Canad. Math. Bull. Vol. **38** (4), 1995 pp. 465–468

ON MINIMAL SETS OF GENERATORS FOR PRIMITIVE ROOTS

FRANCESCO PAPPALARDI

ABSTRACT. A conjecture of Brown and Zassenhaus (see [2]) states that the first $\log p$ primes generate a primitive root $\pmod p$ for almost all primes p. As a consequence of a Theorem of Burgess and Elliott (see [3]) it is easy to see that the first $\log^2 p \log\log^{4+\epsilon} p$ primes generate a primitive root $\pmod p$ for almost all primes p. We improve this showing that the first $\log^2 p/\log\log p$ primes generate a primitive root $\pmod p$ for almost all primes p.

For a given odd prime number p, we define the function κ as

$$\kappa(p) = \min\{r \mid \text{ the first } r \text{ primes generate } \mathbb{F}_p^*\}.$$

In 1969, H. Brown and H. Zassenhaus conjectured in [2] that $\kappa(p) \leq [\log p]$ with probability almost equal to one.

If we denote by $g(p)$ the least primitive root modulo p, then a Theorem of D. A. Burgess and P. D. T. A. Elliott states that

$$\pi(x)^{-1} \sum_{p\leq x} g(p) \ll \log^2 x (\log\log x)^4.$$

If U is the number of primes up to x for which $g(p) \geq T$, then

$$UT \ll \sum_{p\leq x} g(p) \ll \pi(x) \log^2 x (\log\log x)^4.$$

For any $\epsilon > 0$, we choose $T = \log^2 x (\log\log x)^{4+\epsilon/2}$ so that $U = \mathrm{o}\big(\pi(x)\big)$ and since $g(p) \leq T$ is product of primes less that T, we deduce that for almost all primes $p \leq x$,

$$\kappa(p) \leq \log^2 x (\log\log x)^{4+\epsilon/2} \leq \log^2 p (\log\log p)^{4+\epsilon}.$$

We will prove the following:

THEOREM 1. *Let π be the prime counting function. For all but*

$$\mathrm{O}\left(\frac{x}{\exp\left\{\frac{(\log\log\log x)^3 \log x}{4(\log\log x)^3}\right\}}\right)$$

Supported in part by C.N.R.

Received by the editors June 23, 1994; revised October 24, 1994.
AMS subject classification: Primary: 11N56; secondary: 11A07.
Key words and phrases: sieve theory, primitive roots, Riemann hypothesis.

primes $p \leq x$, we have that

$$\kappa(p) \leq \pi\left(\frac{\log^2 p}{e^2}\exp\left\{2\frac{(\log\log\log p)^3}{(\log\log p)^2}\right\}\right).$$

The proof is based on a uniform estimate for the size of the set

$$\mathcal{H}_{m,r}(x) = \#\left\{p \leq x \;\middle|\; |\Gamma_r| = \frac{p-1}{m}\right\}$$

where m and r are given integers strictly greater than one, and

$$\Gamma_r = \langle p_1, \ldots, p_r \pmod p\rangle$$

is the subgroup of $\mathbb{F}_p^*$ generated by the first r primes.

As a subgroup of the cyclic group $\mathbb{F}_p^*$ with index m, Γ_r is the subgroup of m-th powers $(\bmod\, p)$. Hence

$$\mathcal{H}_{m,r}(x) = \{p \leq x \mid p \equiv 1 \pmod m \text{ and } p_i \text{ is an } m\text{-th power} \pmod p\ \forall i = 1, \ldots, r\}.$$

If $n_m(p)$ is the least prime which is not congruent to an m-th power $\pmod p$, then we can also write:

$$\mathcal{H}_{m,r}(x) = \{p \leq x \mid p \equiv 1 \pmod m \text{ and } n_m(p) > p_r\}.$$

We will need to use the large sieve inequality, the proof of which can be found in [1]. That is:

LEMMA 2 (THE LARGE SIEVE). *Let $\mathcal{N}$ be a set of integers contained in the interval $\{1, \ldots, z\}$ and for any prime $p \leq x$, let $\Omega_p = \{h \pmod p) \mid \forall n \in \mathcal{N}, n \not\equiv h \pmod p\}$ and*

$$L = \sum_{q \leq x} \mu^2(q) \prod_{p \mid q} \frac{|\Omega_p|}{p - |\Omega_p|},$$

then

$$|\mathcal{N}| \leq \frac{z + 3x^2}{L}. \qquad \blacksquare$$

In our case, let $\mathcal{N} = \left\{n \leq z \;\middle|\; \forall q | n, q < p_r\right\}$ and note that if $p \in \mathcal{H}_{m,r}(x)$, then

$$\Omega_p \supset \{h \pmod p) \mid h \text{ is not an } m\text{-th power} \pmod p\}$$

therefore, for such p's, $|\Omega_p| \geq p - 1 - (p-1)/m$ and

$$L \geq \sum_{p \in \mathcal{H}_{m,r}(x)} \frac{|\Omega_p|}{p - |\Omega_p|} \geq \frac{m-1}{2}|\mathcal{H}_{m,r}(x)|.$$

If we let $\Psi(s,t)$ denote the number of integers $n \leq s$ free of prime factors exceeding t, then

$$\mathcal{H}_{m,r}(x) \leq \frac{8x^2}{(m-1)\Psi(x^2, p_r)}.$$

Estimating the function $\Psi(z,y)$ is a classical problem in Number Theory. In 1983, R. Canfield, P. Erdős and C. Pomerance (see [4]) proved the following:

