Enumerating Permutation Polynomials over Finite Fields by Degree

Sergei Konyagin[1]

*Department of Mechanics and Mathematics, Moscow State University, Vorobjovy Gory,
119899 Moscow, Russia*
E-mail: kon@mech.math.msu.su,konyagin@ok.ru

and

Francesco Pappalardi

*Dipartimento Di Matematica, Università degli Studi Roma Tre, Largo S. L. Murialdo, 1,
1-00146 Rome, Italy*
E-mail: pappa@mat.uniroma3.it

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We prove an asymptotic formula for the number of permutations for which the associated permutation polynomial has degree smaller than $q - 2$. © 2002 Elsevier Science (USA)

Let $\mathbb{F}_q$ be a finite field with $q = p^f > 2$ elements and let $\sigma \in S(\mathbb{F}_q)$ be a permutation of the elements of $\mathbb{F}_q$. The permutation polynomial f_σ of σ is

$$f_\sigma(x) = \sum_{c \in \mathbb{F}_q} \sigma(c)(1 - (x - c)^{q-1}) \in \mathbb{F}_q[x].$$

f_σ has the property that $f_\sigma(a) = \sigma(a)$ for every $a \in \mathbb{F}_q$ and this explains its name.

For an account of the basic properties of permutation polynomials we refer to the book of Lidl and Niederreiter [5].

[1] To whom correspondence should be addressed.

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From the definition, it follows that for every σ

$$\partial(f_\sigma) \leq q - 2.$$

A variety of problems and questions regarding permutation polynomials have been posed by Lidl and Mullen [3,4]. Among these there is problem of determining the number N_d of permutation polynomials of fixed degree d. In [6,9] Malvenuto and the second author address the problem of counting the permutations that move a fixed number of elements of $\mathbb{F}_q$ and whose permutation polynomials have "low" degree.

Here, we consider all permutations and we want to prove the following:

THEOREM 1. *Let*

$$N = \#\{\sigma \in S(\mathbb{F}_q) \mid \partial(f_\sigma) < q - 2\}.$$

Then,

$$|N - (q - 1)!| \leq \sqrt{2e/\pi} q^{q/2}.$$

This confirms the common belief that *almost all permutation polynomials have degree $q - 2$.*

The first few values of N are listed below:

q	2	3	4	5	7	8	9	11
N	0	0	12	20	630	5368	42 120	3 634 950
$(q-1)!$	1	2	6	24	720	5040	40 320	3 628 800

Proof. The proof uses exponential sums and a similar argument as the one in [2].

By extracting the coefficient of x^{q-2} in $f_\sigma(x)$, we obtain that the degree of $f_\sigma(x)$ is strictly smaller than $q - 2$ if and only if

$$\sum_{c \in \mathbb{F}_q} c\sigma(c) = 0.$$

For a fixed subset S of $\mathbb{F}_q$, we introduce the auxiliary set of functions

$$N_S = \left\{ f \mid f : \mathbb{F}_q \to S, \text{ and } \sum_{c \in S} cf(c) = 0 \right\}$$

and set $n_S = \#N_S$. By inclusion exclusion, it is easy to check that

$$N = \sum_{S \subseteq \mathbb{F}_q} (-1)^{q-|S|} n_S. \tag{1}$$

Now if $e_p(u) = e^{2\pi i u/p}$, consider the identity

$$n_S = \frac{1}{q} \sum_{a \in \mathbb{F}_q} \left(\sum_{f: \mathbb{F}_q \to S} e_p \left(\sum_{c \in \mathbb{F}_q} \mathrm{Tr}(ac f(c)) \right) \right),$$

which follows from the standard property

$$\frac{1}{q} \sum_{a \in \mathbb{F}_q} e_p(\mathrm{Tr}(ax)) = \begin{cases} 1 & \text{if } x = 0, \\ 0 & \text{if } x \neq 0. \end{cases}$$

By exchanging the sum with the product, we obtain

$$n_S = \frac{1}{q} \sum_{a \in \mathbb{F}_q} \left(\prod_{c \in \mathbb{F}_q} \sum_{t \in S} e_p(\mathrm{Tr}(act)) \right).$$

By isolating the term with $a = 0$ in the external sum and noticing that the internal product does not depend on a (for $a \neq 0$), we get

$$n_S = \frac{|S|^q}{q} + \frac{1}{q} \sum_{a \in \mathbb{F}_q^*} \left(\prod_{b \in \mathbb{F}_q} \sum_{t \in S} e_p(\mathrm{Tr}(bt)) \right).$$

Finally,

$$n_S = \frac{|S|^q}{q} + \frac{q-1}{q} \prod_{b \in \mathbb{F}_q} \sum_{t \in S} e_p(\mathrm{Tr}(bt)). \tag{2}$$

Now let us insert Eq. (2) in Eq. (1) and obtain

$$N - \sum_{S \subseteq \mathbb{F}_q} \frac{(-1)^{q-|S|}}{q} |S|^q = \frac{q-1}{q} \sum_{S \subseteq \mathbb{F}_q} (-1)^{q-|S|} \prod_{b \in \mathbb{F}_q} \sum_{t \in S} e_p(\mathrm{Tr}(bt)).$$

