

# An Object-Oriented Approach to Idempotent Analysis: Integral Equations as Optimal Control Problems

Marco Pedicini (Roma Tre University)

Optimal Control and Inverse Problems in PDE Theory  
**Paola Loreti's Festschrift**

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Villa Toeplitz, Varese

# a contribution of mathematics to computer science: dependent types

# Birth of Generic Types

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- Soon the need arose to **reuse code** regardless of the element type—the same **length function** for **List<Int>** and **List<String>**.
- Solution: **generic (parameterised) types**.  
Historical milestones: generics in **Ada** (1983), templates in **C++** (1985), generics in **Java 5** (2004).

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template<typename T> size_t length(const vector<T>& v);  
    becomes length<int>, length<string>, ... depending on the  
    concrete type.
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# Generics $\rightarrow$ Dependent Types

- From parametric polymorphism to **dependent types**: types may depend on *values* (e.g. Vector alpha n carrying its length  $n$  in the type).

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- From parametric polymorphism to **dependent types**: types may depend on *values* (e.g. Vector alpha n carrying its length  $n$  in the type).
- They **express invariants** at the type level, guaranteeing them at compile time.
- Foundation of modern **proof assistants** (Coq, Agda, Lean), where code and proofs live together.

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Nota Bene: the CoC *definitely is not a **civil-engineering statics** handbook!*

# another contribution of mathematics to computer science

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- Flagship achievements: formalisation of the Stone–Weierstrass theorem, the *Liquid Tensor* project (Scholze/Buzzard), and parts of analytic number theory.
- Fields Medalist **Terence Tao** called Lean/Mathlib “one of the most exciting developments in mathematical practice”, encouraging researchers to adopt formalisation.

T. Tao, on the blog *What's new*, 6 April 2020

# Lean Ecosystem

## Lean Community

**Community**

- Zup chat
- Discord
- Blog
- Community information
- Community guidelines
- Teams
- Projects about Lean
- Projects using Lean
- Teaching using Lean
- Events

**Installation**


- Get started
- Online version (no installation)
- Using LaTeX (build system)

**Documentation**

- Learning resources (start here)
- API documentation
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- Can't read?
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- Simplifier
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- Spending on Lean files
- About OWLs
- Glossary

**Library overviews**

- Library overview
- Undergraduate math
- Mathlib's 100 theorems
- 1000+ theorems



## Lean and its Mathematical Library

The Lean theorem prover is a proof assistant developed principally by Leonardo de Moura.

The community recently switched from using Lean 3 to using Lean 4. This website is still being updated, and some pages have outdated information about Lean 3 (these pages are marked with a prominent banner). The old Lean 3 community website has been archived.

The Lean mathematical library, `mathlib`, is a community-driven effort to build a unified library of mathematics formalized in the Lean proof assistant. The library also contains definitions useful for programming. This project is very active, with many regular contributors and daily activity.

You can get a bird's-eye view of what is in the `mathlib` library by reading the [library overview](#), and read about recent additions on our [blog](#). The design and community organization of `mathlib` are described in the 2023 article [The Lean mathematical library](#), although the library has grown by an order of magnitude since that article appeared. You can also have a look at our [repository statistics](#) to see how the library grows and who contributes to it.

<h3>Try it!</h3> <p>You can try Lean in your web browser, install it in an isolated folder, or get the full install. Lean is free, open source software. It works on Linux, Windows, and MacOS.</p> <p>Try the online version of Lean</p>	<h3>Learn to Lean!</h3> <p>You can learn by playing a game, following tutorials, or reading books.</p> <p>Learning resources</p> <p>Theorems Proving in Lean 4</p>	<h3>Meet the community!</h3> <p>Lean has very diverse and active community. It gathers mostly on a Zup chat and on GitHub. You can get involved and join the fun!</p> <p>Meet us</p>
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**Leonardo de Moura** home about me publications talks research

Te is Senior Principal Applied Scientist in the Automated Reasoning Group at [MSR](#)

**Project**

- Lean, Theorem Prover

**Previous Projects**



- Z3 SMT Solver
- Sage 3.8, Z3 SMT Solver
- Coq, the Synthetic Analysis Laboratory

**Teaching**

Here is a list of courses I have taught in the past.

