \mathbb{F}_p[\xi]$, $f(\xi) = 0$, $f \in \mathbb{F}_p[X]$ irreducible, $\partial f = n$;
4. $\mathbb{F}_4 = \mathbb{F}_2[\xi]$, $\xi^2 = 1 + \xi$;
5. $\mathbb{F}_8 = \mathbb{F}_2[\alpha]$, $\alpha^3 = \alpha + 1$ but also $\mathbb{F}_8 = \mathbb{F}_2[\beta]$, $\beta^3 = \beta^2 + 1$, $(\beta = \alpha^2 + 1)$;
6. $\mathbb{F}_{101^{101}} = \mathbb{F}_{101}[\omega], \omega^{101} = \omega + 1$

Algebraic Closure of $\mathbb{F}_q$

1. $\forall n \in \mathbb{N}$, we fix an $\mathbb{F}_{q^n}$
2. We also require that $\mathbb{F}_{q^n} \subseteq \mathbb{F}_{q^m}$ if $n \mid m$
3. We let $\overline{\mathbb{F}}_q = \bigcup_{n \in \mathbb{N}} \mathbb{F}_{q^n}$
4. $\overline{\mathbb{F}}_q$ is *algebraically closed*

If $F(x, y) \in \mathbb{F}_q[x, y]$ a *point of the curve* $F = 0$, means $(x_0, y_0) \in \overline{\mathbb{F}}_q^2$ s.t. $F(x_0, y_0) = 0$.

The j-invariant

Let $E/K : y^2 = x^3 + Ax + B$, $p \geq 5$ and $\Delta_E := 4A^3 + 27B^2$.

$$\begin{cases} x \longleftarrow u^{-2}x \\ y \longleftarrow u^{-3}y \end{cases} \quad u \in K^* \;\Rightarrow\; E \longrightarrow E_u : y^2 = x^3 + u^4Ax + u^6B$$

Definition

The j–invariant of E is $j = j(E) = 1728\frac{4A^3}{4A^3+27B^2}$

Properties of j–invariants

1. $j(E) = j(E_u), \forall u \in K^*$
2. $j(E'/K) = j(E''/K) \;\Rightarrow\; \exists u \in \bar{K}^*$ s.t. $E'' = E'_u$

   if $K = \mathbb{F}_q$ can take $u \in \mathbb{F}_{q^{12}}$
3. $j \neq 0, 1728 \Rightarrow E : y^2 = x^3 + \frac{3j}{1728-j}x + \frac{2j}{1728-j}, j(E) = j$
4. $j = 0 \;\Rightarrow\; E : y^2 = x^3 + B, \quad j = 1728 \;\Rightarrow\; E : y^2 = x^3 + Ax$
5. $j : K \longleftrightarrow \{\bar{K}$–affinely equivalent classes of $E/K\}$.
6. $p = 2, 3$ different definition

Examples of j invariants

From monday $E_1 : y^2 = x^3 + 1$ and $E_2 : y^2 = x^3 + 2$

$$\#E_1(\mathbb{F}_5) = \#E_2(\mathbb{F}_5) = 6 \qquad \text{and} \qquad j(E_1) = j(E_2) = 0$$

$$\begin{cases} x \longleftarrow 2x \\ y \longleftarrow \sqrt{3}y \end{cases}$$

E_1 and E_2 affinely equivalent over $\mathbb{F}_5[\sqrt{3}] = \mathbb{F}_{25}$ (*twists*)

Definition (twisted curve)

Let $E/\mathbb{F}_q : y^2 = x^3 + Ax + B, \mu \in \mathbb{F}_q^* \setminus (\mathbb{F}_q^*)^2$.

$$E_\mu : y^2 = x^3 + \mu^2 Ax + \mu^3 B$$

is called twisted curve.