LEMMA 3. *Let* $u = \frac{\log z}{\log y}$. *There exists an absolute constant* c_1 *such that*

$$\Psi(z,y) \geq z \exp\left\{-u\left(\log u + \log\log u - 1 + \frac{(\log\log u) - 1}{\log u} + c_1 \frac{(\log\log u)^2}{\log^2 u}\right)\right\},$$

for all $z \geq 1$ *and* $u \geq e^e$. ∎

Applying Lemma 3 with $z = x^2$ and $y = p_r$, we get the following:

LEMMA 4. *Let* $u = 2\log x / \log p_r$. *There exists an absolute constant* c_1 *such that*

$$\mathcal{H}_{m,r}(x) \leq \frac{8}{m} \exp\left\{u\left(\log u + \log\log u - 1 + \frac{(\log\log u) - 1}{\log u} + c_1 \frac{(\log\log u)^2}{\log^2 u}\right)\right\},$$

for all $x \geq 1$ *and* $u \geq e^e$. ∎

PROOF OF THEOREM 1. Let us take p_r is the range

$$\log^2 x \geq p_r \geq \frac{\log^2 x}{e^2} \exp\left\{\frac{(\log\log\log x)^3}{(\log\log x)^2}\right\}. \tag{1}$$

If we set $\log_2 x = \log\log x$, $\log_3 x = \log\log\log x$ and $u = 2\frac{\log x}{\log p_r}$, then we can write the estimates:

$$\frac{\log x}{\log_2 x} \leq u \leq \frac{\log x}{\log_2 x - 1 + \log_3^3 x / 2\log_2^2 x};$$

$$\log_2 x - \log_3 x \leq \log u \leq \log_2 x - \log_3 x + \frac{1}{\log_2 x};$$

$$\log_2 u \leq \log_3 x - \frac{\log_3 x}{\log_2 x} + c_2 \frac{\log_3^2 x}{\log_2^2 x};$$

$$\frac{1}{\log_2 x} - \frac{2}{\log_2^3 x} \leq \frac{1}{\log u} \leq \frac{1}{\log_2 x} + c_3 \frac{\log_3 x}{\log_2^2 x}.$$

where c_2 and c_3 are absolute constants.

Now let us apply Lemma 4 and deduce that

$$\begin{aligned} m\mathcal{H}_{m,r}(x) &\ll \exp\left\{\log x \frac{\log_2 x - 1 + c_4 \frac{\log_3^2 x}{\log_2^2 x}}{\log_2 x - 1 + \log_3^3 x / 2\log_2^2 x}\right\} \\ &\ll \exp\left\{\log x\left(1 - \frac{\log_3^3 x}{2\log_2^3 x} + c_5\left(\frac{\log_3^2 x}{\log_2^3 x}\right)\right)\right\} \end{aligned} \tag{2}$$

where c_4 and c_5 are absolute constants.

Now we are ready to estimate

$$\#\{p \leq x \mid [\mathbb{F}_p^* : \Gamma_r] > 1\}.$$

We note that the index $[\mathbb{F}_p^* : \Gamma_r]$ is at most x as it is a divisor of $p - 1$.

Since for all but $O(x/\exp\frac{\log x}{\log\log x})$ primes p, we may assume that

$$p > x/\exp(2\log x/\log\log x),$$

if we set $p_r \geq \frac{\log_2^2 p}{e^2}\exp(2\log_3^3 p/\log_2^2 p)$ then p_r is in the range of (1) and by (2) the number of such primes p for which $[\mathbb{F}_p^* : \Gamma_r] > 1$ is

$$\ll \sum_{m=2}^{x} \mathcal{H}_{m,r}(x) \leq \Big(\sum_{m=2}^{x}\frac{1}{m}\Big)\exp\Big\{\log x\Big(1-\frac{\log_3^3 x}{2\log_2^3 x}+c_5\Big(\frac{\log_3^2 x}{\log_2^3 x}\Big)\Big)\Big\} = O\Big(\frac{x}{\exp\{\frac{\log x\log_3^3 x}{4\log_2^3 x}\}}\Big)$$

and this completes the proof. ■

ACKNOWLEDGMENTS. A version of Lemma 4 has been proven recently also by S. Konyagin and C. Pomerance in [5].

I would like to thank Professor Ram Murty for his suggestions and for a number of interesting observations.

REFERENCES

1. E. Bombieri, *Le grande crible dans la théorie analytique des nombres*, Astérisque **18**(1974).
2. H. Brown and H. Zassenhaus, *Some empirical observation on primitive roots*, J. Number Theory 3(1971) 306–309.
3. D. A. Burgess and P. D. T. A. Elliott, *The average of the least primitive root*, Mathematika **15**(1968), 39–50.
4. E. R. Canfield, P. Erdős and C. Pomerance, *On a problem of Oppenheim concerning "Factorization Numerorum"*, J. Number Theory **17**(1983), 1–28.
5. S. Konyagin and C. Pomerance, *On primes recognizable in deterministic polynomial time*, preprint.

Dipartimento di Matematica
Terza Università degli Studi di Roma
Via Corrado Segre, 4
Roma
00146–Italia
e-mail: pappa@mat.uniroma3.it