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Pairs of integers which are mutually squares

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Abstract We derived an asymptotic formula for the number of pairs of integers which are mutually squares. Earlier results dealt with pairs of integers subject to the restriction that they are both odd, co-prime and squarefree. Here we remove all these restrictions and prove (similar to the best known one with restrictions) that the number of such pair of integers upto a large real X is asymptotic to $\frac{cX^2}{\log X}$ with an absolute constant c which we give explicitly. Our error term is also compatible to the best known one.

Keywords quadratic reciprocity, mutually squares, Jacobi symbol

MSC(2010) 11A15, 11L20

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1 Introduction

Let m and n be integers with $n \neq 0$. We say that m is a *square modulo* n if the equation $X^2 \equiv m \pmod{n}$ is solvable and we write this condition as $m \equiv \square \pmod{n}$. If furthermore n is squarefree, this is equivalent to the fact that for every odd prime p dividing n, we have $(\frac{m}{p}) = 0$ or 1.

In response to a question posed by Serre [9] regarding an upper bound of the number of certain solutions of ternary quadratic forms, a result due to Friedlander and Iwaniec [6, Theorem 1] states that for any fixed $\delta > 0$, uniformly for A, $B \geqslant \exp((\log AB)^{\delta})$, the following asymptotic formula holds:

$$\begin{aligned} F(A,B) &= \sharp\{(a,b) : 1 \leqslant a \leqslant A,\ 1 \leqslant b \leqslant B,\ \mu^2(2ab) = 1,\ a \equiv \square \pmod{b} \text{ and } b \equiv \square \pmod{a}\} \\ &= \frac{AB}{\sqrt{\log A}\sqrt{\log B}}\left(\frac{6}{\pi^3} + O_\delta\left(\frac{1}{\log A} + \frac{1}{\log B}\right)\right). \end{aligned} \tag{1.1}$$

The condition $\mu^2(2ab) = 1$ is equivalent to the requirement that a and b are both odd, coprime and squarefree. The above result can be interpreted in terms of the solvability of a ternary quadratic equation. A similar result is also due to Guo [7] and to Fouvry and Klüners that in [3, Theorem 5] and [4] investigated the solvability of ternary quadratic equations and interpreted their results in terms of the average behavior of the value of the 4-rank of the ideal class group of quadratic fields. This connection has also been exploited more recently by Fouvry et al. [5] to enumerate dihedral and quaternionic extensions of $\mathbb{Q}$.

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Table 1 Numerical experiments

X	$\sharp\mathcal{S}(X,X)$	$f(X)$	$\frac{\sharp\mathcal{S}(X,X)}{f(X)}$	$F(X,X)$	$g(X)$	$\frac{F(X,X)}{g(X)}$
1,000	115,372	97,808.47	1.1796	24,568	28,013.32	0.87701
10,000	8,314,344	7,335,635.35	1.1334	2,003,744	2,100,999.51	0.95371
100,000	648,906,554	586,850,828.54	1.1057	164,863,544	168,079,961.24	0.98086
500,000	14,051,934,364	12,871,861,671.91	1.0917	3,648,637,010	3,686,630,240.18	0.98969
1,000,000	53,139,299,588	48,904,235,712.13	1.0866	13,893,935,102	14,006,663,437.25	0.99195
1,500,000	115,865,614,308	106,897,251,744.10	1.0839	30,407,843,840	30,616,444,685.08	0.99319
2,000,000	201,562,784,584	186,271,396,565.16	1.0821	53,030,177,066	53,349,995,592.04	0.99401
2,500,000	309,769,118,570	286,640,518,154.18	1.0807	81,629,492,250	82,096,718,347.62	0.99431
2,900,000	412,334,047,078	381,856,353,452.19	1.0798		109,367,488,240.97	

The analytic tools appearing in the proofs of the main results in [2–7] are all of the same nature: the use of Jacobi symbols as characters, the Siegel-Walfisz theorem for these characters, and double oscillations theorem. This paper removes the condition $\mu^2(2ab)=1$ of previous results and to this purpose define

$$\mathcal{S}(A,B)=\{(a,b)\in\mathbb{N}^2, a\leqslant A, b\leqslant B \text{ and } a\equiv\square \pmod b, b\equiv\square \pmod a\}.$$

We will prove the following result.

Theorem 1.1. *For any fixed $\delta>0$ and uniformly for A, $B\geqslant\exp((\log AB)^\delta)$, we have the estimate*

$$\sharp\mathcal{S}(A,B)=\frac{AB}{\sqrt{\log A}\sqrt{\log B}}\left(c+O\left(\left(\frac{1}{\log A}+\frac{1}{\log B}\right)(\log\log AB)^5\right)\right),$$

where

$$c=\frac{7\pi^3}{1440}\times\left(1+3\prod_{\substack{p\geqslant 3\\ p \text{ prime}}}\left(1+\frac{1}{p^2}-\frac{1}{p^3}\right)\right)=0.6756369848102110953\cdots.$$

In Table 1, computed with Pari/gp (see [1]), we list the values of $\sharp\mathcal{S}(X,X)$ and of $F(X,X)$ for various small values of X and compare them with $f(x)=cX^2/\log X$ and $g(x)=6X^2/\pi^3\log X$, respectively.

The rest of the paper is organized as follows. Section 2 deals with four lemmas. The first two lemmas are from [6] and will be required to prove the other two lemmas. The last two lemmas provide the combinatorial setup of the proof of Theorem 1.1. In Section 3, we complete the proof of our main result. We first state Lemma 3.1 and apply it to prove Theorem 1.1 and then finally complete the proof of this technical lemma. In Section 4, we give a summary of the results and talk about some possible future research.

2 Lemmas

We refer to the book of Hardy and Wright [8] for classical results from elementary number theory. Two lemmas from [6] will be needed.

Lemma 2.1 (See [6])**.** *Let $(ad,q)=1$, where $q=q_1q_2$ with $(q_1,q_2)=1$. For χ a character modulo q_2 we have*

$$\sum_{\substack{n\leqslant x\\ (n,d)=1\\ n\equiv a \,(\mathrm{mod}\, q_1)}}\mu^2(n)\frac{\chi(n)}{\tau(n)}=\delta_\chi\frac{C(dq)}{\varphi(q_1)}\frac{x}{\sqrt{\log x}}\left\{1+O\left(\frac{(\log\log 3dq)^{3/2}}{\log x}\right)\right\}+O\left(\frac{\tau(d)qx}{(\log x)^D}\right),$$

with any $D>0$. Here $\delta_\chi=1$ if χ is the principal character and $\delta_\chi=0$ otherwise, and

$$C(r)=\frac{1}{\pi^{1/2}}\prod_p\left(1+\frac{1}{2p}\right)\left(1-\frac{1}{p}\right)^{\frac12}\prod_{p\mid r}\left(1+\frac{1}{2p}\right)^{-1}. \tag{2.1}$$

Lemma 2.2 (See [6]). *Let α_m and β_n be complex numbers bounded by one and supported only on odd integers. Then we have*

$$\sum_{|m|\leqslant M}\sum_{n\leqslant N}\alpha_m\beta_n\left(\frac{m}{n}\right)\ll(MN^{\frac{5}{6}}+M^{\frac{5}{6}}N)(\log 3MN)^{\frac{7}{6}},$$

where the implied constant is absolute.

For any integer n, we denote by $v_2(n)$ the highest power of 2 dividing n. Furthermore, for any pair of integers a, b, let

$$\sigma(a;b)=\begin{cases}1, & \text{if } a\equiv\Box \pmod{2^{v_2(b)}},\\ 0, & \text{otherwise,}\end{cases}\quad\text{and}\quad I(a,b)=\begin{cases}1, & \text{if } a\equiv\Box \pmod{b} \text{ and } b\equiv\Box \pmod{a},\\ 0, & \text{otherwise.}\end{cases}$$

Next result provides the combinatorial setup of the proof of Theorem 1.1.

