Some zero-sum constants with weights

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Abstract. For an abelian group G, the Davenport constant $D(G)$ is defined to be the smallest natural number k such that any sequence of k elements in G has a non-empty subsequence whose sum is zero (the identity element). Motivated by some recent developments around the notion of Davenport constant with weights, we study them in some basic cases. We also define a new combinatorial invariant related to $(\mathbb{Z}/n\mathbb{Z})^d$, more in the spirit of some constants considered by Harborth and others and obtain its exact value in the case of $(\mathbb{Z}/n\mathbb{Z})^2$ where n is an odd integer.

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1. Introduction. For an abelian group G, the Davenport constant $D(G)$ is defined to be the smallest natural number k such that any sequence of k elements in G has a non-empty subsequence whose sum is zero (the identity element).

Motivated by some line of investigations taken up in [1], the following generalization of $D(G)$ was considered [3] recently for the particular group $\mathbb{Z}/n\mathbb{Z}$.

Let $n \in \mathbb{N}$ and assume $A \subseteq \mathbb{Z}/n\mathbb{Z}$. Then the function $D_A(n)$ has been defined in [3] to be the least natural number k such that for any sequence $(x_1, \cdots, x_k) \in \mathbb{Z}^k$, there exists a non-empty subsequence $(x_{j_1}, \cdots, x_{j_l})$ and $(a_1, \cdots a_l) \in A^l$ such that

$$\sum_{i=1}^{l} a_i x_{j_i} \equiv 0 \pmod{n}.$$

To avoid trivial cases, one assumes that the weight set A does not contain 0 and it is non–empty.

Subsequently, corresponding generalization for an arbitrary general abelian group has been considered in [2] and [15].

Exact values of $D_A(n)$ are known (see [1], [8], [15]) for several sets $A \subset \mathbb{Z}/n\mathbb{Z} \setminus \{0\}$ of weights.

We consider the particular case, when $n = p$, a prime. For this particular case, we enlist the known cases below.

- For any non-zero element a of $\mathbb{Z}/p\mathbb{Z}$, by pigeonhole principle it follows that
$$D_{\{a\}}(p) = p.$$
- For any non-zero element a of $\mathbb{Z}/p\mathbb{Z}$, it is known (see Lemma 2.1 and an example in the introduction in [1], the introduction in [3] and remarks at the end of [3]) that
$$D_{\{a,p-a\}}(p) = 1 + [\log_2 p].$$
- If $A = \{1, 2 \cdots p-1\}$, then it is easy to see that (see also Fact 1, Fact 2 in the proof of Theorem 6.1 in [1])
$$D_A(p) = 2.$$
- If $A = \{1, 2, \cdots, r\}$, where r is an integer such that $1 < r < p$, then (see [3])
$$D_A(p) = \lceil \frac{p}{r} \rceil,$$
where for a real number x, $\lceil x \rceil$ denotes the smallest integer $\geq x$.
- If A is the set of quadratic residues (or non-residues) $\pmod p$, then ([3], Theorem 3 and remarks at the end of [3])
$$D_A(p) = 3.$$

In this paper, in Section 2, we investigate the constant $D_A(p)$ for many subsets A of the unit group $(\mathbb{Z}/p\mathbb{Z})^*$. More precisely, we prove the following theorems.

Theorem 1. *Let $W \subseteq (\mathbb{Z}/p\mathbb{Z})^*$ such that $|W| \geq \frac{p+2}{3}$, and suppose there exists α in $(\mathbb{Z}/p\mathbb{Z})^*$ such that $\frac{a}{b} \neq \alpha$ for any a, b in W. Then*
$$D_W(p) = 3.$$

Theorem 2. *Let $W \subseteq (\mathbb{Z}/p\mathbb{Z})^*$ such that $|W| = r$, where r is an integer such that $1 < r < p$. Then*
$$D_W(p) \leq \lceil \frac{p}{r} \rceil.$$

In Section 3, we define a new combinatorial invariant related to $(\mathbb{Z}/n\mathbb{Z})^d$, more in the spirit of some constants considered by Harborth [9] and others [4], [10], [13], [7], [12] and obtain its exact value in the case of $(\mathbb{Z}/n\mathbb{Z})^2$ where n is an odd integer.

2. The constant $D_A(p)$. We proceed to prove Theorems 1 and 2. We shall need the following result ([5], [6], [11]).

Theorem A (Cauchy–Davenport inequality). *Let A and B be two non-empty subsets of $\mathbb{Z}/p\mathbb{Z}$. Then*

$$|A+B| \geq \min\{p, |A|+|B|-1\}$$

where

$$A+B = \{x \in \mathbb{Z}/p\mathbb{Z} \mid x = a+b,\ a \in A, b \in B\}$$

and for a subset K of $\mathbb{Z}/p\mathbb{Z}$, $|K|$ denotes the cardinality of K.

