Factoring integers, Producing primes and the RSA cryptosystem

University of Pedagogy

Ho Chi Minh City

Trường Đại Học Sư Phạm

Thành Phố Hồ Chí Minh

December 12, 2005

RSA_{2048} = 25195908475657893494027183240048398571429282126204
032027777137836043662020707595556264018525880784406918290641249
515082189298559149176184502808489120072844992687392807287776735
971418347270261896375014971824691165077613379859095700097330459
748808428401797429100642458691817195118746121515172654632282216
869987549182422433637259085141865462043576798423387184774447920
739934236584823824281198163815010674810451660377306056201619676
256133844143603833904414952634432190114657544454178424020924616
515723350778707749817125772467962926386356373289912154831438167
899885040445364023527381951378636564391212010397122822120720357

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4880842840179742910064245869181719511874612151517265463228221686
9987549182422433637259085141865462043576798423387184774447920739
9342365848238242811981638150106748104516603773060562016196762561
3384414360383390441495263443219011465754445417842402092461651572
3350778707749817125772467962926386356373289912154831438167899885
04044536402352738195137863656439121201039712282212072035 7

RSA_{2048} is a 617 (decimal) digit number

$$
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`http://www.rsasecurity.com/rsalabs/node.asp?id=2093`

$$RSA_{2048} = p \cdot q, \quad p, q \approx 10^{308}$$

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Theorem. If $a \in \mathbb{N} \quad \exists!\ p_1 < p_2 < \cdots < p_k$ *primes*

s.t. $a = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$

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Regrettably: RSAlabs believes that factoring in one year requires:

number	computers	memory
RSA_{1620}	1.6×10^{15}	120 Tb
RSA_{1024}	$342,000,000$	170 Gb
RSA_{760}	215,000	4Gb.

http://www.rsasecurity.com/rsalabs/node.asp?id=2093

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Challenge Number	Prize ($US)
RSA_{576}	\$10,000
RSA_{640}	\$20,000
RSA_{704}	\$30,000
RSA_{768}	\$50,000
RSA_{896}	\$75,000
RSA_{1024}	\$100,000
RSA_{1536}	\$150,000
RSA_{2048}	\$200,000

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Challenge Number	Prize ($US)	Status
RSA_{576}	$10,000	Factored December 2003
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- 1987 Elliptic curves factoring **ECF** (Lenstra)

Carissan's ancient Factoring Machine

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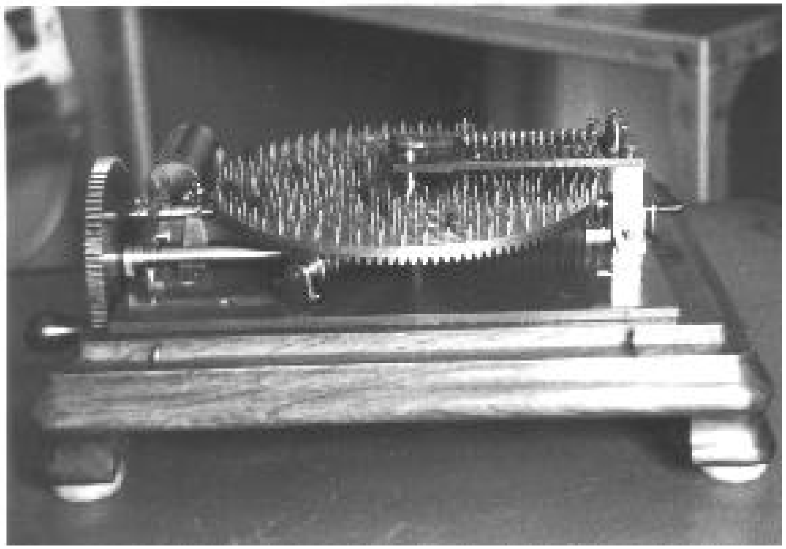