Lemma 2.3. *Let $a,b\in\mathbb{N}$, set $d=\gcd(a,b)$ and write $a=da'$ and $b=db'$. Furthermore, write $d=me^2$, where $m,e\in\mathbb{N}$ are uniquely determined by the requirement that m is squarefree. Denote by $\hat{\omega}(n)$ the number of odd prime divisors of n. Then*

(1) *if $(a'b',m)\neq 1$ then $I(a,b)=0$;*

(2) *if $(a'b',m)=1$ then*

$$I(a,b)=I(ma',mb')=\frac{\sigma(mb';a')\sigma(ma';b')}{2^{\hat{\omega}(a'b')}}\sum_{\substack{a_1a_2=a'\\ b_1b_2=b'}}\mu^2(2a_1b_1)\left(\frac{ma'}{b_1}\right)\left(\frac{mb'}{a_1}\right).$$

Proof. Assume that $I(a,b)=1$. Then there exist $k,\ell\in\mathbb{Z}$ such that $da'=x^2+kdb'$ and $db'=y^2+\ell da'$. So, $me^2=d\mid\gcd(x^2,y^2)$. We deduce that, for some integer M, $me^2M=x^2$ and so $(x/e)^2=mM$. In particular, x/e is an integer divisible by m and since the same conclusion holds for y we deduce that $me\mid\gcd(x,y)$. If we write $x=meX$ and $y=meY$, then

$$a'=mX^2+kb' \quad\text{and}\quad b'=mY^2+\ell a'. \tag{2.2}$$

If $q\mid\gcd(a',m)$, then $q\mid b'$ and this is absurd. Similarly if $q\mid\gcd(b',m)$, then $q\mid a'$ which is also absurd and this implies the claim (1).

In order to prove the second claim, we observe that, multiplying by m the equations in (2.2), we deduce that $ma'=(mX)^2+kmb'$ and $mb'=(mY)^2+lma'$. Hence, if $I(a,b)=1$, then $I(ma',mb')=1$. Vice versa if $I(ma',mb')=1$ then $ma'\equiv\Box \pmod{mb'}$ and $mb'\equiv\Box \pmod{ma'}$. Multiplying these congruences by e^2 we obtain that $a\equiv\Box \pmod{b}$ and $b\equiv\Box \pmod{a}$ so that $I(a,b)=1$. To complete the proof of Lemma 2.3(2), we observe that, $ma'\equiv\Box \pmod{mb'}$ if and only if for all odd $p\mid b'$, $(\frac{ma'}{p})+1\neq 0$ and $\sigma(ma',b')=1$. Finally,

$$\begin{aligned}I(ma',mb')&=\sigma(ma',b')\sigma(mb',a')\prod_{\substack{p\mid a'b'\\ p>2}}\frac{1}{2}\left(1+\left(\frac{ma'}{p}\right)\right)\left(1+\left(\frac{mb'}{p}\right)\right)\\ &=\frac{\sigma(ma',b')\sigma(mb',a')}{2^{\hat{\omega}(a'b')}}\sum_{a_1\mid a',b_1\mid b'}\mu^2(2a_1b_1)\left(\frac{ma'}{b_1}\right)\left(\frac{mb'}{a_1}\right)\end{aligned}$$

and this concludes the proof. □

Lemma 2.4. *Let $L,K,M,q\in\mathbb{N}$ be mutually coprime, let $\chi \pmod{q}$ be a Dirichlet character and let*

$u \in (\mathbb{Z}/qM\mathbb{Z})^*$. *Denote by* ψ *alternatively* $\psi = 1$ *or* $\psi = \mu^2$. *Then, for* X *large enough,*

$$\sum_{\substack{n \leqslant X \\ (n,K)=1 \\ n \equiv u \pmod{M}}} \frac{\psi(n)\chi(n)2^{\omega((n,L))}}{2^{\omega(n)}} = \delta_\chi \frac{C(qKLM)P_\psi(qKM,L)}{\varphi(M)} \frac{X}{\sqrt{\log X}} \times \bigg(1 + O\bigg(\frac{(\log\log 3qKLM)^{\frac{5}{2}} \log\log X}{\log X}\bigg)\bigg) + O_C\bigg(\frac{\tau(LK)qMX \log\log 3L}{\log^C X}\bigg),$$

for any $C > 0$, *where* $\delta_\chi = 1$ *if* χ *is the principal character and otherwise it is zero,* $C(r)$ *is as in* (2.1) *and*

$$P_\psi(a,b) = \begin{cases} \prod_{p \mid b} \frac{p+1}{p}, & \text{if } \psi(n) = \mu(n)^2, \\ \prod_{p \mid b} \frac{2p+1}{2p-1} \prod_{p \nmid a} \Big(1 + \frac{1}{(2p+1)(p-1)}\Big), & \text{if } \psi(n) = 1. \end{cases}$$

Proof. Let S be the sum on the left-hand side of the identity in the statement. Since S does not change if we replace L by the product of its prime factors, we may assume that L is squarefree. Let us write $n = me^2$, where $m, e \in \mathbb{N}$ are uniquely determined by the property that m is squarefree. Note that $(me^2, L) = (m,L)(e,L)/(e,m,L)$. Therefore,

$$S = \sum_{\substack{e \in \mathbb{N} \\ (e,qKM)=1}} \frac{\psi(e^2)\chi(e)^2 2^{\omega((e,L))}}{2^{\omega(e)}} \sum_{\substack{m \leqslant X/e^2 \\ (m,K)=1 \\ m \equiv ue^{-2} \pmod{M}}} \frac{\mu^2(m)\chi(m)2^{\omega((m,L))+\omega((e,m))-\omega((e,m,L))}}{2^{\omega(m)}},$$

where we used the fact that $\omega((me^2, L)) = \omega((m,L)) + \omega((e,L)) - \omega((e,m,L))$ and $\omega(me^2) = \omega(m) + \omega(e) - \omega((e,m))$. Write $s = \gcd(m,L)$ and split the inner sum above according to the value of $s \mid L$ and obtain that

$$S = \sum_{\substack{e \in \mathbb{N} \\ (e,qKM)=1}} \frac{\psi(e^2)\chi(e)^2 2^{\omega((e,L))}}{2^{\omega(e)}} \sum_{s \mid L} \chi(s) \sum_{\substack{m \leqslant X/se^2 \\ (m,KL)=1 \\ m \equiv u(e^2s)^{-1} \pmod{M}}} \frac{\mu^2(m)\chi(m)2^{\omega((e,m))}}{2^{\omega(m)}}. \tag{2.3}$$

First assume that $\psi(n) = \mu(n)^2$ so that $\psi(e^2) = 0$ for $e > 1$ and

$$S = \sum_{s \mid L} \chi(s) \sum_{\substack{m \leqslant X/s \\ (m,KL)=1 \\ m \equiv us^{-1} \pmod{M}}} \frac{\mu^2(m)\chi(m)}{2^{\omega(m)}} = \sum_{\substack{s \mid L \\ s < R}} \chi(s) \sum_{\substack{m \leqslant X/s \\ (m,KL)=1 \\ m \equiv us^{-1} \pmod{M}}} \frac{\mu^2(m)\chi(m)}{2^{\omega(m)}} + O\bigg(\frac{X\tau(L)}{\log^C X}\bigg),$$

where $R = \log^C X$. Next, we apply Lemma 2.1 to the inner sum above so as to obtain that

$$S = \frac{X\delta_\chi C(qKLM)}{\varphi(M)} \sum_{\substack{s \mid L \\ s < R}} \frac{\chi(s)}{s\sqrt{\log(X/s)}} \bigg\{1 + O\bigg(\frac{(\log\log(3qKLM))^{3/2}}{(\log X)}\bigg)\bigg\} + O_C\bigg(\frac{\tau(LK)qMX}{\log^C X} \sum_{s \mid L} \frac{1}{s}\bigg).$$

Now note that $\sum_{s \mid L} \frac{1}{s} \ll \log\log 3L$ and that, since $s < R$,

$$\frac{1}{\sqrt{\log(X/s)}} = \frac{1}{\sqrt{\log X}} \bigg(1 + O\bigg(\frac{\log\log X}{\log X}\bigg)\bigg).$$

Hence, by adding back the terms with $s > R$, with a contribution bounded by the error term, we get

$$S = \frac{X\delta_\chi C(qKLM)}{\varphi(M)\sqrt{\log X}} \sum_{s \mid L} \frac{\chi(s)}{s} \bigg\{1 + O\bigg(\frac{(\log\log X)\log\log(3qKLM)^{5/2}}{\log X}\bigg)\bigg\}$$

$$+ O_C\left(\frac{\tau(LK)qMX}{\log^C X}\log\log 3L\right).$$

The result for $\psi = \mu^2$ follows by noticing:

$$\delta_\chi P_{\mu^2}(qKM, L) = \delta_\chi \sum_{s\,|\,L}\frac{\chi(s)}{s} = \begin{cases}\prod_{p\,|\,L}\frac{p+1}{p}, & \text{if } \chi \text{ is principal},\\ 0, & \text{otherwise}.\end{cases}$$