Exercise: prove that

- $j(E) = j(E_\mu)$
- E and E_μ are $\mathbb{F}_q[\sqrt{\mu}]$–affinely equivalent
- $\#E(\mathbb{F}_{q^2}) = \#E_\mu(\mathbb{F}_{q^2})$
- usually $\#E(\mathbb{F}_q) \neq \#E_\mu(\mathbb{F}_q)$

Determining points of order 2

Let $P = (x_1, y_1) \in E(\mathbb{F}_q) \setminus \{\infty\}$,

$$P \text{ has order } 2 \iff 2P = \infty \iff P = -P$$

So

$$-P = (x_1, -a_1x_1 - a_3 - y_1) = (x_1, y_1) = P \implies 2y_1 = -a_1x_1 - a_3$$

If $p \neq 2$, can assume $E : y^2 = x^3 + Ax^2 + Bx + C$

$$-P = (x_1, -y_1) = (x_1, y_1) = P \implies y_1 = 0, x_1^3 + Ax_1^2 + Bx_1 + C = 0$$

Note

- the number of points of order 2 in $E(\mathbb{F}_q)$ equals the number of roots of $X^3 + Ax^2 + Bx + C$ in $\mathbb{F}_q$
- roots are distinct since discriminant $\Delta_E \neq 0$
- $E(\mathbb{F}_{q^6})$ has always 3 points of order 2 if $E/\mathbb{F}_q$
- $E[2] := \{P \in E(\bar{\mathbb{F}}_q) : 2P = \infty\} \cong C_2 \oplus C_2$

Determining points of order 2 (continues)

- If $p = 2$ and $E : y^2 + a_3 y = x^3 + a_2 x^2 + a_6$

$$-P = (x_1, a_3 + y_1) = (x_1, y_1) = P \implies a_3 = 0$$

  Absurd ($a_3 = 0$) and there are no points of order 2.

- If $p = 2$ and $E : y^2 + xy = x^3 + a_4 x + a_6$

$$-P = (x_1, x_1 + y_1) = (x_1, y_1) = P \implies x_1 = 0, y_1^2 = a_6$$

  So there is exactly one point of order 2 namely $(0, \sqrt{a_6})$

Definition

2–torsion points

$$E[2] = \{P \in E : 2P = \infty\}.$$

In conclusion

$$E[2] \cong \begin{cases} C_2 \oplus C_2 & \text{if } p > 2 \\ C_2 & \text{if } p = 2, E : y^2 + xy = x^3 + a_4 x + a_6 \\ \{\infty\} & \text{if } p = 2, E : y^2 + a_3 y = x^3 + a_2 x^2 + a_6 \end{cases}$$

Elliptic curves over $\mathbb{F}_2, \mathbb{F}_3$ and $\mathbb{F}_5$

Each curve $/\mathbb{F}_2$ has cyclic $E(\mathbb{F}_2)$.

E	$E(\mathbb{F}_2)$	$\lvert E(\mathbb{F}_2)\rvert$
$y^2+xy=x^3+x^2+1$	$\{\infty,(0,1)\}$	2
$y^2+xy=x^3+1$	$\{\infty,(0,1),(1,0),(1,1)\}$	4
$y^2+y=x^3+x$	$\{\infty,(0,0),(0,1),(1,0),(1,1)\}$	5
$y^2+y=x^3+x+1$	$\{\infty\}$	1
$y^2+y=x^3$	$\{\infty,(0,0),(0,1)\}$	3

- $E_1 : y^2 = x^3 + x$ $\qquad$ $E_2 : y^2 = x^3 - x$

  $E_1(\mathbb{F}_3) \cong C_4$ and $E_2(\mathbb{F}_3) \cong C_2 \oplus C_2$

- $E_3 : y^2 = x^3 + x$ $\qquad$ $E_4 : y^2 = x^3 + x + 2$

  $E_3(\mathbb{F}_5) \cong C_2 \oplus C_2$ and $E_4(\mathbb{F}_5) \cong C_4$

- $E_5 : y^2 = x^3 + 4x$ $\qquad$ $E_6 : y^2 = x^3 + 4x + 1$

  $E_5(\mathbb{F}_5) \cong C_2 \oplus C_4$ and $E_6(\mathbb{F}_5) \cong C_8$

Determining points of order 3

Let $P = (x_1, y_1) \in E(\mathbb{F}_q)$

$$P \text{ has order } 3 \iff 3P = \infty \iff 2P = -P$$

So, if $p > 3$ and $E : y^2 = x^2 + Ax + B$

$$2P = (x_{2P}, y_{2P}) = 2(x_1, y_1) = (\lambda^2 - 2x_1, -\lambda^3 + 2\lambda x_1 - \nu)$$

where $\lambda = \frac{3x_1^2+A}{2y_1}, \nu = -\frac{x_1^3-Ax_1-2B}{2y_1}$.

