Expansions in non-integer bases: Fibonacci expansions, a quasi-ergodic approach and Kakeya's method

Marco Pedicini (Roma Tre University)

Recife Meeting on Mathematics
Celebrating the 70th birthday of Vilmos Komornik

21 August 2024, Recife, Brazil

Outline

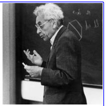
- memories
- expansions
- Kakeya's
- new proofs

MEMORIES

Paul Erdős and his mathematics 1999 - Budapest

A property of the Golden Number
dedicated to the memory of Paul Erdős

Vilmos Komornik, Paola Loreti, Marco Pedicini



Abstract: Let $q > 1$. Initiated by P. Erdős et al. in [?], several authors studied the numbers $l^m(q) = \inf\{y \ : \ y \in \Lambda_m, \ y \neq 0\}$, $m = 1, 2, \ldots$, where Λ_m denotes the set of all finite sums of the form $y = \varepsilon_0 + \varepsilon_1 q + \varepsilon_2 q^2 + \ldots + \varepsilon_n q^n$ with integer coefficients $-m \leq \varepsilon_i \leq m$. It is known ([?], [?], [?], [?]) that q is a Pisot number if and only if $l^m(q) > 0$ for all m. The value of $l^1(q)$ was determined for many particular Pisot numbers [?], but the general case remains widely open. In [?] we determine the value of $l^m(q)$ in other cases. With the help of numerical tests it is possible to generate conjectures that can give suggestions about investigations in number theory. Even if the computer algebra use is very limited by a generally too wide search space, sometimes a suggestion for further theoretical investigations can arrive by running numerical tests.

Problem:

Given a positive real number q and an integer $m \geq 1$, let us denote by $\Lambda = \Lambda_m$ the set of all real numbers y having at least one representation of the form

$$y = \varepsilon_0 + \varepsilon_1 q + \varepsilon_2 q^2 + \ldots + \varepsilon_n q^n$$

with some integer $n > 0$ and $\varepsilon_i \in \{-m, -m+1, \ldots, -1, 0, 1, \ldots, m-1, m\}$, and set

$$l^m(q) = \inf\{|y| \ : \ y \in \Lambda_m, \ y \neq 0\}.$$

It is clear that the sequence $(l^m(q))$ is non-increasing. Furthermore, we have $l^m(q) = 0$ for all m if $q < 1$ and $l^m(q) = 1$ for all m if $q = 1$.

Turning to the case $q > 1$, we recall from [?], [?] and [?] that q is a Pisot number if and only if $l^m(q) > 0$ for all m. (We recall that Pisot numbers are algebraic integers $q > 1$ all of whose conjugates belong to the open unit disk.) See also [?], [?], [?] and their references for other related questions.

Erdős, Joó and Joó determined the precise value of $l^1(q) = 1/q$ for the special Pisot numbers that are solutions of

$$q^r = q^{r-1} + q^{r-2} + \ldots + 1. \tag{1}$$

The smallest $q > 1$ for which this theorem applies is the *Golden number* $A = (\sqrt{5}+1)/2 \approx 1.618$.

The problem we solve is to determine $l^m(A)$ for all m where A is the Golden number.

Numerical Tests:

In order to bound the search space we fix a maximal length for developments: let us denote by Λ_n^m the set of all real numbers y having at least one representation of the form

$$y = \varepsilon_0 + \varepsilon_1 q + \varepsilon_2 q^2 + \ldots + \varepsilon_k q^k$$

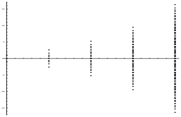
with some integer $0 \leq k \leq n$ and $\varepsilon_i \in \{-m, -m+1, \ldots, -1, 0, 1, \ldots, m-1, m\}$, and set

$$l_n^m(q) = \inf\{|y| \ : \ y \in \Lambda_n^m, \ y \neq 0\}.$$

In order to compute the set Λ_n^m we have to generate $(2m+1)^{n+1}$ combinations of coefficients and compute the real number y with arbitrary precision.

It is possible to improve the algorithm in such a way that its exponential time complexity goes towards exponential space one, but by increasing m or n we reach rapidly the maximal capacity of computational facilities.

Our early numerical tests consisted of running in a computer algebra package (such as Axiom or Mathematica) an algorithm for finding the minimal combinations (by lexicographic order) of the ε_i's corresponding to the value $l_n^m(A)$. In despite of the unfeasible character of the algorithm we arrived to get some information on the behavior of the sequence.