Now suppose $\psi(n) = 1$. The contribution to the sum in (2.3) of those $e \in \mathbb{N}$ such that $se^3 > R$ is

$$O\bigg(\sum_{s\,|\,L}\sum_{e>(R/s)^{1/3}}\frac{X}{se^2}\bigg) = O\bigg(\frac{X\tau(L)}{R^{1/3}}\bigg) = O_C\bigg(\frac{X\tau(L)}{\log^C X}\bigg)$$

if we choose $R = \log^{3C} X$. Therefore,

$$S = \sum_{s\,|\,L}\chi(s)\sum_{\substack{e\in\mathbb{N}, se^3\leqslant R\\ (e,qMK)=1}}\frac{\psi(e^2)\chi(e)^2 2^{\omega((e,L))}}{2^{\omega(e)}}\sum_{\substack{m\leqslant X/se^2\\ (m,KL)=1\\ m\equiv u(e^2s)^{-1} \ (\mathrm{mod}\ M)}}\frac{\mu^2(m)\chi(m)2^{\omega((e,m))}}{2^{\omega(m)}}$$
$$+ O\bigg(\frac{X\tau(L)}{\log^C X}\bigg). \tag{2.4}$$

Let

$$S_{e,s} = \sum_{\substack{m\leqslant X/se^2\\ (m,KL)=1\\ m\equiv u(e^2s)^{-1} \ (\mathrm{mod}\ M)}}\frac{\mu^2(m)\chi(m)2^{\omega((e,m))}}{2^{\omega(m)}}$$

be the most inner sum of (2.4), let $t = \gcd(e, m)$ and split $S_{e,s}$ according to the value of $t \mid e$ to obtain

$$S_{e,s} = \sum_{\substack{t\,|\,e\\ (t,L)=1}}\mu^2(t)\chi(t)\sum_{\substack{m\leqslant X/ste^2\\ (m,KLe)=1\\ m\equiv u(e^2st)^{-1} \ (\mathrm{mod}\ M)}}\frac{\mu^2(m)\chi(m)}{2^{\omega(m)}}.$$

Now apply Lemma 2.1 to the inner sum above and obtain

$$\sum_{\substack{m\leqslant X/ste^2\\ (m,KLe)=1\\ m\equiv u(e^2st)^{-1} \ (\mathrm{mod}\ M)}}\frac{\mu^2(m)\chi(m)}{2^{\omega(m)}} = S_1(s,e,t) + S_2(s,e,t),$$

where

$$S_1(s,e,t) = \delta_\chi C(qKLMe)\frac{X}{\varphi(M)ste^2\sqrt{\log X}},$$
$$S_2(s,e,t) = O\bigg(S_1(s,e,t)\log\log X\frac{(\log\log 3qKLMe)^{3/2}}{(\log X)}\bigg) + O_C\bigg(\frac{\tau(LKe)qMX}{ste^2\log^C X}\bigg). \tag{2.5}$$

Hence,

$$S = \sum_{s\,|\,L}\chi(s)\sum_{\substack{e\in\mathbb{N}, se^3\leqslant R\\ (e,qMK)=1}}\frac{\chi(e)^2 2^{\omega((e,L))}}{2^{\omega(e)}}\sum_{\substack{t\,|\,e\\ (t,L)=1}}\mu^2(t)\chi(t)(S_1(s,e,t) + S_2(s,e,t)). \tag{2.6}$$

We first deal with the main term of (2.6). We may assume that $\chi = \chi_0$ is the principal character modulo q otherwise $S_1(s,e,t) = 0$. Then

$$\sum_{\substack{t\,|\,e\\ (t,L)=1}}\mu^2(t)\chi_0(t)S_1(s,e,t) = S_1(s,e,1)\prod_{\substack{p\,|\,e\\ p\nmid L}}\bigg(1+\frac{1}{p}\bigg).$$

Next, we sum over $e \leqslant (R/s)^{1/3}$ to obtain

$$\begin{aligned}&\sum_{\substack{e\in\mathbb{N}, se^3\leqslant R\\(e,qMK)=1}} \frac{2^{\omega((e,L))}}{2^{\omega(e)}} S_1(s,e,1) \prod_{\substack{p\,|\,e\\p\nmid L}} \left(1+\frac{1}{p}\right)\\ &= S_1(s,1,1) \sum_{\substack{e\in\mathbb{N}, se^3\leqslant R\\(e,qMK)=1}} \frac{2^{\omega((e,L))}}{2^{\omega(e)}e^2} \prod_{\substack{p\,|\,e\\p\nmid L}} \left(1+\frac{1}{p}\right)\left(1+\frac{1}{2p}\right)^{-1}\\ &= S_1(s,1,1) \prod_{p\,|\,L}\left(1-\frac{1}{p^2}\right)^{-1} \prod_{p\nmid qKLM} \left(1+\frac{1}{(2p+1)(p-1)}\right) \times \left(1+O\left(\frac{s^{1/3}}{\log^C X}\right)\right),\end{aligned}$$

where the contribution of $e > (R/s)^{1/3}$ to the sum in the second equation above is $O(s^{1/3}/\log^C X)$. Finally, sum over $s \mid L$ and arrive to the main term

$$\delta_\chi \frac{C(qKLM)X}{\varphi(M)\sqrt{\log X}} \prod_{p\,|\,L}\left(\frac{p}{p-1}\right) \prod_{p\nmid qKLM}\left(1+\frac{1}{(2p+1)(p-1)}\right) \times \left(1+O\left(\frac{\tau(L)}{\log^C X}\right)\right).$$

Note that the above is zero unless $\chi = \chi_0$ and in such a case the main term as in the statement of the lemma is obtained by isolating the factor $\prod_{p\,|\,L}(1+\frac{1}{(2p+1)(p-1)})$. Next, we deal with the error term in (2.6) which is bounded by

$$\sum_{s\,|\,L} \sum_{\substack{e\in\mathbb{N}, se^3\leqslant R\\(e,qMK)=1}} \sum_{\substack{t\,|\,e\\(t,L)=1}} S_2(s,e,t).$$

We split the sum in two parts relative to the terms that form $S_2(s,e,t)$ in (2.5).

$$\begin{aligned}\text{The first part is } &\ll \frac{S_1(1,1,1)}{(\log X)} \sum_{s\,|\,L}\frac{1}{s} \sum_{e} \frac{(\log\log 3qKLMeX)^{3/2}}{e^2} \sum_{t\,|\,e}\frac{1}{t}\\ &\ll S_1(1,1,1)\frac{(\log\log 3qKLMX)^{3/2}}{(\log X)} \log\log 3L \ll S_1(1,1,1)\frac{(\log\log 3qKLMX)^{5/2}}{(\log X)},\end{aligned}$$

while

$$\text{the second part is } \ll \frac{\tau(LK)qMX}{\log^C X} \sum_{\substack{s\,|\,L\\s<R}} \frac{1}{s} \sum_e \frac{\tau(e)}{e^2} \sum_{t\,|\,e}\frac{1}{t} \ll \frac{\tau(LK)qMX\log\log X}{\log^C X}.$$

The proof of the lemma follows. □

3 Proof of Theorem 1.1

We begin by applying Lemma 2.3 to

$$\begin{aligned}\sharp\mathcal{S}(A,B) &= \sum_{a\leqslant A, b\leqslant B} I(a,b) = \sum_{e\in\mathbb{N}} \sum_{\substack{a\leqslant A/e^2\\b\leqslant B/e^2}} \mu^2(\gcd(a,b)) I(a,b)\\ &= \sum_{e,m\in\mathbb{N}} \mu^2(m) S_m(A/me^2, B/me^2) = \sum_{\substack{e\leqslant T\\m\leqslant T}} \mu^2(m) S_m(A/me^2, B/me^2) + O\left(\frac{AB}{T}\right),\end{aligned} \tag{3.1}$$

where $T \geqslant 1$ is a parameter to be chosen later and

$$S_m(A,B) = \sum_{\substack{a\leqslant A, b\leqslant B\\ \gcd(m,ab)=\gcd(a,b)=1}} I(ma,mb). \tag{3.2}$$

The proof is a consequence of the following lemma.