$$P \text{ has order } 3 \iff x_{2P} = x_1$$

$$\text{Substituting } \lambda, \quad x_{2P} - x_1 = \frac{-3x_1^4-6Ax_1^2-12Bx_1+A^2}{4(x_1^3+Ax_1+4B)} = 0$$

Note

- $\psi_3(x) := 3x^4 + 6Ax^2 + 12Bx - A^2$ the 3rd *division* polynomial
- $(x_1, y_1) \in E(\mathbb{F}_q)$ has order 3 $\quad \Rightarrow \psi_3(x_1) = 0$
- $E(\mathbb{F}_q)$ has at most 8 points of order 3
- If $p \neq 3$, $E[3] := \{P \in E : 3P = \infty\} \cong C_3 \oplus C_3$

Determining points of order 3 (continues)

Exercise

Let $E: y^2 = x^3 + Ax^2 + Bx + C, A, B, C \in \mathbb{F}_{3^n}$. Prove that if $P = (x_1, y_1) \in E(\mathbb{F}_{3^n})$ has order 3, then

1. $Ax_1^3 + AC - B^2 = 0$
2. $E[3] \cong C_3$ if $A \neq 0$ and $E[3] = \{\infty\}$ otherwise

Example (from Monday)

If $E: y^2 = x^3 + x + 1$, then $\#E(\mathbb{F}_5) = 9$.

$$\psi_3(x) = (x+3)(x+4)(x^2+3x+4)$$

Hence

$$E[3] = \left\{\infty, (2, \pm 1), (1, \pm\sqrt{3}), (1 \pm 2\sqrt{3}, \pm(1 \pm \sqrt{3}))\right\}$$

1. $E(\mathbb{F}_5) = \{\infty, (2, \pm 1), (0, \pm 1), (3, \pm 1), (4, \pm 2)\} \cong C_9$
2. Since $\mathbb{F}_{25} = \mathbb{F}_5[\sqrt{3}] \quad \Rightarrow \quad E[3] \subset E(\mathbb{F}_{25})$
3. $\#E(\mathbb{F}_{25}) = 27 \quad \Rightarrow \quad E(\mathbb{F}_{25}) \cong C_3 \oplus C_9$

Determining points of order 3 (continues)

Inequivalent curves $/\mathbb{F}_7$ with $\#E(\mathbb{F}_7) = 9$.

E	$\psi_3(x)$	$E[3] \cap E(\mathbb{F}_7)$	$E(\mathbb{F}_7) \cong$
$y^2 = x^3 + 2$	$x(x+1)(x+2)(x+4)$	$\left\{\begin{array}{l}\infty, (0, \pm 3), (-1, \pm 1), \\ (5, \pm 1), (3, \pm 1)\end{array}\right\}$	$C_3 \oplus C_3$
$y^2 = x^3 + 3x + 2$	$(x+2)(x^3 + 5x^2 + 3x + 2)$	$\{\infty, (5, \pm 3)\}$	C_9
$y^2 = x^3 + 5x + 2$	$(x+4)(x^3 + 3x^2 + 5x + 2)$	$\{\infty, (3, \pm 3)\}$	C_9
$y^2 = x^3 + 6x + 2$	$(x+1)(x^3 + 6x^2 + 6x + 2)$	$\{\infty, (6, \pm 3)\}$	C_9

Can one count the number of inequivalent $E/\mathbb{F}_q$ with $\#E(\mathbb{F}_q) = r$?