Result:

In order to formulate our result we introduce the Fibonacci sequence (F_k) defined by

$$F_0 = 0, \ F_1 = 1, \ F_k = F_{k-1} + F_{k-2}, \ k = 2, 3, \ldots$$

We proved the following theorem:

Theorem 1 *If*

$$A^{k-2} < m \leq A^{k-1} \tag{2}$$

for some integer $k \geq 1$, then

$$l^m(A) = |F_k A - F_{k+1}|. \tag{3}$$

The following table specifies the content of theorem 1up to $m = 842$. For example, the line $k = 10$ tells us that for $47 \leq m \leq 76$ we have $l^m(A) = |55A - 89| \approx 0.0081$.

k	F_k	F_{k+1}	A^{k-2}	A^{k-1}	$m \in$	$l^m(A)$
1	1	1	0.618	1.000	$[1, 1]$	0.6180
2	1	2	1.000	1.618		
3	2	3	1.618	2.618	$[2, 2]$	0.2361
4	3	5	2.618	4.236	$[3, 4]$	0.1459
5	5	8	4.236	6.854	$[5, 6]$	0.0902
6	8	13	6.854	11.090	$[7, 11]$	0.0557
7	13	21	11.090	17.944	$[12, 17]$	0.0344
8	21	34	17.944	29.034	$[18, 29]$	0.0213
9	34	55	29.034	46.979	$[30, 46]$	0.0132
10	55	89	46.979	76.013	$[47, 76]$	0.0081
11	89	144	76.013	122.992	$[77, 122]$	0.0050
12	144	233	122.992	199.005	$[123, 199]$	0.0031
13	233	377	199.005	321.997	$[200, 321]$	0.0019
14	377	610	321.997	521.002	$[322, 521]$	0.0012
15	610	987	521.002	842.999	$[522, 842]$	0.0007

Pisot Numbers and Real Number Computations 2001 - Rome

Pisot Numbers and Real Number Computations

Rome, October 3 - 5, 2001

Istituto per le Applicazioni del Calcolo "Mauro Picone", Consiglio Nazionale delle Ricerche

and

Università degli Studi di Roma "La Sapienza", Dipartimento "Metodi e Modelli Matematici per le Scienze Applicate"

Organization

The workshop co-organizers are

- Vilmos Komornik (Universitè Luis Pasteur, Strasbourg),
- Paola Loreti (Università "La Sapienza", Roma),
- Paolo Maroscia (Università "La Sapienza", Roma),
- Marco Pedicini (Istituto per le Applicazioni del Calcolo "Mauro Picone", CNR, Roma),

The meeting is made possible by a financial support of the Istituto per le Applicazioni del Calcolo "Mauro Picone", Consiglio Nazionale delle Ricerche and of the Università degli Studi di Roma "La Sapienza", Dipartimento "Metodi e Modelli Matematici per le Scienze Applicate".

Workshop Focus

The aim of this workshop is to bring together specialists from various research areas, mainly concerned with Pisot numbers and with their applications, and to promote exchanges of ideas and results between the participants working in complementary directions. One section of the meeting is devoted to a series of outlines and surveys mainly on the algebraic results on approximation properties using Pisot numbers as a base to represent real numbers. Another section is devoted to computations based on real numbers. Systems for exact real computation have been built based on alternative representations of real numbers, especially continued fractions. Properties of Pisot numbers have been used to build systems for exact real computations. The state of the art of exact real computing is moving towards exact systems which could be used to replace the traditional approximate systems in VLSI implementations. However, for the full development of exact systems, several areas remain to be understood and implemented. Other areas of applications will be covered as well.

Program

Workshop on Dynamical Aspects of Number System 2008 - Rome

Workshop on Dynamical Aspects of Number System
February 6-8, 2008 Rome Italy

Welcome and Contacts

Welcome and Contacts
Programme and Participants
Venue and Accommodation
Back

The workshop is the follow-up of several past workshops:

- 2001 Workshop Pisot Numbers and Real Number Computations (Komornik, Loreti, Maroscia, Pedicini);
- 2004 Workshop Bernoulli Convolutions and Beta-Expansions (N. Sidorov);
- 2006 Workshop Dynamical aspects of numeration (C. Frougny, W. Steiner).

The workshop organisers:

- Vilmos Komornik, Dep. Mathematique, Universitè de Strasbourg;
- Paola Loreti, Dip. Metodi e Modelli Matematici per le Scienze Applicate, Università degli Studi di Roma "La Sapienza" (Rome);
- Marco Pedicini, Istituto per le Applicazioni del Calcolo "M. Picone", Consiglio Nazionale delle Ricerche (Rome);

The organisers wish to welcome any contribution on the topic of the meeting.