Note that inclusion–exclusion gives

$$\sum_{S \subseteq \mathbb{F}_q} \frac{(-1)^{q-|S|}}{q} |S|^q = (q-1)!$$

Therefore,

$$N - (q-1)! = \frac{q-1}{q} \sum_{S \subseteq \mathbb{F}_q} (-1)^{q-|S|} |S| \prod_{b \in \mathbb{F}_q^*} \sum_{t \in S} e_p(\mathrm{Tr}(bt)).$$

Using the fact that for $b \in \mathbb{F}_q^*$

$$\sum_{t \in S} e_p(\mathrm{Tr}(bt)) = -\sum_{t \notin S} e_p(\mathrm{Tr}(bt))$$

and grouping together the term relative to S and the term relative to $\mathbb{F}_q \setminus S$, we get

$$|N - (q-1)!| \leq \frac{q-1}{2q} \sum_{S \subseteq \mathbb{F}_q} |q - 2|S|| \prod_{b \in \mathbb{F}_q^*} \left| \sum_{t \in S} e_p(\mathrm{Tr}(bt)) \right|. \qquad (3)$$

Now let us also observe that

$$\sum_{b \in \mathbb{F}_q} \left| \sum_{t \in S} e_p(\mathrm{Tr}(bt)) \right|^2 = q|S|,$$

so that

$$\sum_{b \in \mathbb{F}_q^*} \left| \sum_{t \in S} e_p(\mathrm{Tr}(bt)) \right|^2 = (q - |S|)|S|.$$

From the fact that the geometric mean is always bounded by the arithmetic mean (i.e. $(\prod_{i=1}^k |a_i|^2)^{1/k} \leq \frac{1}{k} \sum_{i=1}^k |a_i|^2$), we have that

$$\prod_{b \in \mathbb{F}_q^*} \left| \sum_{t \in S} e_p(\mathrm{Tr}(bt)) \right| \leq \left(\frac{1}{q-1} \sum_{b \in \mathbb{F}_q^*} \left| \sum_{t \in S} e_p(\mathrm{Tr}(bt)) \right|^2 \right)^{(q-1)/2}$$

$$= \left(\frac{(q - |S|)|S|}{q-1} \right)^{(q-1)/2}. \qquad (4)$$

Furthermore, using (3) and (4) we obtain

$$|N - (q-1)!| \leq \frac{q-1}{2q(q-1)^{(q-1)/2}} \sum_{S \subseteq \mathbb{F}_q} |q - 2|S||((q - |S|)|S|)^{(q-1)/2}. \qquad (5)$$

We want to estimate the above sum. Consider the inequality

$$((q - |S|)|S|)^{(q-1)/2} \leq \left(\frac{q}{2} \right)^{q-1}, \qquad (6)$$

and the identity

$$\sum_{S \subseteq \mathbb{F}_q} |q - 2|S|| = 2q \binom{q-1}{[q/2]}, \qquad (7)$$

which holds since

$$2 \sum_{\substack{S \subseteq \mathbb{F}_q, \\ |S| \leq q/2}} (q - 2|S|) = 2 \left[\sum_{j=0}^{[q/2]} \binom{q}{j}(q - j) - \sum_{j=1}^{[q/2]} \binom{q}{j}(j) \right]$$

$$= 2q \left[\sum_{j=0}^{[q/2]} \binom{q-1}{j} - \sum_{j=1}^{[q/2]} \binom{q-1}{j-1} \right] = 2q \binom{q-1}{[q/2]}.$$

From the standard inequality

$$\binom{2n}{n} \leq \sqrt{\frac{2}{\pi}} \frac{2^{2n}}{\sqrt{2n + 1/2}},$$

which can be found for example in [1], we deduce

$$\binom{q-1}{[q/2]} \leq \sqrt{\frac{2}{\pi}} \frac{2^{q-1}}{\sqrt{q - 1/2}}. \tag{8}$$

Therefore, (5), (6), (7) and (8) imply

$$|N - (q - 1)!| \leq \left(\frac{q-1}{\sqrt{q - 1/2} \sqrt{q}} \right) \sqrt{\frac{2}{\pi}} \left(\frac{q}{q-1} \right)^{(q-1)/2} q^{q/2}$$

and in view of the inequalities

$$\frac{q-1}{\sqrt{q - 1/2} \sqrt{q}} < 1, \qquad \left(\frac{q}{q-1} \right)^{(q-1)/2} < \sqrt{e},$$

we finally obtain

$$|N - (q - 1)!| \leq \sqrt{\frac{2e}{\pi}} q^{q/2}$$

and this completes the proof. ∎

CONCLUSION

Computations suggest that a more careful estimate of the sum in (5) would yield to a constant $\sqrt{e/2\pi}$ instead of $\sqrt{2e/\pi}$ as coefficient in $q^{q/2}$ in the statement of Theorem 1. However, we feel that such a minor improvement does not justify the extra work.

The ideas in the proof of Theorem 1 can be used to deal with the analogous problem of enumerating the permutation polynomials that have the ith coefficient equal to 0 and also to the problem of enumerating the permutation polynomials with degree less than $q - k$ (for fixed k). However, the exponential sums that need to be considered are significantly more complicated.

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Note added in proof. Recently the asymptotic estimate in Theorem 1 for a prime q (with slightly weaker remainder term) has been proved in the paper "The number of permutation polynomials of a given degree" by Pinaki Das (to appear in this journal). The author uses algebraic arguments rather than exponential sums.

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