Example (A curve over $\mathbb{F}_4 = \mathbb{F}_2(\xi), \xi^2 = \xi + 1; \quad E : y^2 + y = x^3$)

We know $E(\mathbb{F}_2) = \{\infty, (0,0), (0,1)\} \subset E(\mathbb{F}_4)$.

$E(\mathbb{F}_4) = \{\infty, (0,0), (0,1), (1,\xi), (1,\xi+1), (\xi,\xi), (\xi,\xi+1), (\xi+1,\xi), (\xi+1,\xi+1)\}$

$$\psi_3(x) = x^4 + x = x(x+1)(x+\xi)(x+\xi+1) \Rightarrow E(\mathbb{F}_4) \cong C_3 \oplus C_3$$

Exercise (Suppose $(x_0, y_0) \in E/\mathbb{F}_{2^n}$ has order 3. Show that)

1. $E : y^2 + a_3 y = x^3 + a_4 x + a_6 \;\Rightarrow\; x_0^4 + a_3^2 x_0 + (a_4 a_3)^2 = 0$
2. $E : y^2 + xy = x^3 + a_2 x^2 + a_6 \;\Rightarrow\; x_0^4 + x_0^3 + a_6 = 0$

Determining points of order (dividing) m

Definition (m–torsion point)

Let E/K and let $\bar{K}$ an *algebraic closure of* K.

$$E[m] = \{P \in E(\bar{K}) : \ mP = \infty\}$$

Theorem (Structure of Torsion Points)

Let E/K and $m \in \mathbb{N}$. If $p = \mathrm{char}(K) \nmid m$,

$$E[m] \cong C_m \oplus C_m$$

If $m = p^r m', p \nmid m'$,

$$E[m] \cong C_m \oplus C_{m'} \qquad \text{or} \qquad E[m] \cong C_{m'} \oplus C_{m'}$$

$$E/\mathbb{F}_p \text{ is called } \begin{cases} \textit{ordinary} & \text{if } E[p] \cong C_p \\ \textit{supersingular} & \text{if } E[p] = \{\infty\} \end{cases}$$

Group Structure of $E(\mathbb{F}_q)$

Corollary

Let $E/\mathbb{F}_q$. $\exists n, k \in \mathbb{N}$ are such that

$$E(\mathbb{F}_q) \cong C_n \oplus C_{nk}$$

Proof.

From classification Theorem of finite abelian group

$$E(\mathbb{F}_q) \cong C_{n_1} \oplus C_{n_2} \oplus \cdots \oplus C_{n_r}$$

with $n_i | n_{i+1}$ for $i \geq 1$.
Hence $E(\mathbb{F}_q)$ contains n_1^r points of order dividing n_1. From *Structure of Torsion Theorem*, $\#E[n_1] \leq n_1^2$. So $r \leq 2$ □

Theorem (Corollary of Weil Pairing)

Let $E/\mathbb{F}_q$ and $n, k \in \mathbb{N}$ s.t. $E(\mathbb{F}_q) \cong C_n \oplus C_{nk}$. Then $n \mid q - 1$.

We shall discuss the proof of the latter tomorrow

Sketch of the proof of Structure Theorem of Torsion Points

The division polynomials

The proof generalizes previous ideas and determine the points $P \in E(\mathbb{F}_q)$ such that $mP = \infty$ or equivalently $(m-1)P = -P$.

Definition (Division Polynomials of $E : y^2 = x^3 + Ax + B$ ($p > 3$))

$$
\begin{aligned}
\psi_0 &= 0 \\
\psi_1 &= 1 \\
\psi_2 &= 2y \\
\psi_3 &= 3x^4 + 6Ax^2 + 12Bx - A^2 \\
\psi_4 &= 4y(x^6 + 5Ax^4 + 20Bx^3 - 5A^2x^2 - 4ABx - 8B^2 - A^3) \\
&\ \vdots \\
\psi_{2m+1} &= \psi_{m+2}\psi_m^3 - \psi_{m-1}\psi_{m+1}^3 \qquad \text{for } m \geq 2 \\
\psi_{2m} &= \left(\frac{\psi_m}{2y}\right) \cdot (\psi_{m+2}\psi_{m-1}^2 - \psi_{m-2}\psi_{m+1}^2) \quad \text{for } m \geq 3
\end{aligned}
$$