Workshop on Dynamical Aspects of Number System
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6 February

14.00-14.40 Jean-Louis Verger-Gaugry, On the dichotomy of Perron numbers and beta-conjugates,
14.50-15.30 Wolfgang Steiner, $\beta$-expansions of minimal weight,

15.30-16.00 Break

16.00-16.40 Joerg Thuswaldner, Non-monic digit systems: Part I,
16.50-17.30 Paul Surer, Non-monic digit systems: Part II,

7 february

 9.30-10.10 Nikita Sidorov, Unique expansions in non-integer bases and the Sharkovkii ordering,

10.15-10.45 Break

10.45-11.25 Vilmos Komornik, Unique expansions in non-integer bases,
11.30-12.10 Benedetto Piccoli, Number theoretical problems in control applications

12.15-14.00 Lunch

14.00-14.40 Benoit Loridant, Non disk-like tiles associated to canonical number systems,
14.50-15.30 Michel Rigo, Extensions and restrictions of Wythoff's game preserving Wythoff's sequence as set of $P$ positions,

15.30-16.00 Break

16.00-16.40 Jacques Sakarovitch, On the computation of the successor function,
16.50-17.30 Anna Chiara Lai, Critical constants for general alphabets with three digits,

20.00 -     Social Dinner

8 February

 9.30-10.10 Edita Pelantova, Positional numeration systems with a Parry base: a combinatorial point of view,

10.15-10.40 Break

10.40-11.20 Charlene Kalle, A natural extension for the greedy $\beta$-transformation with three deleted digits,
11.20-11.50 Martijn de Vries, Remarks on univoque sets,
11.50-12.20 Informal discussion and open questions (moderator: Nikita Sidorov)

12.20-14.00 Lunch

14.00-14.40 Karma Dajani, Ergodic Properties of (alpha, beta) expansions,
14.50-15.30 Christiaan van de Woestijne, Number systems with nonzero digits.
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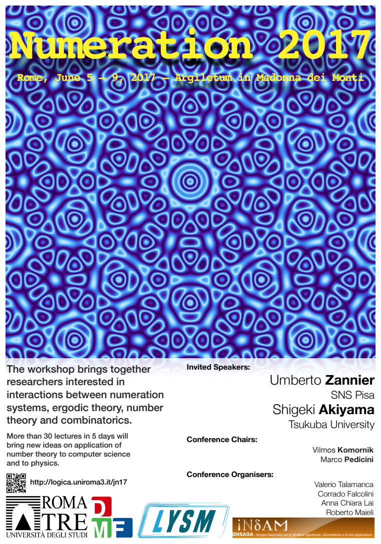
Participants:

- Emilie Charlier, University of Liege;
- Karma Dajani, University of Utrecht;
- Martijn de Vries, University of Utrecht;
- Christiane Frougny, LIAFA and University Paris 8;
- Charlene Kalle, Utrecht University;
- Karel Klouda, Univ, Paris 7;
- Vilmos Komornik, Dep. Mathematique, Universitè de Strasbourg;
- Annachiara Lai, Dip. Metodi e Modelli Matematici per le Scienze Applicate, Università degli Studi di Roma "La Sapienza" (Rome);
- Marion Le Gonidec, University of Liege;
- Paola Loreti, Dip. Metodi e Modelli Matematici per le Scienze Applicate, Università degli Studi di Roma "La Sapienza" (Rome);
- Benoit Loridant, Department of Mathematics and Statistics, University of Leoben, 8700 Leoben, Austria;
- Alessia Marigo, Istituto per le Applicazioni del Calcolo "M. Picone", Consiglio Nazionale delle Ricerche (Rome);
- Marco Pedicini, Istituto per le Applicazioni del Calcolo "M. Picone", Consiglio Nazionale delle Ricerche (Rome);
- Edita Pelantova, Czech Technical University Prague;
- Benedetto Piccoli, Istituto per le Applicazioni del Calcolo "M. Picone", Consiglio Nazionale delle Ricerche (Rome);
- Michel Rigo, University of Liege;
- Jacques Sakarovitch, CNRS/ENST;
- Nikita Sidorov, University of Manchester;
- Wolfgang Steiner, CNRS, Universitè Paris 7;
- Paul Surer, Department of Mathematics and Statistics, University of Leoben, 8700 Leoben, Austria;
- Joerg Thuswaldner, Department of Mathematics and Statistics, University of Leoben, 8700 Leoben, Austria;
- Christiaan van de Woestijne, Institut für Mathematik B Technische Universität Graz;
- Jean-Louis Verger-Gaugry, Institut Fourier Grenoble.

