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ON HOOLEY'S THEOREM WITH WEIGHTS

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Abstract. We adapt Hooley's proof that the Generalized Riemann Hypothesis implies the Artin Conjecture for primitive roots to various other problems. We consider the sum $\sum_{p\leq x} f(i_p)$ where i_p is the index of 2 modulo p and f is a given function. In various cases we establish asymptotic formulas for such a sum and analyse the constants. While we claim no originality, we outline the method to approach this problem in a fairly general case.

1. Introduction

For a fixed prime number p, we denote by i_p the index of 2 (mod p). For a function $f : \mathbb{N} \to \mathbb{C}$, we consider the sum

$$\sum_{p\leq x} f(i_p).$$

We will establish various estimates for such a sum. If $f(1) = 1$ and $f(x) = 0$ for $x \neq 1$, then the famous Artin Conjecture for primitive roots states that

(1) $$\#\{p \leq x \mid i_p = 1\} \sim \delta\pi(x),$$

where δ is the Artin constant,

$$\delta = \prod_{l \text{ prime}} \left(1 - \frac{1}{l(l-1)}\right) = 0.373955813619202\ldots$$

In 1967, C. Hooley (see [5]) proved the Artin Conjecture as a consequence of the Generalized Riemann Hypothesis.

The weaker form of the Artin Conjecture states that *any fixed integer $b > 1$ that is not a perfect square is a primitive root for infinitely many primes.* Heath-Brown [4], Gupta and Murty [3] (see also [9]) solved this form of the Artin conjecture for a very large class of

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numbers b but the asymptotic formula in (1) is still to be proven. For a clear exposition on the Artin Conjecture and its "quasi-resolution" we refer to [8].

Let us set

$$\pi(x,n) = \#\{p \le x \mid n|i_p\}.$$

The intuition of Artin and the subsequent proof of Hooley is based on the identity

$$\#\{p \le x \mid i_p = 1\} = \sum_{n=1}^{\infty} \mu(n)\pi(x,n), \tag{2}$$

which is just the inclusion–exclusion principle.

A natural generalization of (2) is the identity

$$\sum_{p\le x} f(i_p) = \sum_{n=1}^{\infty} g(n)\pi(x,n),$$

where, by Möbius inversion, $g(n) = \sum_{d|n} \mu(n/d)f(d)$ and $f(m) = \sum_{n|m} g(n)$.

If n is a positive integer, we set $K_n = \mathbb{Q}(\zeta_n, 2^{1/n})$, so that

$$k_n = [K_n : \mathbb{Q}] = \begin{cases} n\varphi(n) & \text{if } 8 \nmid n \\ n\varphi(n)/2 & \text{if } 8|n. \end{cases}$$

It is a criterion due to Dedekind that an odd prime p splits completely in K_n if and only n divides i_p. Therefore $\pi(x,n)$ equals the number of primes up to x that are unramified and split completely in K_n.

The Chebotarev Density Theorem provides us with an asymptotic formula for $\pi(x,n)$. The following result is due to Lagarias and Odlyzko [6].

CHEBOTAREV DENSITY THEOREM. *Suppose that the Generalized Riemann Hypothesis (GRH) holds for the Dedekind zeta function of K_n. Then*

$$\pi(x,n) = \frac{1}{k_n}\mathrm{Li}(x) + \mathrm{O}\Big(x^{1/2}\log xn\Big).$$

Unconditionally, there exists an absolute constant A such that if $n \le (\log x)^{1/7}$ then

$$\pi(x,n) = \frac{1}{k_n}\mathrm{Li}(x) + \mathrm{O}\Big(x\exp(-A\sqrt{\log x}/n)\Big).$$

From now on we will suppose that

$$\sum_{n=1}^{\infty} \frac{|g(n)|}{k_n} < \infty. \tag{3}$$

The generalized Artin problem is to establish the asymptotic formula

$$\sum_{p\le x} f(i_p) \sim \sum_{n=1}^{\infty} \frac{g(n)}{k_n}\pi(x).$$

We adopt the following standard notation: $F(x) \lesssim G(x)$ means that for every $\epsilon > 0$ there exists x_ϵ such that if $x > x_\epsilon$, then $F(x) \le (1+\epsilon)G(x)$.