Lemma 3.1. *For any fixed $\delta > 0$ and uniformly for $A, B \geqslant \exp((\log AB)^\delta)$ and m, we have the estimate*

$$S_1(A,B) = \frac{AB}{\pi\sqrt{\log A}\sqrt{\log B}}\bigg(\frac{13}{8} + O\bigg(\bigg(\frac{1}{\log A} + \frac{1}{\log B}\bigg)(\log\log AB)^4\bigg)\bigg)$$

and if $m \geqslant 2$,

$$S_m(A,B) = \frac{AB}{\pi\sqrt{\log A}\sqrt{\log B}}\bigg(\xi_m \times \frac{\varphi(m)}{m} + O\bigg((\log\log AB)^4\bigg(\frac{1}{\log A} + \frac{1}{\log B}\bigg) + \frac{m}{\log^C AB}\bigg)\bigg),$$

where $\xi_m = 19/16$ if m is odd and $\xi_m = 1$ if m is even.

Before proving Lemma 3.1, let us deduce from the statement of Theorem 1.1. Replacing the statement of Lemma 3.1 in (3.1) we obtain that $\sharp\mathcal{S}(A,B)$ equals

$$\begin{aligned}&\sum_{e\leqslant T}\frac{AB}{\pi e^4}\bigg(\frac{1}{\sqrt{\log\frac{A}{e^2}}\sqrt{\log\frac{B}{e^2}}}\bigg(\frac{7}{16} + O\bigg(\bigg(\frac{1}{\log A/e^2} + \frac{1}{\log B/e^2}\bigg)(\log\log AB/e^4)^4\bigg)\bigg)\\&\quad + \sum_{m\leqslant T}\frac{\mu^2(m)}{\sqrt{\log\frac{A}{me^2}}\sqrt{\log\frac{B}{me^2}}}\bigg(\frac{\varphi(m)\xi_m}{m^3} + O\bigg(\frac{(\log\log AB)^4}{m^2}\bigg(\frac{1}{\log\frac{A}{me^2}} + \frac{1}{\log\frac{B}{me^2}}\bigg) + \frac{1}{m\log^C\frac{AB}{me^2}}\bigg)\bigg)\bigg)\\&\quad + O\bigg(\frac{AB}{T}\bigg).\end{aligned}$$

We choose $T = (\log AB)^2$, C large enough and we use the fact that for $e, m \leqslant T$,

$$\frac{1}{\sqrt{\log\frac{A}{me^2}}\sqrt{\log\frac{B}{me^2}}} = \frac{1}{\sqrt{\log A}\sqrt{\log B}}\bigg(1 + O\bigg(\log T\bigg(\frac{1}{\log A} + \frac{1}{\log B}\bigg)\bigg)\bigg).$$

Hence, we deduce that the above sum equals

$$\frac{AB}{\pi\sqrt{\log A}\sqrt{\log B}}\sum_{e\leqslant T}\frac{1}{e^4}\bigg(\frac{7}{16} + \sum_{m\leqslant T}\frac{\mu^2(m)\varphi(m)\xi_m}{m^3} + O\bigg((\log\log AB)^4\bigg(\frac{1}{\log A} + \frac{1}{\log B}\bigg)\bigg)\bigg)$$

and the result follows by noticing that

$$\sum_{e\leqslant T}\frac{1}{e^4} = \frac{\pi^3}{90} + O\bigg(\frac{1}{T^3}\bigg) \quad\text{and}\quad \sum_{m\leqslant T}\frac{\mu^2(m)\varphi(m)\xi_m}{m^3} = \frac{21}{16}\prod_{\substack{p\geqslant 3\\ p\ \text{prime}}}\bigg(1 + \frac{p-1}{p^3}\bigg) + O\bigg(\frac{1}{T}\bigg).$$

Proof of Lemma 3.1. We apply Lemma 2.3 and the quadratic reciprocity to (3.2) to obtain that

$$\begin{aligned}S_m(A,B) &= \sum_{\substack{a_1a_2\leqslant A\\ b_1b_2\leqslant B\\ \gcd(m,ab)=\gcd(a,b)=1}}\sigma(mb;a)\sigma(ma;b)\frac{\mu^2(2a_1b_1)}{2^{\hat{\omega}(ab)}}\bigg(\frac{ma}{b_1}\bigg)\bigg(\frac{mb}{a_1}\bigg)\\&= \sum_{\substack{a_1a_2\leqslant A\\ b_1b_2\leqslant B\\ \gcd(m,ab)=\gcd(a,b)=1}}J(a_1,a_2,b_1,b_2,m),\end{aligned}$$

where

$$J(a_1,a_2,b_1,b_2,m) = (-1)^{\frac{a_1-1}{2}\frac{b_1-1}{2}}\sigma(mb;a_2)\sigma(ma;b_2)\frac{\mu^2(2a_1b_1)}{2^{\hat{\omega}(ab)}}\bigg(\frac{ma_2}{b_1}\bigg)\bigg(\frac{mb_2}{a_1}\bigg),$$

and the sum is over 4-tuples a_1, a_2, b_1 and b_2 with $a_1a_2 = a$ and $b_1b_2 = b$. Observe also that we used the condition $2 \nmid a_1b_1$ to deduce that $\sigma(mb;a) = \sigma(mb;a_2)$ and $\sigma(ma;b) = \sigma(ma;b_2)$. We follow the steps in [6, p. 103]. We fix $V = (\log AB)^{23}$, and split the sum into eight different sums depending on the size of the variables. First we set

$$\mathcal{S}_1 = \{(a_1,a_2,b_1,b_2) \in \mathbb{N}^4 : a_1a_2 = a, b_1b_2 = b,\ a_1 \leqslant V, a_2 \leqslant V, b \leqslant B\},$$

and we obtain

$$\sum_{\substack{\mathcal{S}_1\\ \gcd(a,b)=1\\ \gcd(ab,m)=1}} J(a_1,a_2,b_1,b_2,m) \leqslant \sum_{\substack{a_1\leqslant V\\ a_2\leqslant V}} \sum_{b_1b_2\leqslant B} \frac{\mu^2(b_1)}{2^{\omega(b_1b_2)}} \leqslant BV^2, \tag{3.3}$$

since

$$\sum_{b_1b_2\leqslant B} \frac{\mu^2(b_1)}{2^{\omega(b_1b_2)}} = \sum_{k\leqslant B} \frac{1}{2^{\omega(k)}} \sum_{b_1b_2=k} \mu^2(b_1) = \sum_{k\leqslant B} \frac{2^{\omega(k)}}{2^{\omega(k)}} = [B].$$

The symmetric argument leads to the estimate

$$\sum_{\substack{\mathcal{S}_2\\ \gcd(a,b)=1\\ \gcd(ab,m)=1}} J(a_1,a_2,b_1,b_2,m) \leqslant AV^2, \tag{3.4}$$

where

$$\mathcal{S}_2 = \{(a_1,a_2,b_1,b_2)\in\mathbb{N}^4 : a_1a_2=a, b_1b_2=b,\ b_1\leqslant V, b_2\leqslant V, a\leqslant A\}.$$