Delft 2008

Delft 2008

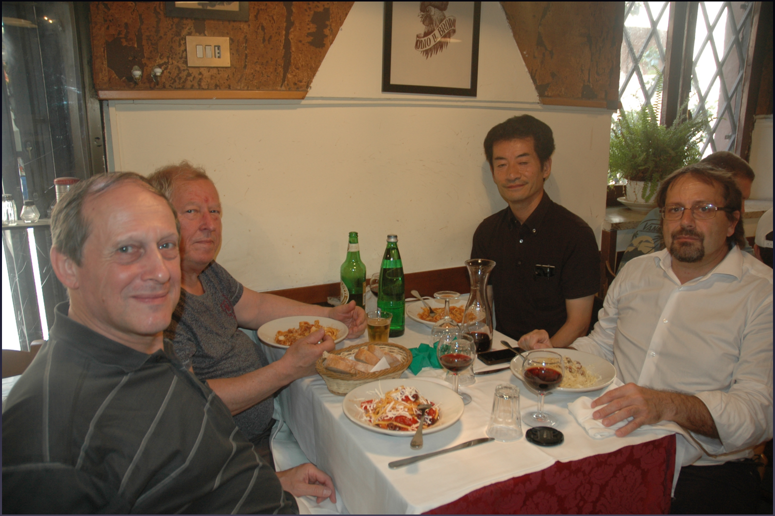
Numeration 2017 - Rome



Numeration 2017 - Rome


ECO
FREE

Numeration 2017



EXPANSIONS

Kakeya sequences

A sequence of real positive numbers $(p_n)_n$ is a **Kakeya sequence** if and only if it satisfies the following properties:

1. $$p_n \underset{n\to\infty}{\to} 0$$
2. $$p_n \leq R_n := \sum_{i>n} p_i$$

We denote by

$$S := \sum_{i>0} p_i$$

the sum of (p_n) and the partial sums by

$$S_n := \sum_{i\leq n} p_i$$

Binary Kakeya expansions

Let us fix the binary alphabet for digits $A := \{0, 1\}$.

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An **expansion of x in base (p_i)** with alphabet A is a binary sequence $(c_i) \in A^{\mathbb{N}}$ such that

$$x = \sum_{i>0} c_i p_i .$$

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Note that only x in the closed interval $[0, S]$ can have an expansion.

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If (p_n) is a Kakeya sequence we say that (c_i) is a **Kakeya expansion of x**.

Existence

Theorem: for each $x \in [0, S]$ there exists a Kakeya expansion of x.

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Proof

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Proof (by a quasi-greedy argument)

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extract from (p_n) two subsequences:

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Consider x and $S - x$

extract from (p_n) two subsequences:

	p_1	p_2	p_3	p_4	p_5	$\dots$	p_{n-1}	p_n	$\dots$
(p_{l_j})	p_1	p_2	$\dots$	$\dots$	p_5	$\dots$	$\dots$	**?**	$\dots$
(p_{r_j})	$\dots$	$\dots$	p_3	p_4	$\dots$	$\dots$	p_{n-1}	**?**	$\dots$

We denote partial sums of the two sequencese S'_n and S''_n

If p_n is such that $S'_{n-1} + p_n < x$ then we assign p_n to the first subsequence, otherwise if $S''_{n-1} + p_n < S - x$ to the second one.

proof

Note that it is impossible that both the last two conditions are not satisfied, otherwise we should have

$$S'_{n-1} + p_n \geq x \qquad \text{and} \quad S''_{n-1} + p_n \geq S - x$$

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and by summing up

$$S'_{n-1} + S''_{n-1} + 2p_n \geq x + S - x = S$$

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but

$$S'_{n-1} + S''_{n-1} = S_{n-1}$$

and $p_n \leq R_n$ so that

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and $p_n \leq R_n$ so that

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but also

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contradiction!

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and $p_n \leq R_n$ so that

$$S_{n-1} + p_n + p_n < S_{n-1} + p_n + R_n = S$$

but also

$$S_{n-1} + 2p_n \geq S$$

contradiction! Therefore the two subsequences are well defined and the limit sum of the first subsequence converges to $S' = x$ (and $S'' = S - x$).

Kind of expansions

Non-integer base expansions: $q > 1$

$$x = \sum_{i>0} \frac{c_i}{q^i}$$

Then by taking $p_n := 1/q^n$, (p_n) obviously converges to 0 and with an extra condition on q we get that it is a Kakeya sequence:

$$p_n = \frac{1}{q^n} < R_n = \sum_{i>n} \frac{1}{q^i} = \frac{1}{q^n} \sum_{i>1} \frac{1}{q^i} = \frac{1}{q^n(q-1)}$$

that is equivalent to

$$1 < \frac{1}{q-1} \Leftrightarrow q-1 < 1 \Leftrightarrow q < 2$$

Existence

Theorem (Rényi 1957) For any $x \in [0, S]$ there exists a non-integer base expansion in a given base $1 < q < 2$.