We have the following

THEOREM 1.

(a) *Suppose that* $g(n) \ge 0$. *Then we have*

$$\sum_{n=1}^{\infty} g(n)\pi(x,n) \gtrsim \sum_{n=1}^{\infty} \frac{g(n)}{k_n}\pi(x).$$

(b) *Suppose that the series* $\sum_{n=1}^{\infty} |g(n)|/\varphi(n)$ *converges and that* $\sum_{n>z} |g(n)|/n = \mathrm{o}\big(\log^{-1} z\big)$. *Then*

$$\sum_{n=1}^{\infty} g(n)\pi(x,n) \sim \sum_{n=1}^{\infty} \frac{g(n)}{k_n}\pi(x).$$

Let us write $H_m(x) = \#\{p \le x \mid i_p = m\}$, so that

$$\sum_{p\le x} f(i_p) = \sum_{m=1}^{\infty} f(m)H_m(x).$$

The function $H_m(x)$ has been studied by L. Murata in [7] where he proved

THEOREM (MURATA). *GRH implies that for every every* $\epsilon > 0$,

$$H_m(x) = \delta_m \pi(x) + \mathrm{O}\left(\frac{m^\epsilon x \log\log x}{\log^2 x}\right)$$

where the implied constant depends only on ϵ. *Furthermore*

$$\delta_m = \sum_{n=1}^{\infty} \frac{\mu(n)}{k_{nm}} = \frac{r_m}{m^2}\,\delta \prod_{\substack{l \text{ prime} \\ l|m}} \frac{l^2-1}{l^2-l-1}$$

with $r_m = 1$ *if* $4\nmid m$, $r_m = 2/3$ *if* $4\| m$, $r_m = 2$ *if* $8|m$ *and* δ *is the Artin constant.*

From now on we will also suppose that

$$\sum_{m=1}^{\infty} |f(m)|\delta_m < \infty.$$

Note that since $\delta_m \ll (\log\log m)/m^2$ we may assume that $f(m) = \mathrm{O}\big(m^2\big)$. By exchanging the order of summation in absolutely convergent series, we have that

$$\sum_{n=1}^{\infty} \frac{g(n)}{k_n} = \sum_{m=1}^{\infty} f(m) \sum_{n=1}^{\infty} \frac{\mu(n)}{k_{nm}} = \sum_{m=1}^{\infty} f(m)\delta_m.$$

We have the following

THEOREM 2.

(a) *Suppose that the series* $\sum_{m=1}^{\infty} |f(m)|/\varphi(m)$ *converges and that* $\max_{m\le y}\{|f(m)|\} \ll y^{1-\tau}$ *for some* $\tau > 0$. *Then*

$$\sum_{m=1}^{\infty} f(m)H_m(x) \lesssim \sum_{m=1}^{\infty} f(m)\delta_m\pi(x).$$

(b) *Assume GRH and suppose that* $f(m) \ge 0$. *Then*

$$\sum_{m=1}^{\infty} f(m)H_m(x) \gtrsim \sum_{m=1}^{\infty} f(m)\delta_m\pi(x).$$

(c) *Assume GRH and suppose that* $\max_{m\le x}\{|f(m)|\} \ll (\log x)^C$. *Then*

$$\sum_{m=1}^{\infty} f(m)H_m(x) =$$

$$= \sum_{m=1}^{\infty} f(m)\delta_m\pi(x) + \mathrm{O}\left(\sum_{\frac{\sqrt{x}}{\log^{C+5} x} \le m \le \sqrt{x}\log^{(C+1)/2} x} \frac{|f(m)|}{\varphi(m)}\pi(x) \right).$$