- [Logic and Algorithms in Real Algebraic Geometry](#), University of Miami, May, May 2015
- [Decision Methods for Arithmetic](#), 3rd, Summer School on Formal Techniques, Miami Park, 2015, [lect.1](#), [lect.2](#), [lect.3](#), [lect.4](#)
- [Semant. Summer School on Formal Techniques](#), Miami Park, 2012
- [First Summer School on Formal Techniques](#), Miami Park, 2011
- [Satisfiability Modulo Theories \(SMT\): Views & Applications](#), University Deusto Bilbao in Miami, May, March 2010 [lect.1](#), [lect.2](#), [lect.3](#), [lect.4](#), [research](#)
- [On Designing and Implementing Satisfiability Modulo Theory Solvers](#), Summer School 2009, Verbalizer, Toulouse, Systems & Applications, Nancy, France [lect.1](#), [lect.2](#)
- [SMT, Solvers, Theory, and Implementation](#), Summer School on Logic and Theorem Proving in Programming Languages, Orange 2008 [lect.1](#)
- [CADE08, Logic Engines of Proof](#), Stanford University, Fall 2008

Leonardo de Moura  
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Education  

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## Math

Mathematics learning Lean by doing

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← Thoughts on the Pythagorean theorem The end of the summer → Search

### Liquid tensor experiment

Posted on December 3, 2023 by alexander

This is a guest post, written by Peter Scholze, explaining a liquid real vector space mathematical formalisation challenge. For a pdf version of the challenge, see [here](#). For comments about formalisation, see section 6. Show over to Peter.

#### 1. The challenge

I want to propose a challenge: Formalise the proof of the following theorem.

**Theorem 1.4 (Clasman-8.3)** Let  $0 < p < 1 \leq p \leq 1$  be real numbers, let  $S$  be a profinite set, and let  $\mathcal{V}$  be a  $p$ -Banach space. Let  $\mathcal{M}_p(S)$  be the space of  $p$ -measures on  $S$ . Then

$$\text{Ext}_{\text{Cont}(S, \mathbb{R})}^1(\mathcal{M}_p(S), \mathcal{V}) = 0$$

$\beta(p) \geq 1$ .

(This is a special case of Theorem 9.1 in [www.math.uzh.ch/bhatt.de/papers/ahabze/AnAbv1.pdf](#), and is the essence of the proof of Theorem 6.5 there.)

Below, I will explain what all the terms in the theorem are, and why I care. I apologise in advance that the background story is a little longer, but I think it's a fascinating story and I tried to be as brief as possible.

#### 2. Getting condensed

## Leonardo de Moura

home about me publications slides research

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- Decision Methods for Arithmetic, 2016, Summer School on Formal Techniques, Helsinki Park, 2016, [lect.1](#), [lect.2](#), [lect.3](#), [lect.4](#), [lect.5](#)
- Second Summer School on Formal Techniques, Helsinki Park, 2012
- First Summer School on Formal Techniques, Helsinki Park, 2011
- Satisfiability Modulo Theories (SMT): Views & Applications, University of Duisburg-Essen, March-2010 [lect.1](#), [lect.2](#), [lect.3](#), [lect.4](#), [lect.5](#), [lect.6](#)
- On Designing and Implementing Satisfiability Modulo Theory Solvers, Summer School 2009, VeriBalkan, Technische Universität München, Munich, France, October 5, 2009
- SMT Solvers, Theory, and Implementations, Summer School on Logic and Theorem Proving in Programming Languages, Orange 2008 (unofficial)
- CS220, Logic, Languages of Proof, Stanford University, Fall 2008

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Documentation 



# BACK TO THE NINETIES

# Tropicalisation of the Real Numbers

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- **idempotent addition**:  $a \oplus a = a$  leads to piecewise-linear “polynomials” and **tropical geometry**.
- Applications in optimisation, combinatorics, and degeneration of algebraic varieties.

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- This flexibility underlies many dualities (e.g. Boolean vs. Heyting algebras) and motivates exploring tropical methods in logic and computer science.

# generics in control theory

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P. Loreti and M. Pedicini, **Idempotent analogue of resolvent kernels for a deterministic optimal control problem**, Mat. Zametki 69 (2001), no. 2, 235–244. MR 2002d:49038

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- Bellman's dynamic programming principle becomes **linear** over  $(\mathbb{R} \cup \{+\infty\}, \min, +)$ .
- This opens the door to algebraic and numerical methods based on idempotent analysis.

# Continuous HJB Problem

We studied the optimal-control HJB equation

$$\max(u(x) - \psi(x), \lambda u(x) + \max_{a \in A} \{-b(x, a) \cdot Du(x) - f(x, a)\}) = 0.$$

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In P. Loreti and M. Pedicini. **An Object-Oriented Approach to Idempotent Analysis: Integral Equations as Optimal Control Problems.**

Contemporary Mathematics, vol. 377, AMS. Proceedings of the Conference on Idempotent Mathematics and Mathematical Physics. Editors G.L. Litvinov and V.P. Maslov (2005)

# Discretised Scheme

The explicit up-wind discretisation reads

$$u_h(x) = \min(\psi(x), \inf_{a \in A} ((1 - \lambda h) u_h(x + hb(x, a)) + hf(x, a))).$$

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Under standard assumptions (Bardi–Capuzzo Dolcetta 1997) the scheme is **monotone** and **convergent**.