Next, we consider

$$\mathcal{S}_3 = \{(a_1,a_2,b_1,b_2)\in\mathbb{N}^4 : a_1a_2=a, b_1b_2=b,\ a_1\leqslant V, b_2\leqslant V, a\leqslant A, b\leqslant B\}$$

and the sum

$$\begin{aligned}\sum_{\substack{\mathcal{S}_3\\ \gcd(a,b)=1\\ \gcd(ab,m)=1}} J(a_1,a_2,b_1,b_2,m) &= \sum_{a_1\leqslant V}\sum_{b_2\leqslant V}\sum_{k\geqslant 0} \sideset{}{^{\text{odd}}}\sum_{\substack{a_2\leqslant \frac{A}{a_1 2^k}\\ b_1\leqslant \frac{B}{b_2}}} J(a_1,2^k a_2,b_1,b_2,m)\\ &\leqslant \sum_{a_1\leqslant V}\sum_{b_2\leqslant V}\sum_{k\geqslant 0} \left| \sideset{}{^{\text{odd}}}\sum_{\substack{a_2\leqslant \frac{A}{a_1 2^k}\\ b_1\leqslant \frac{B}{b_2}}} \alpha_{a_2}\beta_{b_1}\left(\frac{a_2}{b_1}\right)\right|,\end{aligned}$$

where

$$\beta_{b_1} = (-1)^{\frac{a_1-1}{2}\frac{b_1-1}{2}}\left(\frac{m2^k}{b_1}\right)\frac{\mu^2(2a_1b_1)}{2^{\hat{\omega}(b)}}\sigma(mb;2^k), \quad \alpha_{a_2} = \frac{\sigma(m2^k a_1a_2;b_2)}{2^{\hat{\omega}(a_1a_2)}},$$

and the sum over a_2 is restricted to odd integers. We apply Lemma 2.2 to the most inner sum above and obtain

$$\begin{aligned}\sum_{\substack{\mathcal{S}_3\\ \gcd(a,b)=1\\ \gcd(ab,m)=1}} J(a_1,a_2,b_1,b_2,m) &\leqslant \sum_{a_1\leqslant V}\sum_{b_2\leqslant V}\sum_{k\geqslant 0}\left(\frac{A}{2^k a_1}\frac{B}{b_2}((2^k a_1)^{1/6}A^{-1/6} + b_2^{1/6}B^{-1/6})\right)(\log AB)^{7/6}\\ &\ll ABV^{1/6}(A^{-1/6}+B^{-1/6})(\log AB)^{7/6}\log V.\end{aligned}$$

Analogously, if

$$\mathcal{S}_4 = \{(a_1,a_2,b_1,b_2)\in\mathbb{N}^4 : a_1a_2=a, b_1b_2=b,\ a_2\leqslant V, b_1\leqslant V, a\leqslant A, b\leqslant B\},$$

then, by Lemma 2.2, we obtain

$$\sum_{\substack{\mathcal{S}_4\\ \gcd(a,b)=1\\ \gcd(ab,m)=1}} J(a_1,a_2,b_1,b_2,m) \ll ABV^{1/6}(A^{-1/6}+B^{-1/6})(\log AB)^{7/6}\log V.$$

Next, we consider

$$\mathcal{S}_5 = \{(a_1,a_2,b_1,b_2)\in\mathbb{N}^4 : a_1a_2=a, b_1b_2=b,\ a_1> V, b_2> V, a\leqslant A, b\leqslant B\},$$

$$\mathcal{S}_6 = \{(a_1,a_2,b_1,b_2) \in \mathbb{N}^4 : a_1a_2 = a, b_1b_2 = b,\ b_1 > V, a_2 > V, a \leqslant A, b \leqslant B\}.$$

In addition, we apply Lemma 2.2, in a similar fashion as above and obtain

$$\begin{aligned}
&\sum_{\substack{\mathcal{S}_5 \\ \gcd(a,b)=1 \\ \gcd(ab,m)=1}} J(a_1,a_2,b_1,b_2,m) \\
&\leqslant \sum_{\substack{a_2 \leqslant A/V \\ b_1 \leqslant B/V}} \Bigg| \sum_{\substack{a_1 \leqslant A/a_2 \\ b_2 \leqslant B/b_1}} (-1)^{\frac{a_1-1}{2}\frac{b_1-1}{2}} \sigma(mb;a_2)\sigma(ma;b_2) \frac{\mu^2(2a_1b_1)}{2^{\hat{\omega}(ab)}} \left(\frac{mb_2}{a_1}\right) \Bigg| \\
&\ll \sum_{\substack{a_2 \leqslant A/V \\ b_1 \leqslant B/V}} \left(\frac{A}{a_2}\left(\frac{B}{b_1}\right)^{5/6} + \frac{B}{b_1}\left(\frac{A}{a_2}\right)^{5/6} \right) (\log 3AB)^{7/6} \ll \frac{AB}{V^{1/6}} (\log 3AB)^{13/6}.
\end{aligned}$$

Similarly,

$$\sum_{\substack{\mathcal{S}_6 \\ \gcd(a,b)=1 \\ \gcd(ab,m)=1}} J(a_1,a_2,b_1,b_2,m) \ll \frac{AB}{V^{1/6}} (\log 3AB)^{13/6}.$$

We observe that if $\mathcal{S} = \{(a_1,a_2,b_1,b_2) \in \mathbb{N}^4 : a_1a_2 \leqslant A, b_1b_2 \leqslant B\}$, then $\mathcal{S} \setminus (\mathcal{S}_1 \cup \mathcal{S}_2 \cup \mathcal{S}_3 \cup \mathcal{S}_4 \cup \mathcal{S}_5 \cup \mathcal{S}_6) = \mathcal{S}_7 \sqcup \mathcal{S}_8$, where the last union is disjoint and $\mathcal{S}_7 = \{(a_1,a_2,b_1,b_2) \in \mathcal{S} : a_2 \leqslant V, b_2 \leqslant V, a_1 > V, b_1 > V\}$ and $\mathcal{S}_8 = \{(a_1,a_2,b_1,b_2) \in \mathcal{S} : a_1 \leqslant V, b_1 \leqslant V, a_2 > V, b_2 > V\}$. Both the above terms will contribute to the main term.

Let us consider the sum

$$\sum_{\substack{\mathcal{S}_7 \\ \gcd(a,b)=1 \\ \gcd(ab,m)=1}} J(a_1,a_2,b_1,b_2,m) = \sum_{\substack{a_2 \leqslant V, b_2 \leqslant V \\ (a_2,b_2)=1 \\ (a_2b_2,m)=1}} \frac{1}{2^{\hat{\omega}(a_2b_2)}} S_{a_2,b_2} + O(V^2 \log V(A+B)), \tag{3.5}$$

where S_{a_2,b_2} equals

$$\sum_{\substack{a_1 \leqslant A/a_2 \\ b_1 \leqslant B/b_2 \\ (b_1,a_1)=1}} (-1)^{\frac{a_1-1}{2}\frac{b_1-1}{2}} \sigma(ma;b_2)\sigma(mb;a_2) \frac{\mu^2(2a_1)2^{\omega((a_1,a_2))}}{2^{\omega(a_1)}} \frac{\mu^2(2b_1)2^{\omega(b_1,b_2)}}{2^{\omega(b_1)}} \left(\frac{mb_2}{a_1}\right)\left(\frac{ma_2}{b_1}\right).$$

We use the Möbius function to remove the condition $(a_1,b_1) = 1$. This introduces an extra sum over divisors d of (a_1,b_1). Then, we interchange the order of summation and estimate trivially the sum over pairs a_1 and b_1 with common divisor $d > \Delta$ where $\Delta = (\log AB)^4$. We deduce that

$$\begin{aligned}
S_{a_2,b_2} = &\sum_{\substack{d \leqslant \Delta \\ (d,2ma_2b_2)=1}} \frac{\mu(d)}{4^{\omega(d)}} \left(\frac{a_2b_2}{d}\right) \sum_{\substack{a_1 \leqslant A/da_2 \\ (a_1,d)=1}} \sigma(mda_1a_2;b_2) \frac{\mu^2(2a_1)2^{\omega((a_1,a_2))}}{2^{\omega(a_1)}} \left(\frac{mb_2}{a_1}\right)(-1)^{\frac{da_1-1}{2}\frac{d-1}{2}} \\
&\times \sum_{\substack{b_1 \leqslant B/db_2 \\ (b_1,d)=1}} (-1)^{\frac{da_1-1}{2}\frac{b_1-1}{2}} \sigma(mdb_1b_2;a_2) \frac{\mu^2(2b_1)2^{\omega(b_1,b_2)}}{2^{\omega(b_1)}} \left(\frac{ma_2}{b_1}\right) + O\left(\frac{AB}{a_2b_2\Delta}\right).
\end{aligned} \tag{3.6}$$