The polynomial $\psi_m \in \mathbb{Z}[x, y]$ is called the m^{th} *division polynomial*

The division polynomials

Lemma

Let $E : y^2 = x^3 + Ax + B$, $(p > 3)$ and let $\psi_m \in \mathbb{Z}[x, y]$ the m^{th} division polynomial. Then

$$\psi_{2m+1} \in \mathbb{Z}[x] \qquad \text{and} \qquad \psi_{2m} \in 2y\mathbb{Z}[x]$$

Proof is an exercise.

True $\psi_0, \psi_1, \psi_2, \psi_3, \psi_4$ and for the rest apply induction, the identity $y^2 = x^3 + Ax + B \cdots$ and consider the cases m odd and m even. □

Lemma

$$\psi_m = \begin{cases} y(mx^{(m^2-4)/2} + \cdots) & \text{if } m \text{ is even} \\ mx^{(m^2-1)/2} + \cdots & \text{if } m \text{ is odd.} \end{cases}$$

Hence $\psi_m^2 = m^2 x^{m^2-1} + \cdots$

Proof is another exercise on induction:

□

Theorem ($E: Y^2 = X^3 + AX + B$ elliptic curve, $P = (x, y) \in E$)

$$m(x,y) = \left(x - \frac{\psi_{m-1}\psi_{m+1}}{\psi_m^2(x)}, \frac{\psi_{2m}(x,y)}{2\psi_m^4(x)}\right) = \left(\frac{\phi_m(x)}{\psi_m^2(x)}, \frac{\omega_m(x,y)}{\psi_m^3(x,y)}\right)$$

where

$$\phi_m = x\psi_m^2 - \psi_{m+1}\psi_{m-1}, \omega_m = \frac{\psi_{m+2}\psi_{m-1}^2 - \psi_{m-2}\psi_{m+1}^2}{4y}$$

We will omit the proof of the above (see [8, Section 9.5])

Exercise (Prove that after substituting $y^2 = x^3 + Ax + B$)

1. $\phi_m(x) \in \mathbb{Z}[x]$
2. $\phi_m(x) = x^{m^2} + \cdots \qquad \psi_m(x)^2 = m^2 x^{m^2-1} + \cdots$
3. $\omega_{2m+1} \in y\mathbb{Z}[x]$, $\omega_{2m} \in \mathbb{Z}[x]$
4. $\frac{\omega_m(x,y)}{\psi_m^3(x,y)} \in y\mathbb{Z}(x)$
5. $\gcd(\psi_m^2(x), \phi_m(x)) = 1$
   this is not really an exercise!! - see [8, Corollary 3.7]

Lemma

$$\#E[m] = \#\{P \in E(\bar{K}) : mP = \infty\} \begin{cases} = m^2 & \text{if } p \nmid m \\ < m^2 & \text{if } p \mid m \end{cases}$$

Proof.

Consider the homomorphism:

$$[m] : E(\bar{K}) \to E(\bar{K}), P \mapsto mP$$

If $p \nmid m$, need to show that

$$\#\operatorname{Ker}[m] = \#E[m] = m^2$$

We shall prove that $\exists P_0 = (a, b) \in [m](E(\bar{K})) \setminus \{\infty\}$ s.t.

$$\#\{P \in E(\bar{K}) : \ mP = P_0\} = m^2$$

Since $E(\bar{K})$ infinite, we can choose $(a, b) \in [m](E(\bar{K}))$ s.t.