By iterating the above, one obtains the following.

Theorem B. *Let $A_1, A_2, \ldots A_h$ be non-empty subsets of $\mathbb{Z}/p\mathbb{Z}$. Then*

$$|A_1 + A_2 + \cdots + A_h| \geq \min\{p, \sum_{i=1}^{h} |A_i| - h + 1\}.$$

Proof of Theorem 1.

Let (s_1, s_2, s_3) be any sequence with $s_i \in (\mathbb{Z}/p\mathbb{Z})^*$. Let

$$A_i = s_i W, \text{ for } i = 1, 2, 3.$$

By Theorem B,

$$\begin{aligned} |A_1 + A_2 + A_3| &\geq \min\{p, \sum_{i=1}^{3} |A_i| - 2\} \\ &\geq \min\{p, \frac{3(p+2)}{3} - 2\} \\ &= p, \end{aligned}$$

which shows that $D_W(p) \leq 3$.

On the other hand, considering the sequence $\{1, -\alpha\}$, by our assumption we see that $a - \alpha b \neq 0$, for any $a, b \in W$ and this shows that $D_W(p) \geq 3$. □

Remark 1. If $p \geq 7$, then for any non-zero element α ($\neq 1$) in $\mathbb{Z}/p\mathbb{Z}$, choosing one element each from the $\frac{p-1}{2}$ pairs $(a, a\alpha)$ of non-zero elements of $\mathbb{Z}/p\mathbb{Z}$, we see that there are $2^{(p-1)/2}$ subsets W with $\frac{p-1}{2} \geq \frac{p+2}{3}$ elements satisfying the condition that $\frac{a}{b} \neq \alpha$ for all a, b in W. Hence Theorem 1 provides the exact value of $D_W(p)$ for many subsets W of $(\mathbb{Z}/p\mathbb{Z})^*$. Taking $\alpha = -1$, the subset $\{2, 4, \cdots, p-1\}$ could be one such W.

Proof of Theorem 2. By the remarks at the end of [3], if necessary by multiplying the elements of W by a fixed element of $(\mathbb{Z}/p\mathbb{Z})^*$, we may assume, without any loss of generality, that

$$W = \{1, w_1, w_2, \cdots, w_{r-1}\}.$$

Let $\{s_1, s_2, \cdots, s_k\}$ be any sequence of length k with $s_i \in (\mathbb{Z}/p\mathbb{Z})^*$, where $k = \left[\frac{p}{r}\right] + 1$.

Let

$$A_i = s_i W \cup \{0\}, \ \forall i = 1, \cdots, k-1,$$

and

$$A_k = s_k W.$$

Noting that

$$\sum_{i=1}^{k} |A_i| = (k-1)(r+1) + r = kr + k + 1,$$

by Theorem B,

$$\begin{aligned} |A_1 + A_2 + \cdots + A_k| &\geq \min\{p, \sum_{i=1}^{k} |A_i| - k + 1\} \\ &= \min\{p, kr\} \\ &= p. \end{aligned}$$

This implies that there exists a subsequence of $\{s_1, s_2, \cdots, s_k\}$ so that a weighted sum with weights from W vanishes. □

Remark 2. The bound in our Theorem 2 is of a very general nature depending only on the cardinality of the weight set. For instance, in the case when $W = \{1, 2, \cdots, r\}$, as had been observed in [3], the upper bound in the above theorem is the exact value of $D_W(p)$. However, when $W = \{1, -1\}$, then our upper bound is quite off the mark as it is known (see [3]) that $D_W(n) = [\log_2 n] + 1$.

3. A new combinatorial invariant.

Let $n \in \mathbb{N}$ and assume $A \subseteq \mathbb{Z}/n\mathbb{Z} \setminus \{0\}$. Then we define $f_A(n, d)$ to be the smallest positive integer k such that given a sequence $(\overline{x}_1, \cdots, \overline{x}_k)$ of k not necessarily distinct elements of $(\mathbb{Z}/n\mathbb{Z})^d$, there exists a subsequence $(\overline{x}_{j_1}, \cdots, \overline{x}_{j_n})$ of length n and $a_1, \cdots, a_n \in A$ such that

$$\sum_{i=1}^{n} a_i \overline{x}_{j_i} = \overline{0},$$

where $\overline{0}$ is the zero element of the group $(\mathbb{Z}/n\mathbb{Z})^d$ and the multiplication of a vector $\overline{x} = (x_1, \cdots, x_d)$ in $(\mathbb{Z}/n\mathbb{Z})^d$ by an element $a \in A$ is the standard scalar multiplication defined by $a(x_1, \cdots, x_d) = (ax_1, \cdots, ax_d)$.