2. Proofs of Theorems 1 and 2

Proof of Theorem 1. We start by the inequality $\sum_{n=1}^{\infty} g(n)\pi(x,n) \ge \sum_{n\le(\log x)^{1/7}} g(n)\pi(x,n)$. Applying the unconditional version of the Chebotarev Density Theorem, we have that the previous sum equals

$$\left(\sum_{n=1}^{\infty} g(n)\frac{\pi(x)}{k_n}\right) - \mathrm{O}\left(\sum_{n>\log^{1/7} x} \frac{|\ g(n)\ |}{n\varphi(n)}\pi(x) + (\log x)^{3/7} x \exp\left(-A(\log x)^{5/15}\right)\right) =$$

$$= \left(\left(\sum_{n=1}^{\infty} \frac{g(n)}{k_n} \right) - \mathrm{o}(1) \right) \pi(x)$$

since $g(n) \ll n^2$ by hypothesis (3).The statement in (a) follows immediately.

To prove (b) it suffices to show that

$$\sum_{n>(\log x)^{1/7}} g(n)\pi(x,n) = \mathsf{o}(\pi(x)).$$

We split the above sum into the sum for $n > \sqrt{x}$ plus the sum for $n \le \sqrt{x}$. Now

$$\sum_{n>\sqrt{x}} g(n)\pi(x,n) \le \sum_{n>\sqrt{x}} |g(n)| \#\{a \le x \mid n|a-1\} \ll \sum_{n>\sqrt{x}} |g(n)| \frac{x}{n} = \mathsf{o}(\pi(x)),$$

while, by the Brun-Titchmarsh Theorem,

$$\sum_{(\log x)^{1/7} \le n \le \sqrt{x}} g(n)\pi(x,n) \ll \sum_{(\log x)^{1/7} \le n \le \sqrt{x}} |g(n)| \frac{x}{\varphi(n)\log(x/n)} = \mathsf{o}(\pi(x)).$$

This completes the proof. ∎

The result of Murata of Section 1 gives a conditional asymptotic formula for H_m uniform in the range $m < (\log x)^{1/2-\epsilon}$, which is not enough for some of our applications. We can easily prove the following

LEMMA 3. *Let $c < 1/7$. For $m \le \log^c x$ we have unconditionally the upper bound*

$$H_m(x) \le \left(\delta_m + \mathsf{o}\left(\frac{1}{m\varphi(m)}\right)\right)\pi(x).$$

If we assume GRH, then

$$H_m(x) \le \delta_m \pi(x) + \mathsf{O}\left(\frac{x}{m\varphi(m)\log x \log\log x} + x^{1/2}\log^2 x \log m\right).$$

Proof of Lemma 3. Let z be a parameter to be chosen later and let $P(z)$ be the product of the primes up to z. We start from the inequality

$$H_m(x) \le \sum_{n|P(z)} \mu(n)\pi(x,nm).$$

Provided that $nm < \log^{1/7} x$, we can apply the Chebotarev Density Theorem. So the previous sum is

$$\sum_{n|P(z)} \mu(n)\left\{\frac{1}{k_{mn}}\mathrm{Li}(x) + \mathsf{O}\Big(x\exp(-A\sqrt{\log x}/nm)\Big)\right\} =$$

$$= \sum_{n=1}^{\infty} \frac{\mu(n)}{k_{nm}}\pi(x) + \mathsf{O}\left(\sum_{n\ge z} \frac{1}{k_{nm}}\pi(x)\right) + \mathsf{O}\Big(2^{\pi(z)} x\exp(-A\sqrt{\log x}/2^z m)\Big) =$$

$$= \delta_m \pi(x) + \mathrm{O}\left(\frac{x}{m\varphi(m) z \log x} + 2^{\pi(z)} x \exp(-A\sqrt{\log x}/2^z m) \right)$$

and the result follows by choosing $z = \log\log\log x$, say. The estimate assuming GRH is proven in a similar way, therefore we omit it. ∎

Proof of Theorem 2. (a) We write

$$\sum_{m=1}^{\infty} f(m) H_m(x) = \sum_{m \le z} f(m) H_m(x) + \sum_{z<m<y} f(m) H_m(x) + \sum_{m \ge y} f(m) H_m(x).$$