Since $(a_2,b_2) = 1$, we may assume that $2 \nmid a_2$, the symmetric case $2 \nmid b_2$ being analogue. Then, we have $\sigma(mdb_1b_2;a_2) = 1$. We can apply Lemma 2.4 to the most inner sum with the following choice of parameters:

$$\chi = \left(\frac{(-1)^{\frac{da_1-1}{2}} 4ma_2}{\cdot}\right), \quad K = d, \quad M = 1, \quad u = 1 \quad L = \prod_{\ell | b_2, \ell \nmid 2} \ell \quad \text{and} \quad X = B/db_2$$

in such a way that $q=4ma_2$. Furthermore, q,K,M and L are mutually coprime and we obtain as the main term for $2\nmid a_2$,

$$\delta_\chi\times C(4ma_2b_2d)\prod_{\substack{p\mid b_2\\ p\nmid 2md}}\frac{p+1}{p}\times\frac{B/db_2}{\sqrt{\log(B/db_2)}}\bigg(1+O\bigg(\frac{(\log\log 3mda_2b_2)^{5/2}\log\log B/db_2}{\log B/db_2}\bigg)\bigg) \tag{3.7}$$

and as the error term

$$O_C\bigg(\frac{\tau(b_2d)ma_2\log\log(3b_2)B/db_2}{\log^C B/db_2}\bigg). \tag{3.8}$$

Note that the main term is zero unless $m=1$, a_2 is a perfect square and $d\equiv a_1 \pmod 4$. In such a circumstance we have that $\sigma(mda_1a_2;b_2)=\sigma(da_1;b_2)$ and after summing over $a_1\leqslant A/da_2$ and $d\leqslant\Delta$ the error terms in (3.8), one obtains a contribution

$$O_C\bigg(\frac{AB}{\log^C(B/(\Delta b_2))}\frac{\tau(b_2)}{b_2}\log\log b_2\bigg). \tag{3.9}$$

If either a_2 is not a perfect square or if $m\neq 1$, then for $2\nmid a_2$, S_{a_2,b_2} consists only of the error term in (3.8) which, after the summation over $a_2\leqslant V$ and $b_2\leqslant V$ provides a contribution which is

$$O_C\bigg(ABm\frac{V\log^3 V}{\log^C B/\Delta}\bigg)=O_C\bigg(\frac{AB}{\log^{C'}AB}\bigg).$$

For a_2 a prefect square and $m=1$, S_{a_2,b_2}, for $2\nmid a_2$, equals

$$\frac{B}{b_2\sqrt{\log B}}\sum_{\substack{d\leqslant\Delta\\ (d,2a_2b_2)=1}}\frac{\mu(d)}{d4^{\omega(d)}}\bigg(\frac{a_2b_2}{d}\bigg)C(4da_2b_2)\prod_{\substack{p\mid b_2\\ p\nmid 2d}}\frac{p+1}{p}\sum_{\substack{a_1\leqslant A/da_2\\ (a_1,d)=1\\ da_1\equiv 1 \ (\mathrm{mod}\ 4)}}\sigma(da_1;b_2)\frac{\mu^2(a_1)2^{\omega((a_1,a_2))}}{2^{\omega(a_1)}}\bigg(\frac{b_2}{a_1}\bigg)+E_{a_2,b_2}. \tag{3.10}$$

The error term E_{a_2,b_2} is made out of three terms: one in (3.9), one in (3.6) and one obtained summing over $a_1\leqslant A/da_2$ and $d\leqslant\Delta$ the O-terms in (3.7). The latter term is

$$O\bigg(\frac{AB\log\log B\log\log\log(AB)}{\log^{3/2}B}\frac{(\log\log a_2)^{5/2}}{a_2}\frac{\log b_2(\log\log b_2)^{5/2}}{b_2}\bigg), \tag{3.11}$$

where the following bounds have been applied: $C(4da_2b_2)\ll 1$ and $\prod_{p\mid 4b_2}(1+\frac{1}{p})\ll\log b_2$ and the error term produced by the approximation $(\log B/db_2)^{-1/2}=(\log B)^{-1/2}(1+O(\frac{\log db_2}{\log B}))$ is absorbed in (3.11). Hence,

$$\sum_{a_2\leqslant V,b_2\leqslant V}E_{a_2,b_2}=O\bigg(\frac{AB(\log\log AB)^5}{\log^{3/2}B}\bigg). \tag{3.12}$$

Now let us consider the main term in (3.5) which, by (3.10), will be

$$\frac{B}{\sqrt{\log B}}\sum_{\substack{(a_2,2b_2)=1\\ a_2=\square\\ a_2\leqslant V,b_2\leqslant V}}\frac{1}{b_22^{\hat\omega(a_2b_2)}}\sum_{\substack{(d,2a_2b_2)=1\\ d\leqslant\Delta}}\frac{\mu(d)}{d4^{\omega(d)}}\bigg(\frac{a_2b_2}{d}\bigg)C(4da_2b_2)\prod_{\substack{p\mid b_2\\ p\neq 2}}\frac{p+1}{p}S_{a_2,b_2,d}, \tag{3.13}$$

where

$$S_{a_2,b_2,d}=\sum_{\substack{a_1\leqslant\frac{A}{da_2}\\ (a_1,d)=1\\ a_1\equiv d \ (\mathrm{mod}\ 4)}}\frac{\mu^2(a_1)2^{\omega((a_1,a_2))}}{2^{\omega(a_1)}}\sigma(da_1;b_2)\bigg(\frac{b_2}{a_1}\bigg). \tag{3.14}$$

We now split the sum in (3.14) depending on b_2.

Case 1. $2 \nmid b_2$. In this case $\sigma(da_1; b_2) = 1$. Moreover, since $a_1 \equiv d \pmod 4$, we have $(\frac{b_2}{a_1}) = (-1)^{\frac{a_1-1}{2}\frac{b_2-1}{2}}(\frac{a_1}{b_2}) = (-1)^{\frac{d-1}{2}\frac{b_2-1}{2}}(\frac{a_1}{b_2})$. Hence,

$$S_{a_2,b_2,d} = (-1)^{\frac{d-1}{2}\frac{b_2-1}{2}} \sum_{\substack{a_1 \leqslant \frac{A}{da_2} \\ (a_1,d)=1 \\ a_1 \equiv d \ (\mathrm{mod}\ 4)}} \frac{\mu^2(d) 2^{\omega((a_1,a_2))}}{2^{\omega(a_1)}} \left(\frac{a_1}{b_2}\right), \tag{3.15}$$

and we can apply Lemma 2.4 with $q = b_2$, $\chi = (\frac{\cdot}{b_2})$, $K = d$, $L = \prod_{p \mid a_2} p$, and $M = 4$, $u = d$, to get a contribution to the main term, only when b_2 is a perfect square, which is given by

$$T_{a_2,b_2,d} = \frac{C(4a_2b_2d)}{2da_2} \prod_{p \mid a_2} \frac{p+1}{p} \frac{A}{\sqrt{\log A}}. \tag{3.16}$$

Plugging this into (3.13), and neglecting the terms for $d > \Delta$, $a_2 > V$, or $b_2 > V$, we obtain

$$\begin{aligned}
&\frac{AB}{\sqrt{\log A}\sqrt{\log B}} \sum_{\substack{(a_2,b_2)=1 \\ a_2=b_2=\square \\ (a_2b_2,2)=1}} \frac{1}{2a_2b_2 2^{\omega(a_2b_2)}} \sum_{\substack{(d,2)=1 \\ d \leqslant \Delta}} \frac{\mu(d)}{d^2 4^{\omega(d)}} \left(\frac{a_2b_2}{d}\right) C^2(4da_2b_2) \prod_{p \mid b_2} \frac{p+1}{p} \prod_{p \mid a_2} \frac{p+1}{p} \\
&= \frac{AB}{\sqrt{\log A}\sqrt{\log B}} \frac{16\mathfrak{C}}{25} \sum_{\substack{(a_2,b_2)=1 \\ a_2=b_2=\square \\ (a_2b_2,2)=1}} \frac{1}{2a_2b_2 2^{\omega(a_2b_2)}} \prod_{p \mid a_2b_2} \frac{4p(p+1)}{(2p+1)^2} \sum_{\substack{(d,2)=1 \\ d \leqslant \Delta}} \frac{\mu(d)}{d^2 4^{\omega(d)}} \left(\frac{a_2b_2}{d}\right) \prod_{p \mid d} \frac{4p^2}{(2p+1)^2} \\
&= \frac{AB}{\sqrt{\log A}\sqrt{\log B}} \frac{16\mathfrak{C}}{25} \sum_{\substack{(a_2,b_2)=1 \\ a_2=b_2=\square \\ (a_2b_2,2)=1}} \frac{1}{2a_2b_2 2^{\omega(a_2b_2)}} \prod_{p \mid a_2b_2} \frac{4p(p+1)}{(2p+1)^2} \prod_{p \nmid 2a_2b_2} \left(1 - \frac{1}{(2p+1)^2}\right) \\
&= \frac{AB}{\sqrt{\log A}\sqrt{\log B}} \frac{\mathfrak{C}}{3} \prod_{p} \frac{4p(p+1)}{(2p+1)^2} \sum_{\substack{(a_2,b_2)=1 \\ a_2=b_2=\square \\ (a_2b_2,2)=1}} \frac{1}{a_2b_2 2^{\omega(a_2b_2)}},
\end{aligned}$$