1. $ab \neq 0$
2. $\forall x_0 \in \bar{K} : (\phi'_m \psi_m - 2\phi_m \psi'_m)(x_0)\psi_m(x_0) = 0 \Rightarrow a \neq \frac{\phi_m(x_0)}{\psi_m^2(x_0)}$

   if $p \nmid m$, conditions imply that $\phi_m(x) - a\psi_m^2(x)$ has $m^2 = \partial(\phi_m(x) - a\psi_m^2(x))$ distinct roots in fact $\partial\phi_m(x) = m^2$ and $\partial\psi_m^2(x) = m^2 - 1$ $\square$

Proof continues.

Write

$$mP = m(x,y) = \left(\frac{\phi_m(x)}{\psi_m^2(x)}, \frac{\omega_m(x,y)}{\psi_m(x)^3}\right) = \left(\frac{\phi_m(x)}{\psi_m^2(x)}, yr(x)\right)$$

The map

$$\{\alpha \in \bar{K} : \phi_m(\alpha) - a\psi_m(\alpha)^2 = 0\} \leftrightarrow \{P \in E(\bar{K}) : mP = (a,b)\}$$
$$\alpha_0 \mapsto (\alpha_0, br(\alpha_0)^{-1})$$

is a well defined bijection.

Hence there are m^2 points $P \in E(\bar{K})$ with $mP = (a,b)$

So there are m^2 elements in $\mathrm{Ker}[m]$.

If $p \mid m$, the proof is the same except that $\phi_m(x) - a\psi_m(x)^2$ has multiple roots!!
In fact $\phi_m'(x) - a\psi_m'(x)^2 = 0$ $\square$

From Lemma, Theorem follows:

If $p \nmid m$, apply classification Theorem of finite Groups:

$$E[m] \cong C_{n_1} \oplus C_{n_2} \oplus \cdots C_{n_k},$$

$n_i \mid n_{i+1}$. Let $\ell \mid n_1$, then $E[\ell] \subset E[m]$. Hence $\ell^k = \ell^2 \Rightarrow k = 2$. So

$$E[m] \cong C_{n_1} \oplus C_{n_2}$$

Finally $n_2 \mid m$ and $n_1 n_2 = m^2$ so $m = n_1 = n_2$.

If $p \mid m$, write $m = p^j m'$, $p \nmid m'$ and

$$E[m] \cong E[m'] \oplus E[p^j] \cong C_{m'} \oplus C_{m'} \oplus E[p^j]$$

The statement follows from:

$$E[p^j] \cong \begin{cases} \{\infty\} \\ C_{p^j} \end{cases} \quad \text{and} \quad C_{m'} \oplus C_{p^j} \cong C_{m' p^j}$$

which is done by induction.

From Lemma, Theorem follows (continues)

Induction base:

$$E[p] \cong \begin{cases} \{\infty\} \\ C_p \end{cases} \qquad \text{if follows from } \ \#E[p] < p^2$$

- If $E[p] = \{\infty\} \ \Rightarrow \ E[p^j] = \{\infty\} \ \forall j \geq 2$:
  In fact if $E[p^j] \neq \{\infty\}$ then it would contain some element of order p(contradiction).
- If $E[p] \cong C_p$, then $E[p^j] \cong C_{p^j} \ \forall j \geq 2$:
  In fact $E[p^j]$ is cyclic (otherwise $E[p]$ would not be cyclic!)

Fact: $[p] : E(\bar{K}) \ \rightarrow \ E(\bar{K})$ is surjective (to be proven tomorrow)

If $P \in E$ and $\operatorname{ord} P = p^{j-1} \ \Rightarrow \ \exists Q \in E$ s.t. $pQ = P$ and $Q = p^j$.
Hence $E[p^j] \cong C_{p^j}$ since it contains an element of order p^j.

