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ON THE EXPONENT OF THE GROUP OF POINTS OF AN ELLIPTIC CURVE OVER A FINITE FIELD

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Abstract. We present a lower bound for the exponent of the group of rational points of an elliptic curve over a finite field. Earlier results considered finite fields $\mathbb{F}_{q^m}$ where either q is fixed or $m = 1$ and q is prime. Here, we let both q and m vary; our estimate is explicit and does not depend on the elliptic curve.

1. Introduction

Let $\mathbb{F}_q$ be a finite field with $q = p^m$ elements and let E be an elliptic curve defined over $\mathbb{F}_q$. It is well known (see for example the book of Washington [7]) that the group of rational point of E over $\mathbb{F}_q$ satisfies

$$E(\mathbb{F}_q) \cong \mathbb{Z}_n \times \mathbb{Z}_{nk},$$

where $n, k \in \mathbb{N}$ are such that $n \mid q - 1$. The *exponent* of $E(\mathbb{F}_q)$ is

$$\exp(E(\mathbb{F}_q)) = nk.$$

The problem of studying $\exp(E(\mathbb{F}_q))$ is a natural one and was started by Schoof [6] in 1989. He proved that if E is an elliptic curve over $\mathbb{Q}$ without complex multiplication, then for every prime $p > 2$ of good reduction for E, one has the estimate

$$\exp(E(\mathbb{F}_p)) > C_E\sqrt{p}\frac{\log p}{(\log\log p)^2},$$

where $C_E > 0$ is a constant depending only on E.

In 2005, Luca and Shparlinski [4] considered the case when q is fixed and they proved that if $E/\mathbb{F}_q$ is ordinary, there exists an effectively computable constant $\vartheta(q)$ depending only on q such that

$$\exp(E(\mathbb{F}_{q^m})) > q^{m/2+\vartheta(q)m/\log m} \tag{1}$$

holds for all positive integers $m > 1$.

Other lower bounds that hold for families of primes (resp. for families of powers of fixed primes) with density one were proven by Duke in [2] (resp. by Luca and Shparlinski in [4]).

Here we let both p and m vary, and we prove the following:

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Theorem. *Let E be any elliptic curve over $\mathbb{F}_{p^m}$ where $m \geq 3$. Then either $m = 2r$ is even and*

$$E(\mathbb{F}_{p^{2r}}) \cong \mathbb{Z}_{p^r \pm 1} \times \mathbb{Z}_{p^r \pm 1}$$

or

$$\exp(E(\mathbb{F}_{p^m})) \geq 2^{-46} p^{m/2} \frac{m^{1/3}}{(\log m)^{8/3} (\log\log m)^{1/3}}.$$

Note that the result also applies to supersingular elliptic curves and that it improves on that in (1) for values of m which are small with respect to p.

2. Lemmas

The proof is based on estimates for the distance between perfect powers due to Bugeaud. More precisely, we will apply the following result from [1]:

Lemma 1. *Let $f \in \mathbb{Z}[X]$ be a monic polynomial of degree $d \geq 2$ without multiple roots. Let H be the maximum of the absolute values of its coefficients and D be its discriminant. Let a, x, y, and m be rational integers satisfying $a \neq 0$, $|y| \geq 2$, $m \geq 2$, $f(x) = ay^m$. Denote by $\log_2$ the logarithm in base 2 and write $\log_* x$ for $\max\{\log x, 1\}$. The following inequality holds:*

$$m < \max\left\{ d \log_2(2H+3), 2^{15(d+6)} d^{7d} |D|^{3/2} (\log|D|)^{3d} (\log_* |a|)^2 \log_* \log_* |a| \right\}.$$

We need the following elementary lemma:

Lemma 2. *If q is a prime power and E is an elliptic curve defined over $\mathbb{F}_q$ such that $E(\mathbb{F}_q) \cong \mathbb{Z}_n \times \mathbb{Z}_{nk}$, then $q = n^2k + n\ell + 1$ for some integer ℓ that satisfies $|\ell| \leq 2\sqrt{k}$.*

Proof. By the Hasse bound, we can write $n^2k = q + 1 - a_q$ for some integer a_q that satisfies $a_q^2 \leq 4q$. Using the Weil pairing one also sees that $q \equiv 1 \pmod{n}$. Hence $a_q = 2 + n\ell$ for some integer ℓ and $q = n^2k + n\ell + 1$. Finally

$$n^2\ell^2 + 4n\ell + 4 = a_q^2 \leq 4q = 4n^2k + 4n\ell + 4,$$

and the result follows. □

We will also need the classical characterizations of the group structures due to Waterhouse (see [7, Theorem 4.3, page 98]) which describes possible cardinalities $\#E(\mathbb{F}_q)$ of the set of $\mathbb{F}_q$-rational points of elliptic curves over $\mathbb{F}_q$.