where we have used the notation $\mathfrak{C} = \frac{1}{\pi} \prod_p \frac{(2p+1)^2(p-1)}{4p^3}$. Splitting the last sum into the two parameters, a_2 and b_2, we get

$$\begin{aligned}
\sum_{\substack{(a2,b_2)=1 \\ a_2=b_2=\square \\ (a_2b_2,2)=1}} \frac{1}{2a_2b_2 2^{\omega(a_2b_2)}} &= \sum_{(a_2,2)=1} \frac{1}{a_2^2 2^{\omega(a_2)}} \sum_{(b_2,2a_2)=1} \frac{1}{b_2^2 2^{\omega(b_2)}} \\
&= \sum_{(a_2,2)=1} \frac{1}{a_2^2 2^{\omega(a_2)}} \prod_{p \nmid 2a_2} \left(1 + \frac{1}{2(p^2-1)}\right) \\
&= \prod_{p \neq 2} \left(1 + \frac{1}{2(p^2-1)}\right) \sum_{(a_2,2)=1} \frac{1}{a_2^2 2^{\omega(a_2)}} \prod_{p \mid a_2} \left(1 + \frac{1}{2(p^2-1)}\right)^{-1} \\
&= \prod_{p \neq 2} \left(1 + \frac{1}{2(p^2-1)}\right) \prod_{p \neq 2} \left(1 + \frac{1}{2(p^2-1)}\right) = \prod_{p \neq 2} \frac{p^2}{(p^2-1)}.
\end{aligned}$$

Now, using the definition of $\mathfrak{C}$, we obtain that the contribution to (3.13) with b_2 odd is

$$\frac{AB}{\sqrt{\log A}\sqrt{\log B}} \frac{1}{4\pi}. \tag{3.17}$$

Case 2. $2 \,\|\, b_2$. Let $b_2 = 2\hat{b}_2$, with $\hat{b}_2$ an odd number. In this case $\sigma(da_1; b_2) = 1$. Now $(\frac{b_2}{a_1}) = (\frac{2}{a_1})(\frac{\hat{b}_2}{a_1}) = (\frac{2}{a_1})(-1)^{\frac{a_1-1}{2}\frac{\hat{b}_2-1}{2}}(\frac{a_1}{\hat{b}_2}) = (\frac{2}{a_1})(-1)^{\frac{d-1}{2}\frac{\hat{b}_2-1}{2}}(\frac{a_1}{\hat{b}_2})$. Hence, splitting the sum in terms on the congruence of $a_1 \pmod 8$, and noting that $(\frac{2}{a_1})$ has opposite sign when $a_1 \equiv d \pmod 8$ or $a_1 \equiv d+4 \pmod 8$, we get

$$S_{a_2,b_2,d} = \pm(-1)^{\frac{d-1}{2}\frac{b_2-1}{2}}\Bigg(\sum_{\substack{a_1\leqslant \frac{A}{da_2}\\ (a_1,d)=1\\ a_1\equiv d \ (\mathrm{mod}\ 8)}} \frac{\mu^2(d)2^{\omega((a_1,a_2))}}{2^{\omega(a_1)}}\left(\frac{a_1}{\hat{b}_2}\right) - \sum_{\substack{a_1\leqslant \frac{A}{da_2}\\ (a_1,d)=1\\ a_1\equiv d+4 \ (\mathrm{mod}\ 8)}} \frac{\mu^2(d)2^{\omega((a_1,a_2))}}{2^{\omega(a_1)}}\left(\frac{a_1}{\hat{b}_2}\right)\Bigg). \tag{3.18}$$

Applying Lemma 2.4, and noting that it is independent of the value of u in the congruence, we see that the main terms in (3.18) cancel and, in particular, there is no contribution to the main term in (3.13) when $2\,||\,b_2$.

Case 3. $4\,||\,b_2$. Let us call $b_2 = 4\hat{b}_2$. Again in this case $\sigma(da_1; b_2) = 1$. Hence, we change b_2 by $4\hat{b}_2$ everywhere in (3.13) to obtain a contribution 1/4 of the quantity in (3.17). In particular, the contribution to (3.13) when $4\,||\,b_2$ is

$$\frac{AB}{\sqrt{\log A}\sqrt{\log B}}\frac{1}{16\pi}. \tag{3.19}$$

Case 4. $8 \mid b_2$. Let $b_2 = 2^e\hat{b}_2$, where $2 \nmid \hat{b}_2$ and $e \geqslant 3$. In this case $\sigma(da_1; b_2) = 1$ only if $a_1 \equiv d \pmod 8$ and is zero otherwise. Moreover, $(\frac{b_2}{a_1}) = (\frac{2}{a_1})^e(\frac{\hat{b}_2}{a_1})$. Hence,

$$S_{a_2,b_2,d} = \sum_{\substack{a_1\leqslant \frac{A}{da_2}\\ (a_1,d)=1\\ a_1\equiv d \ (\mathrm{mod}\ 8)}} \frac{\mu^2(a_1)2^{\omega((a_1,a_2))}}{2^{\omega(a_1)}}\left(\frac{b_2}{a_1}\right) = \left(\frac{2}{d}\right)^e \sum_{\substack{a_1\leqslant \frac{A}{da_2}\\ (a_1,d)=1\\ a_1\equiv d \ (\mathrm{mod}\ 8)}} \frac{\mu^2(a_1)2^{\omega((a_1,a_2))}}{2^{\omega(a_1)}}\left(\frac{\hat{b}_2}{a_1}\right). \tag{3.20}$$

The only term that affects M, the main term above is 1/2 of that of (3.16). Hence, substituting b_2 by $2^e\hat{b}_2$ everywhere in (3.13) (note that $(\frac{2}{d})^e$ disappears with $(\frac{b_2}{d})$), we get that the contribution to (3.13) from this case is

$$\frac{AB}{\sqrt{\log A}\sqrt{\log B}}\frac{1}{8\pi}\sum_{e\geqslant 3}\frac{1}{2^e} = \frac{AB}{\sqrt{\log A}\sqrt{\log B}}\frac{1}{32\pi}. \tag{3.21}$$

Adding all the terms together, we get a main term when a_2 is odd of

$$\frac{AB}{\sqrt{\log A}\sqrt{\log B}}\frac{11}{32\pi}. \tag{3.22}$$

In order to get the total contribution to (3.5) we will add the case in which b_2 is odd, and subtract when both are odd numbers for not to count. In particular, we get

$$\sum_{\substack{\mathcal{S}_7\\ \gcd(a,b)=1\\ \gcd(ab,m)=1}} J(a_1,a_2,b_1,b_2,m) = \frac{AB}{\sqrt{\log A}\sqrt{\log B}}\frac{7}{16\pi} + E_7. \tag{3.23}$$

In order to bound E_7, we note that the bound (3.11) is still valid when applying Lemma 2.4 in (3.15) and, hence, we have a total contribution as in (3.12). Moreover, when considering $a_2 > V$, $b_2 > V$ or $d\Delta$, we note that when a_2 and b_2 are perfect squares, the sums in those three variables are convergent and, in particular, we will get an error term bounded by $O(\frac{AB}{(\log AB)^{9/2}})$, which is absorbed by (3.12).