Remark:

- $E[2m+1] \setminus \{\infty\} = \{(x, y) \in E(\bar{K}) : \ \psi_{2m+1}(x) = 0\}$
- $E[2m] \setminus E[2] = \{(x, y) \in E(\bar{K}) : \ y^{-1}\psi_{2m}(x) = 0\}$

Theorem (Hasse)

Let E be an elliptic curve over the finite field $\mathbb{F}_q$. Then the order of $E(\mathbb{F}_q)$ satisfies

$$|q+1-\#E(\mathbb{F}_q)| \le 2\sqrt{q}.$$

So $\#E(\mathbb{F}_q) \in [(\sqrt{q}-1)^2, (\sqrt{q}+1)^2]$ the *Hasse interval* $\mathcal{I}_q$

Example (Hasse Intervals)

q	$\mathcal{I}_q$
2	$\{1, 2, 3, 4, 5\}$
3	$\{1, 2, 3, 4, 5, 6, 7\}$
4	$\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$
5	$\{2, 3, 4, 5, 6, 7, 8, 9, 10\}$
7	$\{3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13\}$
8	$\{4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14\}$
9	$\{4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16\}$
11	$\{6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18\}$
13	$\{7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21\}$
16	$\{9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 25\}$
17	$\{10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26\}$
19	$\{12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28\}$
23	$\{15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33\}$
25	$\{16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36\}$
27	$\{18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38\}$
29	$\{20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40\}$
31	$\{21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43\}$
32	$\{22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44\}$

Theorem (Waterhouse)

Let $q = p^n$ and let $N = q + 1 - a$.

$$\exists E/\mathbb{F}_q \text{ s.t.} \#E(\mathbb{F}_q) = N \Leftrightarrow |a| \le 2\sqrt{q} \text{ and}$$

one of the following is satisfied:

(i) $\gcd(a, p) = 1$;

(ii) *n even and one of the following is satisfied:*

1. $a = \pm 2\sqrt{q}$;
2. $p \not\equiv 1 \pmod 3$, *and* $a = \pm\sqrt{q}$;
3. $p \not\equiv 1 \pmod 4$, *and* $a = 0$;

(iii) *n is odd, and one of the following is satisfied:*

1. $p = 2$ *or* 3, *and* $a = \pm p^{(n+1)/2}$;
2. $a = 0$.

Example (q prime $\forall N \in I_q, \exists E/\mathbb{F}_q, \#E(\mathbb{F}_q) = N$. q not prime:)

q	$a \in$
$4 = 2^2$	$\{-4, -3, -2, -1, 0, 1, 2, 3, 4\}$
$8 = 2^3$	$\{-5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5\}$
$9 = 3^2$	$\{-6, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, 6\}$
$16 = 2^4$	$\{-8, -7, -6, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, 6, 7, 8\}$
$25 = 5^2$	$\{-10, -9, -8, -7, -6, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$
$27 = 3^3$	$\{-10, -9, -8, -7, -6, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$
$32 = 2^5$	$\{-11, -10, -9, -8, -7, -6, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$

Theorem (Rück)

Suppose N is a possible order of an elliptic curve $/\mathbb{F}_q$, $q = p^n$. Write

$$N = p^e n_1 n_2, \quad p \nmid n_1 n_2 \quad \text{and} \quad n_1 \mid n_2 \ (\text{possibly } n_1 = 1).$$

There exists $E/\mathbb{F}_q$ s.t.

$$E(\mathbb{F}_q) \cong C_{n_1} \oplus C_{n_2 p^e}$$

if and only if

1. *$n_1 = n_2$ in the case (ii).1 of Waterhouse's Theorem;*
2. *$n_1 | q - 1$ in all other cases of Waterhouse's Theorem.*

Example

- If $q = p^{2n}$ and $\#E(\mathbb{F}_q) = q + 1 \pm 2\sqrt{q} = (p^n \pm 1)^2$, then
  $$E(\mathbb{F}_q) \cong C_{p^n \pm 1} \oplus C_{p^n \pm 1}.$$
- Let $N = 100$ and $q = 101 \Rightarrow \exists E_1, E_2, E_3, E_4/\mathbb{F}_{101}$ s.t.
  $$E_1(\mathbb{F}_{101}) \cong C_{10} \oplus C_{10} \qquad E_2(\mathbb{F}_{101}) \cong C_2 \oplus C_{50}$$
  $$E_3(\mathbb{F}_{101}) \cong C_5 \oplus C_{20} \qquad E_4(\mathbb{F}_{101}) \cong C_{100}$$

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