Quantum Entanglement and the Bell Matrix

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Abstract

In Quantum Computing (QC) entangled states are important since it is widely recognised that capacity to compute exponentially faster than classic systems is a consequence of entangled states.

On the other hand entangled states in case of mixed systems in which sub-systems interact are not characterised in a unique way. One of the method used to analyse quantum entanglement is via Schmidt decomposition and a variant of von Neumann entropy to measure the entanglement of parties.

In fact, we adopted this point of view to prove that states obtained as the transformation of canonical base states (pure states) by our generalisation of the binary Bell matrix are indeed maximally entangled states for any dimension *n*.

References

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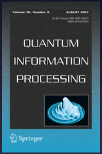


BITS

Quantum bits

Definition (Quantum bits)

A *quantum bit* is any element of a bidimensional Hilbert space $\mathbb{C}^2$ and it is expressed with respect an orthonormal base $|u\rangle$, $|u_\perp\rangle$ as a unitary linear combination:

$$|\psi\rangle = \alpha\,|u\rangle + \beta\,|u_\perp\rangle$$

where $\alpha, \beta \in \mathbb{C}$ and $\alpha^2 + \beta^2 = 1$.

Direct Sums and Tensors, quantum bits can be combined by the Kronecker operations: **Kronecker Sum** and **Kronecker Product**.

$$|\psi_1\rangle \oplus |\psi_2\rangle \in \mathbb{C}^{2^{n_1}+2^{n_2}} \qquad |\psi_1\rangle \otimes |\psi_2\rangle \in \mathbb{C}^{2^{n_1+n_2}}$$

if $|\psi_1\rangle \in \mathbb{C}^{2^{n_1}}$ and $|\psi_2\rangle \in \mathbb{C}^{2^{n_2}}$

Quantum Registers

The **Kronecker Product** is taken to build quantum registers by combining several quantum bits:

$$\begin{aligned}|\psi_1\rangle \otimes |\psi_2\rangle =&(\alpha_1 |u_1\rangle + \beta_1 |u_{1\perp}\rangle) \otimes (\alpha_2 |u_2\rangle + \beta_2 |u_{2\perp}\rangle) = \\ =&(\alpha_1\alpha_2 |u_1\rangle \otimes |u_2\rangle + \alpha_1\beta_2 |u_1\rangle \otimes |u_{2\perp}\rangle + \\ &+ \beta_1\alpha_2 |u_{1\perp}\rangle \otimes |u_2\rangle + \beta_1\beta_2 |u_{1\perp}\rangle \otimes |u_{2\perp}\rangle\end{aligned}$$

The four combinations of the two bases form a base for the new space:

$$|u_1\rangle \otimes |u_2\rangle, |u_1\rangle \otimes |u_{2\perp}\rangle, |u_{1\perp}\rangle \otimes |u_2\rangle, |u_{1\perp}\rangle \otimes |u_{2\perp}\rangle$$

with the canonical base $|0\rangle$ an $|1\rangle$ the four elements are more conveniently denoted by

$$|00\rangle, |10\rangle, |01\rangle, |11\rangle.$$

ENTANGLEMENT

Measurement and Entanglement

Observables of quantum states are obtained by **measurements**, which can be performed with respect to an element of the base, essentially as a scalar product by the “bra” of one component of the base:

$$\langle u| \left(\alpha \left|u\right\rangle + \beta \left|u_\perp\right\rangle\right) = \alpha \left\langle u|u\right\rangle + \beta \left\langle u|u_\perp\right\rangle = \alpha$$

Entanglement of a state $|\phi\rangle$ is verified when with respect to a measurement it is impossible to separate contributions from one copy of the first space and from the other one, it is the case in the EPR states, for instance in

$$|\psi\rangle = \frac{1}{\sqrt{2}} \left(|00\rangle + |11\rangle\right)$$

Bell States

They are called **entangled** since they cannot be expressed as a tensor of two quantum bit: if there exist then

$$(a_1 |0\rangle + b_1 | |1\rangle) \otimes (a_2 |0\rangle + b_2 |1\rangle) =$$
$$= a_1 a_2 |00\rangle + a_1 b_2 |01\rangle + b_1 a_2 |10\rangle + b_1 b_2 |11\rangle$$

and therefore $a_1 b_2 = a_2 b_1 = 0$ but in this way one of the two a_1 or b_2 are 0 and also a_2 or b_1 is zero therefore we have impossibility for annihilating components $|01\rangle$ and $|10\rangle$ of the tensor product and at the same time not to annihilate both remaining components $|00\rangle$ and $|11\rangle$.