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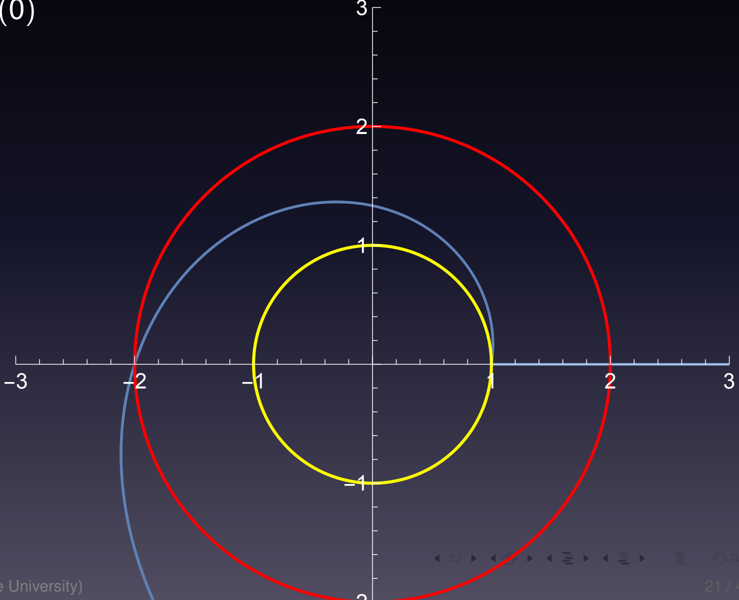
Observe that $\left(\frac{1}{q^n}\right)$ is a Kakeya sequence, therefore a non-integer base expansion is indeed a Kakeya expansion.

Critical Values

Qualitative analysis by projecting the interval $[0, S]$ on the circle of radius $q)$

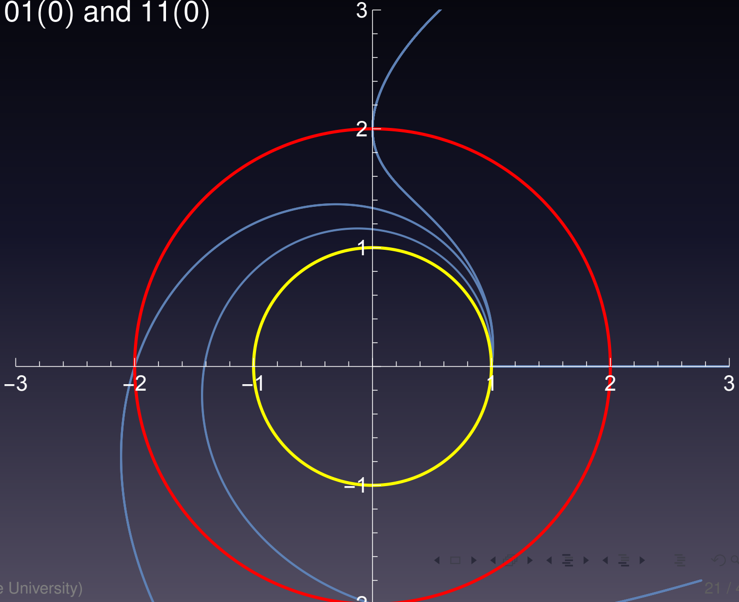
(0) and 1(0)



Critical Values

Qualitative analysis by projecting the interval [0, S] on the circle of radius q)

(0), 1(0), 01(0) and 11(0)