First note that since $H_m(x) = 0$ if $m > 0$,

$$\sum_{m \ge y} f(m) H_m(x) \ll x^{1-\tau} \# \left\{ p \mid \frac{p-1}{i_p} \le \frac{x}{y} \right\} \ll x^{1-\tau} \frac{x^2}{y^2 \log(x/y)}$$

by a similar argument as in the Theorem of Hooley ([5], pages 211-212). So, if we choose $y = x^{1-\tau/3}$, we get that the above sum is $\ll x^{1-\tau/3}$.

Further, if z goes to infinity as x goes to infinity,

$$\sum_{z<m<y} f(m) H_m(x) \le \sum_{z<m<y} |f(m)| \pi(x, m, 1) \ll \sum_{m>z} |f(m)| \frac{x}{\varphi(m) \log x} = \mathrm{o}(\pi(x))$$

by the Brun-Titchmarsh Theorem and the hypothesis that $\sum |f(m)|/\varphi(m)$ converges. Finally

$$\sum_{m \le z} f(m) H_m(x) \le \sum_{m \le z} f(m) \left\{ \delta_m \pi(x) + \mathrm{o}\left(\frac{\pi(x)}{m\varphi(m)} \right) \right\} =$$

$$= \sum_{m=1}^{\infty} f(m) \delta_m \pi(x) + \mathrm{o}(\pi(x))$$

by Lemma 1, choosing $z = \log^{1/8} x$. This completes the proof of (a).

(b) Note that since $f(n) \ge 0$,

$$\sum_{m=1}^{\infty} f(m) H_m(x) \ge \sum_{m \le (\log x)^{1/4}} f(m) H_m(x).$$

Now we apply the result of Murata and deduce that the previous sum is

$$\sum_{m \le (\log x)^{1/4}} f(m) \left(\delta_m - \mathrm{O}\left(\frac{m^{\epsilon} \log\log x}{\log x} \right) \right) \pi(x)$$

$$= \left(\sum_{n=1}^{\infty} f(m) \delta_m - \mathrm{o}(1) \right) \pi(x) - \mathrm{O}\left(\frac{x \log\log x}{\log^2 x} \sum_{m \le (\log x)^{1/4}} f(m) m^{\epsilon} \right)$$

and this completes the proof since $\sum_{m \le t} f(m) m^{\epsilon} \ll t^{3+2\epsilon}$.

(c) As we did in the proof of (a) we split the sum

$$(4)\qquad \sum_{m=1}^{\infty} f(m)H_m(x) = \sum_{m\le z} f(m)H_m(x) + \sum_{z<m<y} f(m)H_m(x) + \sum_{m\ge y} f(m)H_m(x).$$

By Lemma 3

$$\sum_{m\le z} f(m)H_m(x) \le$$

$$\le \sum_{m\le z} f(m)\left(\delta_m\pi(x) + \mathrm{O}\left(\frac{x}{m\varphi(m)\log x\log\log x} + x^{1/2}\log^2 x\log m\right)\right)$$

$$= \sum_{m=1}^{\infty} f(m)\delta_m\pi(x) + \mathrm{o}(\pi(x)) + \mathrm{O}\left(z\log^{C+1} z\, x^{1/2}\log^2 x\right).$$

If we set $z = x^{1/2}/\log^{5+C} x$, the error term in the above sum is $\mathrm{o}(\pi(x))$.

To deal with the last sum in (4) we proceed as we did in the proof of (a) and we get

$$\sum_{m\ge y} f(m)H_m(x) \le \max_{m\ge y} |\, f(m)\,| \frac{x^2}{y^2\log(x/y)} \ll \log^C x \frac{x^2}{y^2\log(x/y)}$$

which is $\mathrm{o}(\pi(x))$ if we set $y = x^{1/2}\log^{(C+1)/2} x$.