Entanglement of multiple qubits

We consider elements of Hilbert spaces $|\psi\rangle \in \mathbb{C}^{2^n}$ which are *pure quantum states*, i.e., they are complex (column) vectors of unit Euclidean norm:

$$|\psi\rangle = (\psi_1, \ldots, \psi_{2^n})^T \qquad \text{and} \qquad \sum_{j=1}^{2^n} |\psi_j|^2 = 1.$$

Definition (Globally entangled state)

A state $|\psi\rangle$ is *globally entangled* if for any $|\phi_1\rangle$ and $|\phi_2\rangle$ we have $|\psi\rangle \neq |\phi_1\rangle \otimes |\phi_2\rangle$.

We use the symbol I_{2^n} to denote the 2^n-dimensional identity matrix:

$$I_{2^n} := \underbrace{I_2 \otimes \cdots \otimes I_2}_{n-\text{times}}.$$

being $I_{2^n} = (1)$ if $n = 0$.

Operators

The **expectation value** of the operator A in the state ψ is denoted by

$$\langle A \rangle_\psi := \langle \psi | A | \psi \rangle.$$

Let us denote by σ_y the **Pauli matrix**

$$\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$

Two more operators are used in the next definition, the first one pays a crucial role in our entanglement criterion

$$M_{2^n} := \sigma_y \otimes I_{2^{n-2}} \otimes \sigma_y$$

and

K_{2^n} is the **conjugation operator**.

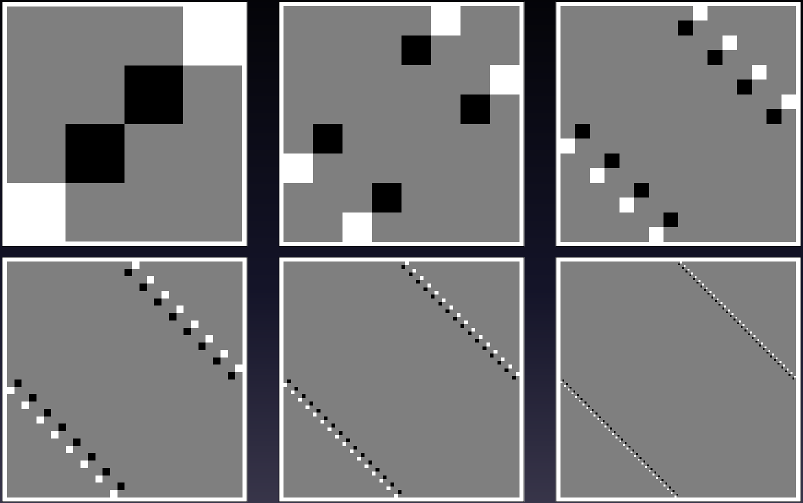
Example

We explicitly compute M_4:

$$M_4 := \sigma_y \otimes I_{2^{2-2}} \otimes \sigma_y = \sigma_y \otimes (1) \otimes \sigma_y = \sigma_y \otimes \sigma_y =$$

$$= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \otimes \sigma_y = \begin{pmatrix} 0 & -i\sigma_y \\ i\sigma_y & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

For a pictorial representation of matrices M_{2^n} with $n \geq 2$ see the next slide.



We show in this picture the matrices M_{2^n} for $n = 2, \ldots, 7$. Entries 0 are shown in grey color, entries $+1$ by black color and entries -1 by white color.