At last, let us consider the sum

$$\sum_{\substack{\mathcal{S}_8\\ \gcd(a,b)=1\\ \gcd(ab,m)=1}} J(a_1,a_2,b_1,b_2,m) = \sum_{\substack{a_1\leqslant V, b_1\leqslant V\\ (a_1,b_1)=1\\ (a_1b_1,m)=1}} (-1)^{\frac{a_1-1}{2}\frac{b_1-1}{2}}\frac{\mu^2(2a_1b_1)}{2^{\omega(a_1b_1)}}\left(\frac{m}{a_1b_1}\right)T_{a_1,b_1}(m) + O(V^2\log V(A+B)), \tag{3.24}$$

where

$$T_{a_1,b_1}(m)=T_{a_1,b_1}(A,B,m)=\sum_{\substack{a_2\leqslant\frac{A}{a_1},b_2\leqslant\frac{B}{b_1}\\(a_2b_2,m)=1,(b_2,a_2)=1}}\frac{2^{\omega((a_1,a_2))}}{2^{\hat{\omega}(a_2)}}\frac{2^{\omega((b_1,b_2))}}{2^{\hat{\omega}(b_2)}}\sigma(mb;a_2)\sigma(ma;b_2)\left(\frac{a_2}{b_1}\right)\left(\frac{b_2}{a_1}\right).$$

First we consider the sum $T^{\alpha}_{a_1,b_1}(A,B,m)$ restricted to the pairs a_2 and b_2 such that $2^{\alpha}\|a_2$ so that b_2 is odd, $\sigma(ma;b_2)=1$ and $\sigma(mb;a_2)=\sigma(mb;2^{\alpha})=1$ if and only if $mb\equiv 1\ (\mathrm{mod}\ 2^{\min\{\alpha,3\}})$. Also note that if $\alpha>0$ then m is odd. Furthermore,

$$T_{a_1,b_1}(A,B,m)=T^0_{a_1,b_1}(A,B,m)+\sum_{\alpha\geqslant 1}(T^{\alpha}_{a_1,b_1}(A,B,m)+T^{\alpha}_{b_1,a_1}(B,A,m)).$$

We introduce a Möbius function to split the condition $(a_22^{-\alpha},b_2)=1$. Hence $T^{\alpha}_{a_1,b_1}(A,B,m)$ equals

$$\sum_{\substack{d\leqslant\Delta\\(d,2m)=1}}\frac{\mu(d)}{4^{\omega(d)}}2^{\omega((d,a_1b_1))}\left(\frac{d}{a_1b_1}\right)\left(\frac{2^{\alpha}}{b_1}\right)\sum_{\substack{a_2\leqslant\frac{A}{2^{\alpha}da_1}\\(a_2,m)=1\\a_2\equiv 1\ (\mathrm{mod}\ 2)}}\frac{2^{\omega(da_1,a_2)}}{2^{\omega(a_2)}}\left(\frac{a_2}{b_1}\right)\sum_{\substack{b_2\leqslant\frac{B}{db_1}\\(b_2,2m)=1\\b_2\equiv dmb_1\ (\mathrm{mod}\ 2^{\min\{\alpha,3\}})}}\frac{2^{\omega(db_1,b_2)}}{2^{\omega(b_2)}}\left(\frac{b_2}{a_1}\right).$$

Apply Lemma 2.4 to both the inner sum above. For the sum over b_2 make the following choice of parameters:

$$\chi=\left(\frac{\cdot}{a_1}\right),\quad M=\max\{2,2^{\min\{\alpha,3\}}\},\quad u=\frac{dmb_1}{(2,m)}\ (\mathrm{mod}\ M),$$
$$K=\frac{m}{(m,2)},\quad L=\frac{db_1}{(a_1,d)}\quad\text{and}\quad X=\frac{B}{db_1}.$$

For the sum over a_2, make the analogous choice. In all cases, q,K,M and L are mutually coprime and we obtain that $T^{\alpha}_{a_1,b_1}(m)$ equals

$$\frac{AB}{2^{\alpha+\kappa}a_1b_1\sqrt{\log A}\sqrt{\log B}}\sum_{\substack{d\in\mathbb{N}\\(d,2m)=1}}\Bigg(\frac{\mu(d)2^{\omega((d,a_1b_1))}}{4^{\omega(d)}d^2}\left(\frac{d}{a_1b_1}\right)\left(\frac{2^{\alpha}}{b_1}\right)\delta_{(\frac{\cdot}{a_1})}\delta_{(\frac{\cdot}{b_1})}$$
$$\times C(2a_1b_1md)^2P_1\left(2a_1m,\frac{db_1}{(d,a_1)}\right)P_1\left(2b_1m,\frac{da_1}{(d,b_1)}\right)\Bigg)+E'''_{a_1,b_1},\tag{3.25}$$

where $\kappa=0$ if $\alpha\leqslant 1$, $\kappa=1$ if $\alpha=2$ and $\kappa=2$ if $\alpha\geqslant 3$ and

$$\sum_{\alpha\geqslant 0}\sum_{a_1\leqslant V,b_1\leqslant V}|E'''_{a_1,b_1}(m)|=O\left(\frac{AB(\log\log mAB)^4}{\sqrt{\log A}\sqrt{\log B}}\left(\frac{1}{\log A}+\frac{1}{\log B}+\frac{m}{\log^C AB}\right)\right).$$

The main term in (3.25) is non zero if and only if $a_1=b_1=1$ and

$$T^{\alpha}_{1,1}(m)=\frac{AB}{2^{\alpha+\kappa}\sqrt{\log A}\sqrt{\log B}}\sum_{\substack{d\in\mathbb{N}\\(d,2m)=1}}\frac{\mu(d)}{4^{\omega(d)}d^2}C(2md)^2P_1(2m,d)^2+E'''_{1,1}(m).$$

If m is an odd number, we have, using $\mathfrak{C}=\frac{1}{\pi}\prod_p\frac{(2p+1)^2(p-1)}{4p^3}$ as before,

$$\begin{aligned}&\sum_{(d,2m)=1}\frac{\mu(d)}{d^24^{\omega(d)}}C^2(2md)P_1^2(2m,d)\\&=\mathfrak{C}\prod_{p\mid 2m}\frac{4p^2}{(2p+1)^2}\prod_{p\nmid 2m}\frac{p^2(2p-1)^2}{(2p+1)^2(p-1)^2}\sum_{(d,2m)=1}\frac{\mu(d)}{d^24^{\omega(d)}}\prod_{p\mid d}\frac{4p^2}{(2p-1)^2}\\&=\mathfrak{C}\prod_{p}\frac{4p^2}{(2p+1)^2}\prod_{p\nmid 2m}\frac{(2p-1)^2}{4(p-1)^2}\prod_{p\nmid 2m}\frac{4p(p-1)}{(2p-1)^2}\end{aligned}$$

$$= \frac{1}{\pi} \prod_{p \mid 2m} \frac{(p-1)}{p},$$

while if m is even, we need to substitute $2m$ by m everywhere in the last identity. In particular we get for any m,

$$T_{1,1}^{\alpha}(m) = \frac{AB}{2^{\alpha+\kappa}\pi\sqrt{\log A}\sqrt{\log B}} \frac{\gcd(2,m)\varphi(m)}{2m} + E_{1,1}'''(m).$$

Summing over all α, and noting that for even m there is only contribution for $\alpha = 0$, we get that (3.24) equals

$$\frac{AB}{\pi\sqrt{\log A}\sqrt{\log B}}\bigg(\xi_m \times \frac{\varphi(m)}{m} + O\bigg((\log\log mAB)^4\bigg(\frac{1}{\log A} + \frac{1}{\log B}\bigg) + \frac{m}{\log^C A}\bigg)\bigg),$$

where $\xi_m = 19/16$ if m is odd and $\xi_m = 1$ if m is even. This concludes the proof. □

4 Conclusion

Let m and n be two integers, coprime or not, such that n is non zero. We say that m is an *r-th power modulo n* if and only if the equation $X^r \equiv m \pmod{n}$ is solvable. Furthermore we say that m and n, both non zero, are *(r,s)-mutual powers* if the equations

$$X^r \equiv m \pmod{n}, \quad X^s \equiv n \pmod{m} \tag{4.1}$$

are both solvable. Finally, we define

$$\mathcal{S}_{r,s}(A,B) = \{(a,b) \in \mathbb{N}^2, a \leqslant A, b \leqslant B, \text{ and } a \text{ and } b \text{ are } (r,s)\text{-mutual powers}\}.$$

With this terminology, $S(A,B) = \mathcal{S}_{2,2}(A,B)$. It is plausible that the techniques of the present paper can be employed to obtain an asymptotic formula for $\mathcal{S}_{r,s}(A,B)$. A number of extra technical issues will have to be considered.

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