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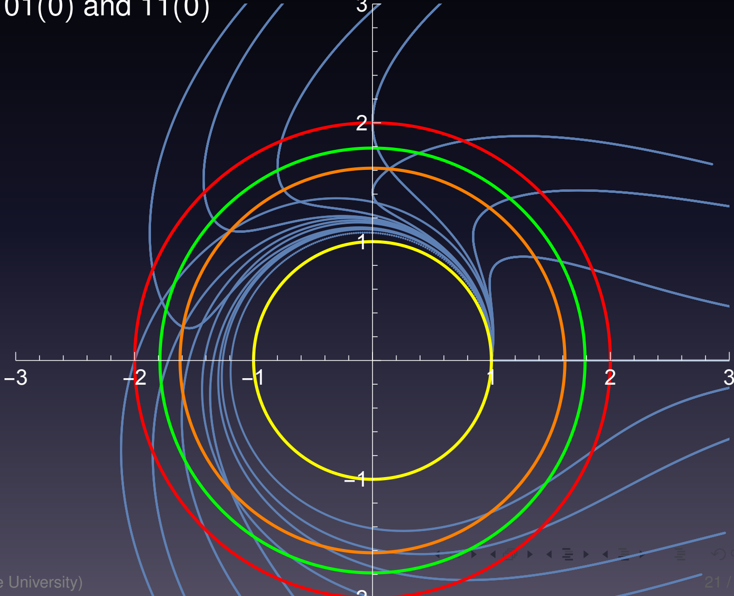
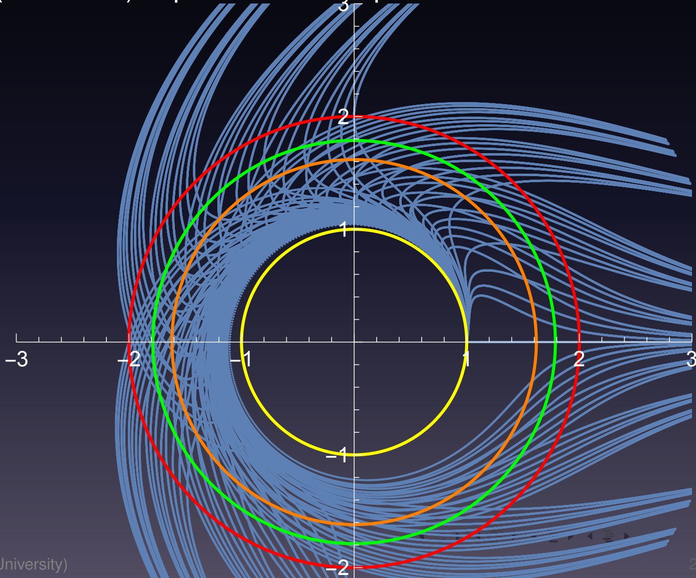


Critical Values

Qualitative analysis by projecting the interval [0, S] on the circle of radius q)
only finite (and short) expansions are represented!



Fibonacci expansions

The Golden ratio: $\varphi := \dfrac{1+\sqrt{5}}{2} > 1$ is closely related to the Fibonacci sequence, defined as

$$F_0 := 0, \quad F_1 := 1, \quad F_{n+1} := F_n + F_{n-1}, \text{ for } n > 1.$$

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Then by taking $p_n := 1/F_n$, we have that (p_n) obviously converges to 0 ($F_n \to \infty$)
By recurrence, it can be easily proven that $F_{n+j} \leq 2^j F_n$ for all $n \geq 1$ and $j \geq 1$, therefore it is a Kakeya sequence:

$$R_n := \sum_{j>0} \frac{1}{F_{n+j}} > \sum_{j>0} \frac{1}{2^j F_n} = \frac{1}{F_n} = p_n.$$

Existence

Theorem: For any $x \in [0, S]$ there exists a Fibonacci expansion of x.

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There is another dimension in this kind of research, by considering general alphabets (not only the binary one).

$$A := \{0, 1, \dots, M\} \quad \text{or} \quad A := \{0, 1, m\}$$

or

$$A := \{a_1 < a_2 < \cdots < a_k\}$$

the same kind of questions arise: we have an interval between the smallest number that can be represented and the greatest one:

$$\sum_{n>0} a_1 p_i \qquad \text{and} \qquad \sum_{n>0} a_k p_i$$

and conditions to have that any number $x \in \left[\sum_{n>0} a_1 p_i, \sum_{n>0} a_k p_i\right]$ admits an expansion.

The number of expansions

Every $x \in [0, 1]$ has one or two binary expansions.

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Since the Golden ratio φ satisfies the equality $q^2 = q + 1$, we have

$$1 = \frac{1}{q} + \frac{1}{q^2}$$

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Note that the first one is the greedy expansion, and that all these expansions are finite.