Testing entanglement

We now define a special operator which permits to express a **sufficient** condition for **entanglement**:

Definition

Let us denote by $\mathcal{F} : \mathbb{C}^{2^n} \to \mathbb{C}$ the function which associates to a state $|\psi\rangle$ the expectation value of the operator $M_{2^n}K_{2^n}$ in the state $|\psi\rangle$, namely:

$$\mathcal{F}(|\psi\rangle) := \langle M_{2^n}K_{2^n}\rangle_\psi \quad (1)$$

Note that

$$\mathcal{F}(|\psi\rangle) := \langle M_{2^n}K_{2^n}\rangle_\psi = \langle\psi|M_{2^n}K_{2^n}|\psi\rangle = \langle\psi|M_{2^n}|\bar{\psi}\rangle$$

where $|\bar{\psi}\rangle$ denotes the complex conjugate of $|\psi\rangle$.

Expectation value of the operator on entangled states

We now show that $M_{2^n} K_{2^n}$ has zero expectation value on product states.

Proposition (1)

If $|\psi\rangle$ is not a globally entangled state then $\mathcal{F}(|\psi\rangle) = 0$.

Thus we may use this value to test entanglement:

$$\mathcal{F}(|\psi\rangle) \neq 0 \implies \exists \phi_1, \phi_2 \text{ such that } |\psi\rangle = |\phi_1\rangle \otimes |\phi_2\rangle$$

Meyer-Wallach measure

+

von Neumann's Entropy

=

Maximal Entropy

Maximal Entanglement

Next result shows that $\mathcal{F}$ also provides a sufficient condition for maximal entanglement. It is useful to recall the following

Definition (Schmidt decomposition)

Let $n_1, n_2 \in \mathbb{N}$ such that $n_1 + n_2 = n$ and let $A = \mathbb{C}^{2^{n_1}}$ and $B = \mathbb{C}^{2^{n_2}}$ so that $\mathbb{C}^{2^n} = A \otimes B$. Then any state $|\psi\rangle \in \mathbb{C}^{2^n}$ can be written in the form

$$|\psi\rangle = \sum_{k=1}^{K} c_k \left|\phi_k^A\right\rangle \otimes \left|\phi_k^B\right\rangle$$

where $K = \min\{dim(A), dim(B)\} = \min\{2^{n_1}, 2^{n_2}\}$, $c_k \geq 0$ and $\{|\phi_k^A\rangle\}$, $\{|\phi_k^B\rangle\}$ are two orthonormal subsets of A and B, respectively.

Sub-systems and decomposition

Consider the decomposition $\mathbb{C}^{2^n} = A \otimes B$ and let $\rho_{A,\psi}$ be the density operator of the state $|\psi\rangle$ on the subsystem A.

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As a consequence,

$$Tr[\rho_{A,\psi}] = \sum_{k=1}^{K} c_k^2 = 1 \text{ and } Tr[\rho_{A,\psi}^2] = \sum_{k=1}^{K} c_k^4.$$

Proposition (2)

If $|\mathcal{F}(|\psi\rangle)| = 1$ then $|\psi\rangle$ is ***maximally entangled*** *with respect to MW measure.*

pre-conclusion

Above results relate the value of $|\mathcal{F}(|\psi\rangle)|$ to a measure of entanglement of the state $|\psi\rangle$.

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In particular
if $|\mathcal{F}(|\psi\rangle)|$ is minimal, i.e., $|\mathcal{F}(|\psi\rangle)| = 0$, then $|\psi\rangle$ is not entangled
while
if $|\mathcal{F}(|\psi\rangle)|$ is maximal, i.e., $|\mathcal{F}(|\psi\rangle)| = 1$ then $|\psi\rangle$ is maximally entangled.

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However the condition $|\mathcal{F}(|\psi\rangle)| = 0$ (respectively $|\mathcal{F}(|\psi\rangle)| = 1$) is a sufficient but not necessary condition to have $|\psi\rangle$ unentangled (resp. maximally entangled).