Hình 1: Conservatoire Nationale des Arts et Métiers in Paris

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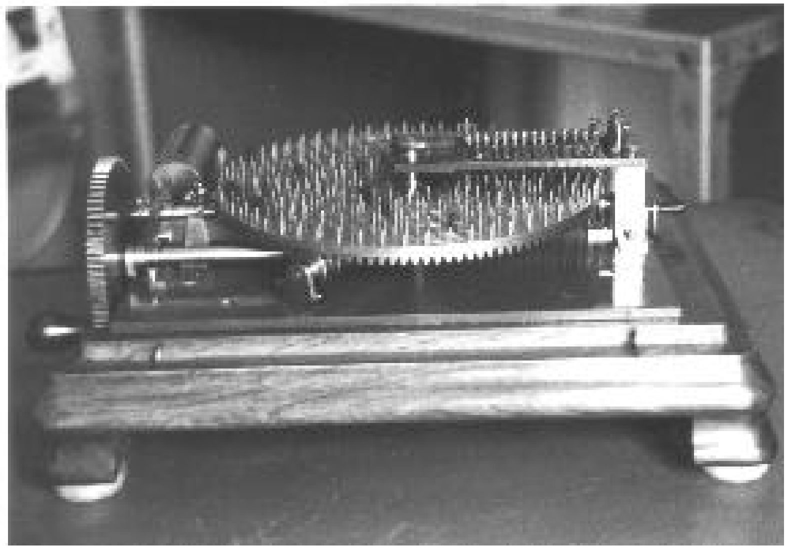

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http://www.math.uwaterloo.ca/ shallit/Papers/carissan.html

Hình 2: Lieutenant Eugène Carissan

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$$225058681 = 229 \times 982789 \qquad \text{2 minutes}$$

$$3450315521 = 1409 \times 2418769 \qquad \text{3 minutes}$$

$$3570537526921 = 841249 \times 4244329 \qquad \text{18 minutes}$$

Contemporary Factoring 1/2

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1. 1994, Quadratic Sieve (QS): (8 months, 600 voluntaries, 20 countries)
   D.Atkins, M. Graff, A. Lenstra, P. Leyland

$$
\begin{aligned}
RSA_{129} &= 114381625757888867669235779976146612010218296721242362562561842935706\\
&\quad 935245733897830597123563958705058989075147599290026879543541 =\\
&= 3490529510847650949147849619903898133417764638493387843990820577\times\\
&\quad 32769132993266709549961988190834461413177642967992942539798288533
\end{aligned}
$$

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❷ (February 2 1999), Number Fields Sieve (NFS): (160 Sun, 4 months)

$$\begin{aligned} RSA_{155} &= 1094173864157052742180970732204035761200373294544920599091384213147634998428 \\ &\quad 893478471799725789126733249762575289978183379707653724402714674353159335433897 = \\ &= 102639592829741105772054196573991675900716567808038066803341933521790711307779 \times \\ &\quad 106603488380168454820927220360012878679207958575989291522270608237193062808643 \end{aligned}$$

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❸ (December 3, 2003) (NFS): J. Franke et al. (174 decimal digits)

$$RSA_{576} = 1881988129206079638386972394616504398071635633794173827007633564229888597152346\\65485319060606504743045317388011303396716199692321205734031879550656996221305168759307650257059 =\\= 398075086424064937397125500550386491199064362342526708406385189575946388957261768583317\times\\472772146107435302536223071973048224632914695302097116459852171130520711256363590397527$$

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4. (May 9,2005) (NFS): F. Bahr, et al (663 binary digits)

$$RSA_{200} = 27997833911221327870829467638722601621070446786955428537560009929326128400107609345671052955360 8\\56061822351910951365788637105954482006576775098580557613579098734950144178863178946295187237869221823983 =\\3532461934402770121272604978198464368671197400197625023649303468776121253679423200058547956528088349 \times\\7925869954478333033347085841480059687737975857364219960734330341455767872818152135381409304740185467$$

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For updates see Paul Zimmerman's "*Integer Factoring Records*":

`http://www.loria.fr/ zimmerma/records/factor.html`

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Update on "factorization of Fermat Numbers":

`http://www.prothsearch.net/fermat.html`

Last Minute News

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Date: Thu, 10 Nov 2005 22:07:26 -0500
From: Jens Franke <franke@math.uni-bonn.de>
To: NMBRTHRY@LISTSERV.NODAK.EDU

We have factored RSA640 by GNFS. The factors are

16347336458092538484431338838650908598417836700330
92312181110852389333100104508151212118167511579

and

19008712816648221131268515739354139754718967899685
15493666638539088027103802104498957191261465571

We did lattice sieving for most special q between 28e7 and 77e7 using factor base bounds of 28e7 on the algebraic side and 15e7 on the rational side. The bounds for large primes were 2 34. This produced 166e7 relations. After removing duplicates 143e7 relations remained. A filter job produced a matrix with 36e6 rows and columns, having 74e8 non-zero entries. This was solved by Block-Lanczos.