Lemma 3. *Let $q = p^m$ be a power of a prime p and let $N = q + 1 - a$. There is an elliptic curve E defined over $\mathbb{F}_q$ such that $\#E(\mathbb{F}_q) = N$ if and only if $|a| \leq 2\sqrt{q}$ and a satisfies one of the following:*

(i) $\gcd(a, p) = 1$;
(ii) *m even and* $a = \pm 2\sqrt{q}$;
(iii) *m is even,* $p \not\equiv 1 \pmod{3}$, *and* $a = \pm\sqrt{q}$;
(iv) *m is odd,* $p = 2$ *or* 3, *and* $a = \pm p^{(m+1)/2}$;
(v) *m is even,* $p \not\equiv 1 \pmod{4}$, *and* $a = 0$;
(vi) *m is odd and* $a = 0$.

For each admissible cardinality, Rück (see Washington [7, Theorem 4.4, page 98]) describes the possible group structures.

Lemma 4. *Let N be an integer that occurs as the order of an elliptic curve over a finite field $\mathbb{F}_q$, where $q = p^m$ is a power of a prime p. Write $N = p^e n_1 n_2$ with $p \nmid n_1 n_2$ and $n_1 \mid n_2$ (possibly $n_1 = 1$). There is an elliptic curve E over $\mathbb{F}_q$ such that*

$$E(\mathbb{F}_q) \cong \mathbb{Z}_{p^e} \times \mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2}$$

if and only if

(1) $n_1 = n_2$ *in case* (ii) *of Lemma* 3*;*
(2) $n_1 | q-1$ *in all other cases of Lemma* 3.

Finally we need the following numerical statement:

Lemma 5. *Assume that α and β are real numbers with $\alpha > 4$ and $\beta \geq 4$. If*

$$\alpha \leq \beta^{3/2} \cdot (\log \beta)^8 \cdot \log\log \beta,$$

then

$$\beta \geq \frac{\alpha^{2/3}}{(\log \alpha)^{16/3}(\log\log \alpha)^{2/3}}.$$

Proof. If $\alpha \geq \beta \geq 4$, then

$$\beta \geq \left(\frac{\alpha}{(\log \beta)^8 \log\log \beta}\right)^{2/3} \geq \left(\frac{\alpha}{(\log \alpha)^8 \log\log \alpha}\right)^{2/3}.$$

If $4 < \alpha \leq \beta$,

$$\beta \geq \alpha \geq \frac{\alpha^{2/3}}{(\log \alpha)^{16/3}(\log\log \alpha)^{2/3}}.$$

□

3. Proof of the Theorem

Assume that $E(\mathbb{F}_{p^m}) \cong \mathbb{Z}_n \times \mathbb{Z}_{nk}$. Then, by Lemma 2, we have that

$$p^m = kn^2 + \ell n + 1 \qquad \text{for some } \ell \text{ with } |\ell| \leq 2\sqrt{k}.$$

If $\ell = \pm 2\sqrt{k}$, then k must be a perfect square, and we write $k = M^2$ so that $\ell = \pm 2M$. Therefore in the above identity we have

$$p^m = (Mn \pm 1)^2,$$

which implies that $m = 2r$ is even and that $Mn = p^r \mp 1$. Furthermore, in this case

$$p^m + 1 - \#E(\mathbb{F}_{p^m}) = \ell n + 2 = \pm 2Mn + 2 = \pm 2p^{m/2}.$$

This happens precisely in case (ii) of Lemma 3. Note also that in this case $p \nmid \#E(\mathbb{F}_{p^m})$. Hence, by case (1) in Lemma 4, we have that $n = nk$ so that $k = 1$.

We conclude that if $l = \pm 2\sqrt{k}$, then $k = 1$, $n = p^r \mp 1$ and finally

$$E(\mathbb{F}_{p^{2r}}) \cong \mathbb{Z}_{p^r \pm 1} \times \mathbb{Z}_{p^r \pm 1}.$$

From now on, we can assume that $|\ell| < 2\sqrt{k}$. We apply Lemma 1 with the following data:

$$f(X) = X^2 + \ell X + k, \quad d = 2, \quad |D| = 4k - \ell^2, \quad H = k,$$
$$x = kn, \quad y = p \quad \text{and} \quad a = k.$$

Note that since $|\ell| < 2\sqrt{k}$, we have that $D \neq 0$ so that f has two distinct roots.

From the identity $kp^m = (kn)^2 + \ell(kn) + k$ and from Lemma 1, it follows that

$$m \le \max\{2\log_2(2k+3), 2^{134}(4k)^{3/2}(\log 4k)^6(\log_* k)^2(\log_* \log_* k)\}.$$

Since we can assume that $k \ge 2$, it follows that

$$m \le 2^{134}(4k)^{3/2}(\log 4k)^8 \log\log 4k.$$

If $4 \le m \le 2^{136}$, then $m^{1/3}/(2^{45}(\log m)^{8/3}(\log\log m)^{1/3}) < 1/4$, and the statement of the Theorem is vacuous since $\exp(E(\mathbb{F}_q)) \ge \sqrt{q} - 1$ for every q.