Finally the middle sum in (4) is

$$\sum_{z<m<y} f(m)H_m(x) \le \sum_{z<m<y} |f(m)|\,\pi(x,m,1) \ll \sum_{z<m<y} \frac{|f(m)|}{\varphi(m)}\,\pi(x)$$

by the hypothesis on f and the Brun-Titchmarsh Theorem. This completes the proof. ∎

2. Applications

Suppose $S \subset \mathbb{N}$ and χ_S is the characteristic function of S. If we let

$$\pi_S(x) = \#\{p \le x \mid i_p \in S\} = \sum_{m=1}^{\infty} \chi_S(m)H_m(x),$$

then by the Möbius inversion formula,

$$\pi_S(x) = \sum_{n=1}^{\infty} \tilde{\chi}_S(n)\pi(x,n)$$

where

$$\tilde{\chi}_S(n) = \sum_{m\mid n} \mu(n/m)\chi_S(m) = \sum_{m\mid n,\, m\in S} \mu(n/m).$$

If we assume GRH then, by Theorem 2.b, we have the lower bound

$$\pi_S(x) \gtrsim \sum_{n=1}^{\infty} \frac{\tilde{\chi}_S(n)}{k_n} \pi(x).$$

If the sum over $s \in S$ of $1/\varphi(s)$ converges, we have, by Theorem 2.a, the (unconditional) upper bound

$$\pi_S(x) \lesssim \sum_{n=1}^{\infty} \frac{\tilde{\chi}_S(n)}{k_n} \pi(x).$$

EXAMPLE 1. Suppose $S = \mathbb{P}$ is the **set of all rational primes**. Then, if $\nu(n)$ is the number of prime divisors of n,

$$\tilde{\chi}_{\mathbb{P}}(n) = \sum_{l|n,\, l \text{ prime}} \mu(n/l) = \begin{cases} -\mu(n)\nu(n) & \text{if } n \text{ is square free} \\ \mu(n) & \text{if } \exists! p \in \mathbb{P} \text{ with } p^2 || n \\ 0 & \text{otherwise} \end{cases}$$

Applying Theorem 2.b we get that (on GRH)

$$\pi_{\mathbb{P}}(x) \geq (\delta_{\mathbb{P}} + \mathrm{o}(1))\, \pi(x),$$

where a quick calculation shows that

$$\delta_{\mathbb{P}} = \sum_{n=1}^{\infty} \frac{\sum_{l|n} \mu(n/l)}{k_n} = \delta \sum_{l \text{ prime}} \frac{l^2-1}{l^4-l^3-l^2}.$$

Furthermore, note that by Mertens' Theorem

$$\sum_{\sqrt{x}(\log x)^{-C-5} \leq p \leq \sqrt{x}(\log x)^{(C+1)/2}} \frac{1}{p-1} \ll \frac{\log\log x}{\log x}.$$

Therefore, applying Theorem 2.c, we have the following

COROLLARY 4. *Assume GRH. Then* $\pi_{\mathbb{P}}(x) \sim \delta_{\mathbb{P}} \pi(x)$.

The PARI 1.37 command (see [2])

```
A=0.;forprime(l=2,100000,q=l^2;A=A+(q-1)/(q*(q-l-1)));A*0.3739558
```

gives an approximation for $\delta_{\mathbb{P}}$

```
%1 = 0.38700258336604991820189507 57
```

while the PARI 1.37 command

```
C=0;forprime(l=3,200000,if(isprime((l-1)/order(mod(2,l))),C=C+1,));C
```

calculates the number of odd primes up to 200000 such that i_p is prime and its output is

```
%2 = 7019
```

Since $\pi(200000) = 17984$, we have

$$\frac{\pi_{\mathbb{P}}(200000)}{\pi(200000)} = 0.39029137010676156583629893\,23$$

EXAMPLE 2. Suppose that S is a set of natural numbers such that $\chi_S(n)$ is a multiplicative function. Then

$$\tilde{\chi}_S(l^a) = \begin{cases} 1 & \text{if } a = 0 \\ \chi_S(l^a) - \chi_S(l^{a-1}) & \text{otherwise.} \end{cases}$$