Sieving has been done on 80 2.2 GHz Opteron CPUs and took 3 months. The matrix step was performed on a cluster of 80 2.2 GHz Opterons connected via a Gigabit network and took about 1.5 months.

Calendar time for the factorization (without polynomial selection) was 5 months.

More details will be given later.

F. Bahr, M. Boehm, J. Franke, T. Kleinjung

RSA

Adi Shamir, Ron L. Rivest, Leonard Adleman (1978)

The RSA cryptosystem

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$$\boxed{\begin{array}{ccc} A\ (Alice) & \longrightarrow & B\ (Bob) \\ & \uparrow & \\ & C\ (Charles) & \end{array}}$$

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1. KEY GENERATION — Bob has to do it
2. ENCRYPTION — Alice has to do it
3. DECRYPTION — Bob has to do it
4. ATTACK — Charles would like to do it

Bob: Key generation

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- He chooses randomly p and q primes $(p, q \approx 10^{100})$
-
-
-
-

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Problem: How does Bob do all this?- We will go came back to it!

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$$E(\texttt{SAIGON}) = 226366440^{65537} \pmod{79537397720925283289}$$

$$= 71502481501746956206 = \mathcal{C} = \texttt{ZPOYWXZXDNCGUBA}$$

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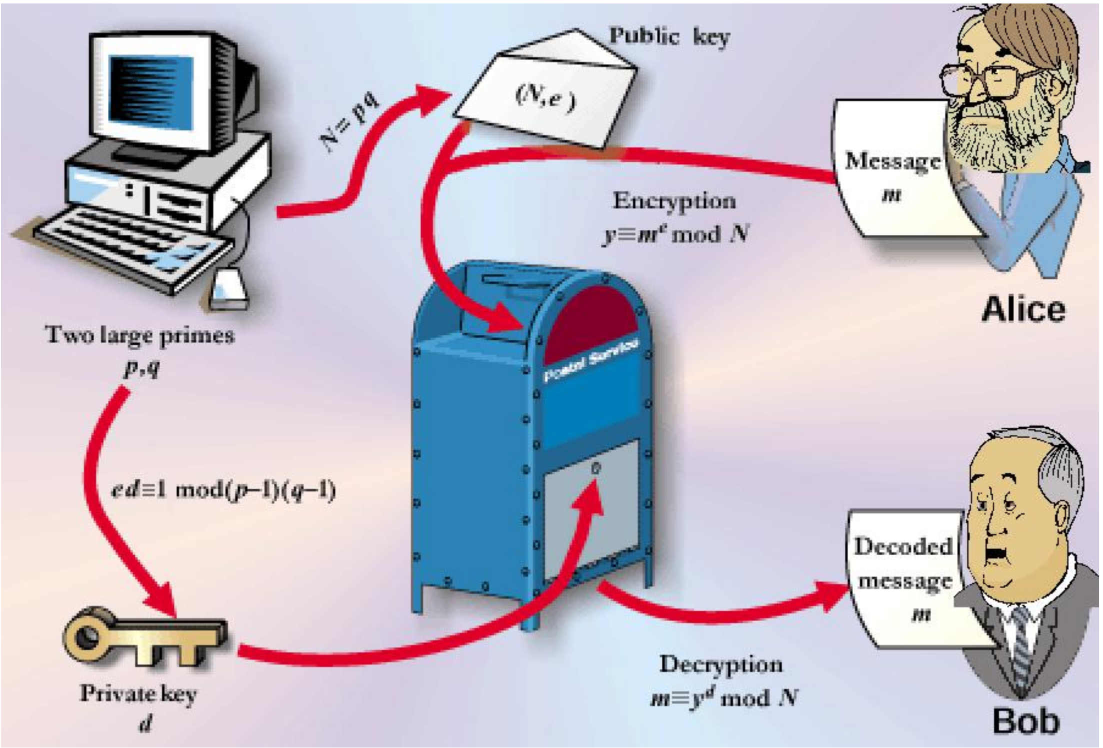
Example(cont.): $d = 65537^{-1} \bmod \varphi(9049465727 \cdot 8789181607) = 57173914060643780153$

$D(\texttt{ZPOYWXZXDNCGUBA}) =$

$71502481501746956206^{57173914060643780153} (\bmod 79537397720925283289) =$ SAIGON

RSA at work


Public key
(N,e)
N=pq
Message
m
Encryption
y≡m^e mod N
Alice
Two large primes
p,q
ed≡1 mod(p−1)(q−1)
Postal Service
Decoded
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m
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Bob

Repeated squaring algorithm

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Problem: How does one compute $a^b \bmod c$?