We here consider the particular case $A = \{1, -1\}$; for this case, in [1] it had already been established that

$$f_{\{1,-1\}}(n, 1) = n + [\log_2 n].$$

Here we prove the following

Theorem 3. *For an odd integer n, we have*

$$f_{\{1,-1\}}(n, 2) = 2n - 1.$$

Proof. First we note that it is enough to establish Theorem 3 for any odd prime p. Indeed, we can proceed by induction on the number of prime factors (counted with multiplicity) of n. Therefore, given a sequence of vectors $v_1, v_2, \cdots, v_n$ in $\mathbb{Z}^2$, if $n > 1$ is not a prime, we write $n = mp$ where p is prime and assume that the result is true for all integers with number of prime factors less than that of n.

By our assumption, each subsequence of $2p - 1$ members of the sequence $v_1, v_2, \cdots, v_{2n-1}$ has a subsequence $v_i, i \in I$ of p elements such that there exists a weighted sum $\sum_{j \in I} a_j v_j$, $a_j \in \{1, -1\}$ with

$$\sum_{j \in I} a_j v_j \equiv (0, 0) \pmod{p}.$$

From the original sequence we go on repeatedly omitting such subsequences of p elements having sum equal to $(0, 0)$. Even after $2m - 2$ such sequences are omitted, we are left with $2pm - 1 - (2m - 2)p = 2p - 1$ elements and we can have one more subsequence of p elements with the property that sum of its elements is congruent to $(0, 0)$ modulo p.

Thus we have found $2m - 1$ pairwise disjoint subsets $I_1, I_2, \cdots, I_{2m-1}$ of $\{1, 2, \cdots, 2mp - 1\}$ with $|I_i| = p$ and $\sum_{j \in I_i} a_j v_j \equiv (0, 0) \pmod{p}$ for each i. Writing $x_i = \sum_{j \in I_i} a_j v_j$, we now consider the sequence $\frac{1}{p}x_1, \frac{1}{p}x_2, \cdots, \frac{1}{p}x_{2m-1}$ in $\mathbb{Z}^2$. By the induction hypothesis, this new sequence has a subsequence of m elements, a suitable weighted sum of which is divisible by m. Observing that the weight set $\{1, -1\}$ is closed under multiplication, the union of the corresponding sets I_i will supply the desired subsequence of $mp = n$ elements.

We now proceed to prove that for an odd prime p,

$$f_{\{1,-1\}}(p, 2) = 2p - 1.$$

Let $m = 2p - 1$ and $v_1 = (c_1, d_1)$, $v_2 = (c_2, d_2), \cdots$, $v_m = (c_m, d_m)$, be any sequence of vectors in $(\mathbb{Z}/p\mathbb{Z})^2$.

We consider the following system of equations in $2p-1$ variables x_i over $F_p = \mathbb{Z}/p\mathbb{Z}$:

$$\begin{aligned}\sum_{i=1}^{2p-1} c_i x_i^{(p-1)/2} &= 0,\\ \sum_{i=1}^{2p-1} d_i x_i^{(p-1)/2} &= 0,\\ \sum_{i=1}^{2p-1} x_i^{p-1} &= 0.\end{aligned}$$

Since $2(p-1) < 2p-1$ and $x_1 = x_2 = \cdots = x_{2p-1} = 0$ is a solution, by Chevalley's Theorem (see [14], for instance), there is another solution. Let $J \subset \{1, 2, \cdots, 2p-1\}$ be the set of all indices of the non-zero entries of such a solution.

From the first two equations it follows that

$$\sum_{i \in J} a_i v_i = (0,0), \text{ in } (\mathbb{Z}/p\mathbb{Z})^2,$$

where $a_i \in \{1, -1\}$. From the third equation we have $|J| = p$.

This proves that

$$f_{\{1,-1\}}(p,2) \leq 2p-1.$$

In the other direction, we consider a sequence of $2p-2$ elements where each of the elements $(1,0)$ and $(0,1)$ is repeated $(p-1)$ times. Since p is an odd prime, we observe that the sequence where each of the elements $(1,0)$, $(0,1)$, $(-1,0)$ and $(0,-1)$ is repeated $(p-1)$ times, does not have any subsequence of p elements summing to zero and hence

$$f_{\{1,-1\}}(p,2) \geq 2p-1.$$

Hence the theorem. □

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