Applying Theorem 2.b we get that (on GRH) $\pi_S(x) \gtrsim \delta_S \pi(x)$, where a quick calculation shows that

$$\delta_S = \sum_{n=1}^{\infty} \frac{\tilde{\chi}_S(n)}{k_n} =$$

$$= \left(\frac{1}{2} - \frac{3\chi_S(2) + \chi_S(4)}{8} + 3 \sum_{\substack{i \geq 1 \\ 2^i \in S}} \frac{1}{2^{2i}} \right) \prod_{l \text{ odd prime}} \left(1 - \frac{1}{l(l-1)} + \frac{l+1}{l} \sum_{\substack{i \geq 1 \\ l^i \in S}} \frac{1}{l^{2i}} \right).$$

i) If F_k is the set of k-**free numbers** then

$$\delta_{F_k} = \left(\rho_k + 1 - \frac{1}{4^{k-1}} \right) \prod_{l \text{ odd prime}} \left(1 - \frac{1}{l^{2k-1}(l-1)} \right),$$

where

$$\rho_k = \begin{cases} 1/2 & \text{if } k = 1 \\ 1/8 & \text{if } k = 2 \\ 0 & \text{if } k \geq 3. \end{cases}$$

In this case we note that $\tilde{\chi}_{F_k}(n) = \mu(m)$ if $n = m^k$ and 0 otherwise. Hence the series

$$\sum_{n=1}^{\infty} \frac{\tilde{\chi}_{F_k}(n)}{n} = \sum_{m=1}^{\infty} \frac{\mu(m)}{m^k}$$

converges. Therefore, by Theorem 1.b, we have the following

COROLLARY 5. *The asymptotic formula* $\pi_{F_k}(x) \sim \delta_{F_k} \pi(x)$ *holds unconditionally.*

The PARI 1.37 command

```
F=0.875;forprime(l=3,100000,F=F*(1-1/((l-1)*l^3)));F
```

gives an approximation for δ_{F_2}

```
%3 = 0.8565404448535421984682105482
```

while the PARI 1.37 command

```
A=0;forprime(l=3,200000,if(issqfree((l-1)/order(mod(2,l))),A=A+1,));A
```

calculates the number of odd primes up to 200000 such that i_p is square-free and its output is

```
%4 = 15430
```

Since $\pi(200000) = 17984$, we have

$$\frac{\pi_{F_2}(200000)}{\pi(200000)} = 0.857984875444839857651245551 5.$$

ii) If G_k is the set of k-**full numbers** then

$$\delta_{G_k} = \left(\tau_k + \frac{1}{4^{k-1}}\right) \prod_{l \text{ odd prime}} \left(1 - \frac{l^{2k-2}-1}{l^{2k-1}(l-1)}\right),$$

where

$$\tau_k = \begin{cases} 0 & \text{if } k = 1 \\ 3/8 & \text{if } k = 2 \\ 1/2 & \text{if } k \geq 3. \end{cases}$$

Since the sum of the reciprocals of k-full numbers converges (for $k \geq 2$), by Theorem 2.a we have

COROLLARY 6. *We have $\pi_{G_k}(x) \gtrsim \delta_{G_k}\pi(x)$ on GRH, and $\pi_{G_k}(x) \lesssim \delta_{G_k}\pi(x)$ unconditionally.*

iii) If P_k is the set of k-**powers** ($k \geq 1$) then

$$\delta_{P_k} = \left(\tau_k + \frac{3}{4^k - 1}\right) \prod_{l \text{ odd prime}} \left(1 - \frac{l^{2k-1} - l}{(l^{2k}-1)(l-1)}\right)$$

where τ is as above. Since the sum of the reciprocals of k-powers converges (for $k \geq 2$), by Theorem 2.b we have

COROLLARY 7. *We have $\pi_{P_k}(x) \gtrsim \delta_{P_k}\pi(x)$ on GRH, and $\pi_{P_k}(x) \lesssim \delta_{P_k}\pi(x)$ unconditionally.*