# Linear Form over $\mathbb{R}_{\min,+}$

- Define  $S(x) = u_h(x)$ ,  $F(x) = \psi(x)$ , and the (min-plus) linear operator

$$(HS)(x) = \inf_{a \in A} ((1 - \lambda h) S(x + hb(x, a)) + hf(x, a)).$$

- Then **Bellman equation**:  $S = HS \oplus F$  with  $\oplus = \min$  and  $\odot = +$ .
- Hence HJB Equation is **linear** in the idempotent semiring  $\mathbb{R}_{\min,+}$ .

# Idempotent Integral Equation

Using the **idempotent integral**  $\int^\oplus$ , we rewrite

$$u_h(x) = \psi(x) \oplus \tilde{\lambda} \int_{B(x,A)}^\oplus u_h(\xi) \odot G_h(x, \xi) d\xi,$$

# Idempotent Integral Equation

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$$u_h(x) = \psi(x) \oplus \tilde{\lambda} \int_{B(x,A)}^\oplus u_h(\xi) \odot G_h(x, \xi) d\xi,$$

where  $\tilde{\lambda} = 1 - \lambda h$ ,  $B(x, a) = x + hb(x, a)$  and the kernel

$$G_h(x, \xi) = \frac{h}{1 - \lambda h} \min\{f(x, a) : a \in A, \xi = x + hb(x, a)\}.$$

# Idempotent Integral Equation

Using the **idempotent integral**  $\int^\oplus$ , we rewrite

$$u_h(x) = \psi(x) \oplus \tilde{\lambda} \int_{B(x,A)}^\oplus u_h(\xi) \odot G_h(x, \xi) d\xi,$$

where  $\tilde{\lambda} = 1 - \lambda h$ ,  $B(x, a) = x + hb(x, a)$  and the kernel

$$G_h(x, \xi) = \frac{h}{1 - \lambda h} \min\{f(x, a) : a \in A, \xi = x + hb(x, a)\}.$$

This is a **Volterra-type** equation in the tropical semiring.

# Resolvent Kernels and $p$ -Step Approximation

- **Iterated kernels**  $K^{(n)}$  defined by tropical convolution yield the series

$$u_h(x) = \psi(x) \oplus \bigoplus_{n=1}^{\infty} \tilde{\lambda}^n \int_{B^n(x,A)}^{\oplus} \psi(t) \odot K^{(n)}(t, x) dt.$$

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$$u_h(x) = \psi(x) \oplus \bigoplus_{n=1}^p \tilde{\lambda}^n \int_{B^n(x,A)}^{\oplus} \psi(t) \odot K^{(n)}(t,x) dt \oplus \bigoplus_{n=p+1}^{\infty} \tilde{\lambda}^n \int_{B^n(x,A)}^{\oplus} u_h(t) \odot K^{(n)}(t,x) dt.$$

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- Matches the classical value-iteration algorithm.

# Generic Implementation

We provide **two generic templates** required for the implementation of integral kernels. These templates are automatically adapted when used with usual real numbers or within min-plus algebra.

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Then, we provide two modules defining operations and constants on the real numbers structure, and functions used for the discretization of intervals. This part is introduced to realize the proper instantiation of the arithmetical structure.

The second part of the implementation consists just in the definition of the particular problem we aim to solve. So, we specify the integral equation of second type by the **kernel** and its **constant term function** (in the analog of the optimal control problem, they can be derived from dynamics, the cost function and the stopping time cost function).

# Solution

At the very end of the program, the approximate **solution of the integral equation** is computed (and in the analog of the optimal control problem, this corresponds to an approximation of the value function).

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We also compared the approximate solution with viscosity solutions (the classic approximation). In some problems they coincide.

# Idempotent Equations and Applications

Reformulated analogues of integral/differential equations:

- HJB has an idempotent formulation for optimal trajectory computation.

## Applications:

- Optimization (e.g. scheduling, timed networks)
- Control theory (Bellman–Hamilton–Jacobi)
- Game theory and decision
- Tropical mathematics (degenerations of classical structures)
- Automata theory (quantitative semantics)
- Classical limits of quantum mechanics (Maslov's idempotent mechanics)

# Concluding Remarks

- Idempotent analysis offers a combinatorial view of HJB
- Tropical solutions often match viscosity solutions
- Min-plus convolutions provide a practical computational method
- Useful in discrete models, numerical schemes, graph optimization

**Thanks!**

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