Continuing this construction indefinitely, we obtain an infinite expansion:

$$1 = \frac{1}{q} + \frac{1}{q^3} + \frac{1}{q^5} + \frac{1}{q^7} + \frac{1}{q^9} + \frac{1}{q^{11}} + \cdots .$$

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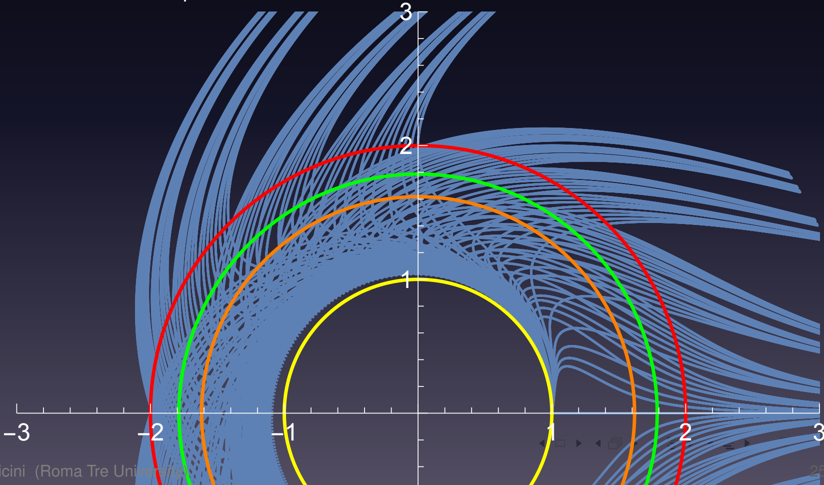

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Proof. Since

$$0 < x < \frac{1}{q} + \frac{1}{q^2} + \cdots, \quad \text{and} \quad 1 < \frac{1}{q^2} + \frac{1}{q^3} + \cdots$$

because $q < \varphi$, we may fix a large integer k such that

$$1 \leq \frac{1}{q^2} + \cdots + \frac{1}{q^k} \quad \text{and} \quad \frac{1}{q^k} + \frac{1}{q^{2k}} + \cdots \leq x \leq \sum_{k \not| j} \frac{1}{q^j}$$

(j runs over the positive integers which are not multiples of k).

Continuum many expansions

Since there are continuum many choices of the digits $c_k, c_{2k}, c_{3k}, \ldots \in \{0, 1\}$, the proof will be completed if we show that for each such choice we can find suitable digits $c_j \in \{0, 1\}$ for all $k \not| j$ such that

$$x - \left(\frac{c_k}{q^k} + \frac{c_{2k}}{q^{2k}} + \frac{c_{3k}}{q^{3k}} + \cdots\right) = \sum_{k \not| j} \frac{c_j}{q^j}.$$

This follows by applying Kakeya's above mentioned theorem with $(p_i) = (q^{-j})_{k \not| j}$. This is possible because $p_n \to 0$, and our choice of k implies the following inequalities:

$$p_n \leq p_{n+1} + \cdots + p_{n+k}$$

for every $n \geq 1$, and

$$0 \leq x - \left(\frac{c_k}{q^k} + \frac{c_{2k}}{q^{2k}} + \frac{c_{3k}}{q^{3k}} + \cdots\right) \leq p_1 + p_2 + \cdots .$$

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Theorem (C. Baiocchi, V. Komornik, P. Loreti, 2022): Every $0 < x < \sum_{i=1}^{\infty} \frac{1}{F_i}$ has a continuum of Fibonacci expansions.

Special Kakeya Expansions

Multiple Kakeya expansions

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Theorem (Komornik–Loreti–Pedicini, 2024) Let us consider an alphabet $A = \{0, 1, \dots, M\}$. If (p_n) is a sequence of positive numbers converging to 0 and such that

$$p_n \leq (M+1)p_{n+1} \qquad \text{for any } n$$

$$p_n < \left\lfloor \frac{M-1}{2} \right\rfloor p_{n+1} + \sum_{i>n+1} Mp_i \qquad \text{for infintely many indices}$$

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Then every $x \in (0, S)$ has a continuum of expansions with base (p_i) and alphabet A:

$$x = \sum_{n > 0} c_n p_n \qquad \text{where } (c_i) \in A^{\mathbb{N}}.$$

Generalised Golden Ratios

Given in Example 4.4 (ii) of (De Vries–Komornik–Loreti 2022) it is possible apply the new version of the Kakeya's Theorem to generalized Golden ratios:

$$\varphi_M := \begin{cases} (m+\sqrt{m^2+4m})/2 & \text{if } M = 2m-1 \text{ is odd,} \\ m+1 & \text{if } M = 2m \text{ is even,} \end{cases} \qquad m = 1, 2, \ldots,$$

Theorem (Baker, 2014) every $0 < x < \frac{M}{q-1}$ has a continuum of expansions of the form

$$x = \sum_{i=1}^{\infty} \frac{c_i}{q^i}, \quad (c_i) \in \{0, 1, \ldots, M\}^{\mathbb{N}}$$

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Proof

we observe that

$$1 = \frac{m-1}{\varphi_M} + \sum_{i=2}^{\infty} \frac{M}{\varphi_M^i}. \tag{1}$$