The PARI 1.37 command

```
G=0.575;forprime(l=3,100000,G=G*(1-(l^2+l)/(l^4-1)));G
```

gives an approximation for δ_{G_2}

```
%5 = 0.4398154555775779797707734332
```

while the PARI 1.37 command

```
B=0;forprime(l=3,200000,if(issquare((l-1)/order(mod(2,l))),B=B+1,));B
```

calculates the number of odd primes up to 200000 such that i_p is a perfect square and its output is

```
%6 = 7898
```

Since $\pi(200000) = 17984$, we have

$$\frac{\pi_{G_2}(200000)}{\pi(200000)} = 0.43916814946619217081850533380.$$

EXAMPLE 3. Suppose that $S(b,a)$ is the **arithmetic progression** $\{a, a+b, a+2b, \ldots\}$ where for simplicity we assume a and b coprime. By Theorem 2.b, we have on GRH the lower bound

$$\#\{p \le x \mid i_p \equiv a \ (\mathrm{mod}\ b)\} \gtrsim \delta_{a,b}\pi(x)$$

where

$$\delta_{a,b} = \sum_{\substack{m=1,\\ m\equiv a \ (\mathrm{mod}\ b)}}^{\infty} \delta_m.$$

To deduce the upper bound it suffices to write

$$\#\{p \le x \mid i_p \equiv a \ (\mathrm{mod}\ b)\}$$

$$= \pi(x) - 1 - \sum_{c \ (\mathrm{mod}\ b),\ (c,b)=1,\ c\neq b} \#\{p \le x \mid i_p \equiv c \ (\mathrm{mod}\ b)\} - \sum_{d|b} \pi(x,d)$$

and apply Theorem 2.b and the Chebotarev Density Theorem to the right hand side. Hence, performing the computation, we can deduce the following

COROLLARY 8. *On GRH we have the asymptotic formula*

$$\#\{p \le x \mid i_p \equiv a \ (\mathrm{mod}\ b)\} \sim \delta_{a,b}\pi(x) \qquad \text{with} \qquad \delta_{a,b} = \frac{1}{\varphi(b)} \sum_{\chi \ (\mathrm{mod}\ b)} \overline{\chi(a)} \cdot \delta_\chi,$$

where the sum is extended to all the Dirichlet characters (mod b) and

$$\delta_\chi = \left(\frac{1}{2} - \frac{\chi(2)(\chi^2(2) - \chi(2) + 12)}{8(4 - \chi(2))}\right) \prod_{l \text{ odd prime}} \left(1 - \frac{l(1-\chi(l))}{(l-1)(l^2 - \chi(l))}\right).$$

Note that if χ_0 is the principal character (mod b), then

$$\delta_{\chi_0} = \prod_{l|b} \left(1 - \frac{1}{l(l-1)}\right).$$

If $b = 3$ then $\delta_{\chi_0} = 5/6$ and the non-principal character is

$$\chi_1(n) = \begin{cases} 0 & \text{if } 3|n \\ 1 & \text{if } n \equiv 1 (\mathrm{mod} 3) \\ -1 & \text{if } n \equiv 2 (\mathrm{mod} 3) \end{cases}$$

so that

$$\delta_{\chi_1} = \frac{3}{20} \prod_{l \text{ odd prime } l \equiv 2 (\mathrm{mod}\ 3)} \left(1 - \frac{2l}{(l-1)(l^2+1)}\right).$$

Finally

$$\delta_{1,3} = \frac{5}{12} + \frac{1}{2}\delta_{\chi_1} \text{ and } \delta_{2,3} = \frac{5}{12} - \frac{1}{2}\delta_{\chi_1}.$$

Using PARI we get approximations

$$\delta_{1,3} = 0.4819 \text{ and } \delta_{2,3} = 0.3514$$

while

$$\frac{\pi_{S(3,1)}(200000)}{\pi(200000)} \sim 0.4693 \text{ and } \frac{\pi_{S(3,2)}(200000)}{\pi(200000)} \sim 0.3645.$$

EXAMPLE 4. Suppose that $f(n) = \log n$. E. Bach, R. Lukes, J. Shallit and H. C. Williams in [1] consider the sum $\sum_{p \le x} \log i_p$.