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$$7150248150174695620 6^{5717391406064378015 3} \pmod{7953739772092528328 9}$$

✍

✍

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$$715024815017469562 06^{571739140606437801 53} \pmod{79537397720925283289}$$

- Compute the binary expansion $b = \sum_{j=0}^{[\log_2 b]} \epsilon_j 2^j$
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Problem: How does one compute a^b mod c?

$$715024815017469562 06^{57173914060643780153} \pmod{79537397720925283289}$$

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  57173914060643780153=110001100101110010100010111110101011110011011000100100011000111001

- Compute recursively a^{2^j} mod $c, j = 1, \ldots, [\log_2 b]$:

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- Multiply the a^{2^j} mod c with $\epsilon_j = 1$

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$$a^b \bmod c = \left(\prod_{j=0,\epsilon_j=1}^{[\log_2 b]} a^{2^j} \bmod c\right) \bmod c$$

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Pseudo code: $e_c(a, b) = a^b \bmod c$

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PSEUDO CODE: $e_c(a,b) = a^b \bmod c$

$e_c(a,b)$	`=`	`if`	$b=1$	`then`	$a \bmod c$
		`if`	$2 \mid b$	`then`	$e_c(a, \frac{b}{2})^2 \bmod c$
		`else`			$a * e_c(a, \frac{b-1}{2})^2 \bmod c$

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To encrypt with $e = 2^{16}+1$, only 17 operations in $\mathbb{Z}/M\mathbb{Z}$ are enough

Key generation

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Probabilistic algorithm (type Las Vegas)
1. Let $p = \textsc{Random}(10^{100})$
2. If ISPRIME(p)=1 then OUTPUT=p else goto 1

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False Metropolitan Legend: Check primality is equivalent to factoring

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Therefore

$$0.0043523959267 < Prob\left((\textsc{Random}(10^{100}) = \texttt{prime}\right) < 0.004371422086$$

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$$0.98365832 < P'_{250} < 0.98395199$$

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$$2^{93960} \equiv 1 \pmod{93961} \quad \text{but} \quad 93961 = 7 \times 31 \times 433$$

Strong pseudo primes

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From now on $m \equiv 3 \bmod 4$ (just to simplify the notation)

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Definition. $m \in \mathbb{N}$, $m \equiv 3 \bmod 4$, composite is said strong pseudo prime (SPSP) in base a if

$$a^{(m-1)/2} \equiv \pm 1 \pmod{m}.$$

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MILLER RABIN ALGORITHM WITH k ITERATIONS
$N = (m-1)/2$
`for` $j = 0$ `to` k `do` $a =$`Random`(m)
`if` $a^N \not\equiv \pm 1 \bmod m$ `then OUPUT=(`m `composite): END`
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In the real world, software uses Miller Rabin with $k = 10$

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Theorem. (Miller, Bach) If m is composite, then

$$\textbf{GRH} \Longrightarrow \ \exists a \leq 2\log^2 m \text{ s.t. } \ a^{(m-1)/2} \not\equiv \pm 1 \pmod{m}.$$

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It runs in $O(\log^5 m)$ operations in $\mathbb{Z}/m\mathbb{Z}$.

Certified prime records

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Top 10 Largest primes:

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1	$2^{25964951}-1$	7816230	Nowak	2005	Mersenne	42?
2	$2^{24036583}-1$	7235733	Findley	2004	Mersenne	41?
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- For more see

http://primes.utm.edu/primes/

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http://www.cse.iitk.ac.in/news/primality.html

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- $q|r-1$;
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Many simplifications and improvements: Bernstein, Lenstra, Pomerance.....

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  The two problems are polynomially equivalent

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- **Private key (or symmetric)**
  - Lucifer
  - DES
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  - Lucifer
  - DES
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- **Public key**
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  - Diffie–Hellmann
  - Knapsack
  - NTRU