If $m > 2^{136}$, we apply Lemma 5 with $\alpha = m/2^{134} > 4$ and $\beta = 4k$, and we obtain

$$k \ge \frac{1}{4} \cdot \frac{(m/2^{134})^{2/3}}{(\log_* \frac{m}{2^{134}})^{\frac{16}{3}}(\log_* \log_* \frac{m}{2^{134}})^{\frac{2}{3}}} \ge \frac{1}{2^{\frac{274}{3}}} \cdot \frac{m^{2/3}}{(\log m)^{16/3}(\log\log m)^{2/3}},$$

and so

$$\exp(E(\mathbb{F}_{p^m}) = nk \ge (\sqrt{p^m} - 1)\sqrt{k} \ge p^{m/2}\frac{m^{1/3}}{2^{46}(\log m)^{8/3}(\log\log m)^{1/3}}.$$

This concludes the proof of the Theorem. $\square$

The constant 2^{-46} can be slightly improved with a more careful analysis, but this is not too important.

4. Conclusion

To construct curves with a small exponent one can consider a recent result of Matomäki in [5] that states that, for any $\epsilon > 0$, there exist infinitely many primes p of the form $p = an^2 + 1$ with $a < p^{1/2+\epsilon}$.

Let $p > 3$ be one such prime. Since $p + 1 - an^2 = 2$ and p is odd, part (i) of Lemma 3 assures that there exists an ordinary elliptic curve E over $\mathbb{F}_p$ with $\#E(\mathbb{F}_p) = an^2$ points. Furthermore, since $p \equiv 1 \bmod n$, part (2) of Lemma 4 assures that one can choose E in such a way that

$$E(\mathbb{F}_p) \cong \mathbb{Z}_n \times \mathbb{Z}_{na}.$$

This implies that there exists an infinite sequence of primes p, each with an ordinary elliptic curve $E/\mathbb{F}_p$ such that

$$\exp(E(\mathbb{F}_p)) = an < p^{3/4+\epsilon}.$$

One can also consider, for a prime p, the identity

$$p^3 + 1 - (p+2)(p-1)^2 = 3p - 1.$$

Since $3p-1$ is coprime to p and $3p-1 \le 2\sqrt{p^3}$, part (i) of Lemma 3 can be applied with $q = p^3$ and $N = (p+2)(p-1)^2$. It follows that there exists an elliptic curve E over $\mathbb{F}_{p^3}$ with $\#E(\mathbb{F}_{p^3}) = (p+2)(p-1)^2$ points. Furthermore, if $p \equiv 7 \bmod 9$, we can write $N = n_1 n_2$, where $n_1 = 3(p-1)$ and $n_2 = \frac{(p+2)(p-1)}{3}$. It is clear that $n_1 \mid n_2$ and that $n_1 \mid p^3 - 1$, so part (2) of Lemma 4 can be applied. It follows that for every prime $p \equiv 7 \bmod 9$, there exists an ordinary elliptic curve over $\mathbb{F}_{p^3}$ such that

$$E(\mathbb{F}_{p^3}) \cong \mathbb{Z}_{3p-3} \times \mathbb{Z}_{\frac{(p+2)(p-1)}{3}}.$$

We immediately conclude that there exists a infinite sequence of distinct q with an elliptic curve $E/\mathbb{F}_q$ such that

$$\exp(E(\mathbb{F}_q)) = \frac{q^{2/3}}{3}(1 + o(1)). \tag{2}$$

This should be compared on one side with Schoof's result in [6] that (assuming GRH) if E is an elliptic curve over $\mathbb{Q}$, there exists a constant c_E such that $\exp(E(\mathbb{F}_p)) < c_E p^{7/8} \log p$ for infinitely many primes p and on another side with Luca, McKee and Shparlinski's results in [3] that there exists an absolute constant $\rho > 0$ such that if $E/\mathbb{F}_q$ is a fixed elliptic curve, the inequality

$$\exp(E(\mathbb{F}_{q^m})) < q^m \exp\left(-m^{\rho/\log\log m}\right)$$

holds for infinitely many positive integers m.

We wonder if, for every $\epsilon > 0$, one can construct an infinite family of prime powers q, each with an elliptic curve $E/\mathbb{F}_q$ such that

$$E(\mathbb{F}_q) \ncong \mathbb{Z}_{\sqrt{q}\pm 1} \times \mathbb{Z}_{\sqrt{q}\pm 1}$$

and

$$\exp(E(\mathbb{F}_q)) \ll_\epsilon q^{1/2+\epsilon}$$

or if the 2/3 in (2) can be improved.

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