Indeed, the right hand side is equal to

$$\frac{m-1}{\varphi_M} + \frac{M}{\varphi_M(\varphi_M - 1)} = \frac{(m-1)(\varphi_M - 1) + M}{\varphi_M(\varphi_M - 1)},$$

and it remains to show that the numerator is equal to the denominator in the last fraction. or equivalently that

$$\varphi_M^2 = m\varphi_M + (M + 1 - m).$$

The last equality is obvious if $M = 2m$ and $\varphi_M = m + 1$. If $M = 2m - 1$, then the relation takes the form $\varphi_M^2 - m\varphi_M - m = 0$, and φ_M is clearly a solution of this second-order equation.

cont. Proof

Now we show that the sequence $p_n := q^{-n}$ satisfies the conditions of the Theorem for every fixed $1 < q < \varphi_M$.

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Now we show that the sequence $p_n := q^{-n}$ satisfies the conditions of the Theorem for every fixed $1 < q < \varphi_M$.

Indeed, $p_n \to 0$ because $q > 1$, and $p_n \leq (M+1)p_{n+1}$ for every n because

$$\frac{p_n}{p_{n+1}} = q < \varphi_M,$$

and $\varphi_M \leq m + 1 \leq M + 1$ from the definition of φ_M.

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Finally, the crucial condition follows from (1) for every n because $\left[\frac{M-1}{2}\right] = m - 1$ and $1 < q < \varphi_M$.

Lucas Sequences

Let $M = 2m - 1$, $m \geq 1$, and introduce a sequence of integers F_i by the formulas

$$F_0 = 0, \quad F_1 = 1, \quad \text{and} \quad F_n = m(F_{n-1}+F_{n-2}) \quad \text{for} \quad n = 2, 3, \ldots.$$

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Thanks to the new special conditions on Kakeya expansions we show a new shorter proof of

Theorem 1.1 in [KomLorPed2022]: every $0 < x < \sum_{i=1}^{\infty} \frac{2m-1}{F_i}$ has a continuum of expansions of the form

$$x = \sum_{i=1}^{\infty} \frac{c_i}{F_i}, \quad (c_i) \in \{0, 1, \ldots, 2m-1\}^{\mathbb{N}} .$$

We recall from [KomLorPed2022] that $F_{n+1}/F_n \to \varphi_M$.

Proof.

The sequence $p_n := 1/F_n$ satisfies the conditions of the Theorem. Indeed, $p_n \to 0$ because $F_n \geq n - 1$ for every n by an easy induction, and $p_n \leq (M+1)p_{n+1}$ because $F_{n+1} \leq (m+1)F_n$ for every $n \geq 1$ by an easy induction, and hence

$$\frac{p_n}{p_{n+1}} = \frac{F_{n+1}}{F_n} \leq m + 1 \leq M + 1$$

for all $n \geq 1$.

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for all $n \geq 1$.

Finally, the crucial relation holds because

$$\frac{1}{F_n} < \frac{m-1}{F_{n+1}} + \sum_{i=n+2}^{\infty} \frac{2m-1}{F_i}$$

for every odd index n by (Komornik-Loreti-Pedicini 2022, Lemma 2.1).

Another variant of Kakeya Sequences

Theorem Let (p_n) be a sequence of positive numbers, converging to zero, and $m \geq 2$ a real number. Assume that

$$(m-1)p_n \leq (2m-1)p_{n+1} \quad \text{for every} \quad n \geq 1, \tag{2}$$

and

$$(m-1)p_n < m \sum_{i=n+2}^{\infty} p_i \quad \text{for infinitely many indices} \quad n. \tag{3}$$

Then every

$$x \in \left(0, m\sum_{i=1}^{\infty} p_i\right)$$

has a continuum of expansions of the form

$$x = \sum_{i=1}^{\infty} c_i p_i, \quad (c_i) \in \{0, 1, m\}^{\mathbb{N}}.$$

Continuum of expansions

This version can be applied to non-integer base expansions with alphabet $A = \{0, 1, m\}$, that is expansions of the form

$$x = \sum_{i=1}^{\infty} \frac{c_i}{q^i}, \quad (c_i) \in \{0, 1, m\}^{\mathbb{N}} \tag{4}$$

with $m \geq 2$ and $q > 1$.
Theorem If $m \geq 2$, $q > 1$, and

$$q(q-1) < \frac{m}{m-1}, \tag{5}$$

then every $x \in \left(0, \frac{m}{q-1}\right)$ has a continuum of expansions of the form (4).

THANKS!!