Since $\sum_{d|n} \mu(d) \log(n/d)$ is the von Mangoldt function $\Lambda(n)$, by Theorem 1.a we have unconditionally the lower bound

$$\sum_{p \le x} \log i_p \gtrsim \delta_B \pi(x) \tag{5}$$

where

$$\delta_B = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{k_n} = \sum_{m=1}^{\infty} (\log m)\delta_m.$$

Furthermore, note that

$$\sum_{n=1}^{\infty} \frac{\Lambda(n)}{k_n} = \sum_{q \text{ prime}} \frac{q \log q}{(q-1)(q^2-1)} + \frac{1}{24} \log 2$$

as predicted in [1].

We are unable to deduce an upper bound of the type $c\pi(x)$ for the sum in (5) even on GRH. However

PROPOSITION 9. *Assume GRH. Then*

$$\sum_{p \le x} \log i_p \ll \frac{x \log \log x}{\log x}.$$

Proof. We have seen that

$$\sum_{p\le x}\log i_p=\sum_{n\le x}\Lambda(n)\pi(x,n)=$$

$$=\sum_{n\le\sqrt{x}/\log^3 x}\Lambda(n)\pi(x,n)+\sum_{\sqrt{x}/\log^3 x\le n\le x}\Lambda(n)\pi(x,n).$$

By the Chebotarev Density Theorem the first sum above is (on GRH)

$$\ll\sum_{n\le\sqrt{x}/\log^3 x}\log n\left\{\frac{x}{n^2\log x}+\mathrm{O}\big(\sqrt{x}\log xn\big)\right\}\ll\pi(x),$$

while the second sum equals

$$\sum_{q>\sqrt{x}/\log^3 x}\log q\;\pi(x,q)+\sum_{q^\alpha>\sqrt{x}/\log^3 x,\alpha>1}\log q\;\pi(x,q^\alpha). \tag{6}$$

The second sum in (6) is

$$\le\sum_{q^\alpha>\sqrt{x}/\log^3 x,\alpha>1}\log q\;\frac{x}{q^\alpha}\ll x\sum_{q>\sqrt{x}/\log^3 x}\frac{\log q}{q^2}\ll\pi(x),$$

while the first sum in (6) equals

$$\sum_{\sqrt{x}/\log^3 x<q\le\sqrt{x}\log x}\log q\;\pi(x,q)+\sum_{q>\sqrt{x}\log x}\log q\;\pi(x,q). \tag{7}$$

We bound the second sum in (7) with

$$\#\left\{q\;\middle|\;q\text{ divides}\prod_{m\le\sqrt{x}/\log x}(2^m-1)\right\}\cdot\;\log x\ll\pi(x).$$

Finally the first sum in (7), by the Brun-Titchmarsh Theorem and the Mertens' formula, is

$$\le\sum_{\sqrt{x}/\log^3 x<q\le\sqrt{x}\log x}\log q\;\pi(x,q,1)\ll\frac{x}{\log x}\sum_{\sqrt{x}/\log^3 x<q\le\sqrt{x}\log x}\frac{\log q}{q}\ll\frac{x\log\log x}{\log x}$$

and this ends the proof. ∎

We mention that Theorem 2.c implies that if $f(m)=\mathrm{o}\big(\log^{-1}(m)\big)$, then (on GRH)

$$\sum_{p\le x}f(i_p)\sim\left(\sum_{m=1}^{\infty}f(m)\delta_m\right)\pi(x).$$

3. Conclusion

The results in the present paper can be generalized to the case where $i_p = ind_p(a)$ with a any integer. The computation would be affected by the corresponding formula for $K_n = [\mathbb{Q}(\zeta_n, a^{1/n}), \mathbb{Q}]$.

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