Greenberger-Horne-Zeilinger Example

The *Greenberger-Horne-Zeilinger* state is

$$|GHZ_n\rangle := \frac{1}{\sqrt{2}}(|0_n\rangle + |1_n\rangle).$$

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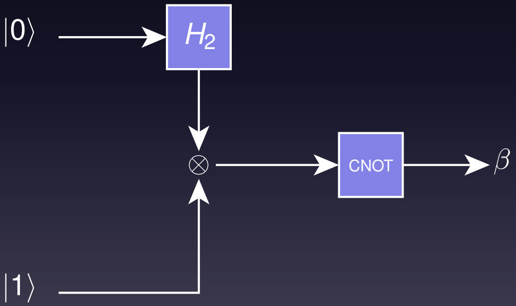
Furthermore, for all $n \geq 2$, the state $|GHZ_n\rangle$ is maximally entangled with respect to MW measure and $\mathcal{F}(|\psi\rangle) \neq 1$, thus also the inverse implication of Proposition 2 (that is, $\mathcal{F}(|\psi\rangle) = 1$ implies $|\psi\rangle$ is maximally entangled) in general is not true.

CIRCUITS

Bell's solution

1 qubit

Bell Circuit: entanglement of two elements of the canonical basis $|0\rangle$ and $|1\rangle$



Hadamard matrix

- The Hadamard matrix is $H_2 := \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ and its definition can be extended inductively to any n

$$H_{2^n} := \underbrace{H_2 \otimes \cdots \otimes H_2}_{n \text{ times}}$$

  is its 2^n-dimensional generalisation.

- Pauli's matrices:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

- Orthogonal projectors:

$$L := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad R := \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

CIRCUITS

our solution

Bell at any number of qubits

The cnot gate

With matrices of the previous slide cnot gate satisfies the equality

$$\text{cnot} := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} = L \otimes I_2 + R \otimes \sigma_x$$

while the columns of the matrix

$$B_4 := \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & -1 & 0 \end{pmatrix} = \text{cnot}(H_2 \otimes I_2)$$

are the coordinate vectors of the Bell states in the standard base.

Main Construction

We extend the above definitions of cnot and of B_2 to an arbitrary number of qubits as follows

Definition

For $n \geq 2$ we set

$$\text{cnot}_{2^n} := L \otimes I_{2^{n-1}} + R \otimes \underbrace{\sigma_x \otimes \cdots \otimes \sigma_x}_{n \text{ times}} \tag{2}$$

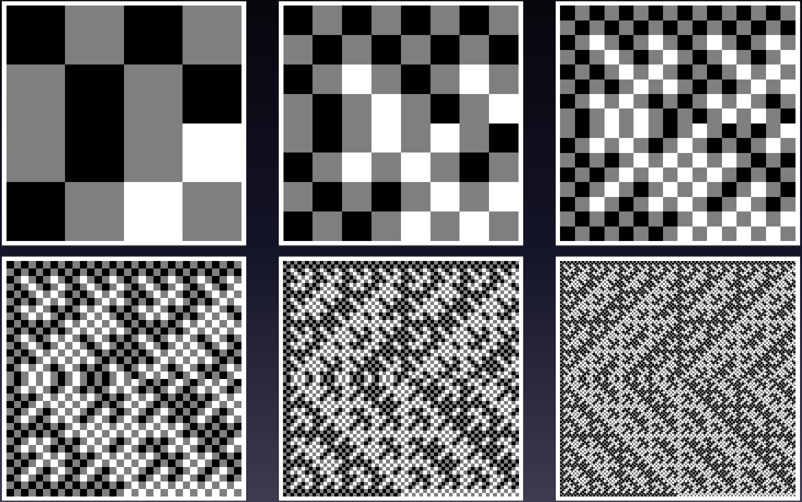
and

$$B_{2^n} := \text{cnot}_{2^n}(H_{2^{n-1}} \otimes I_2). \tag{3}$$

We define 2^n*-dimensional Bell state* any state

$$|b_k\rangle := B_{2^n}|k\rangle$$

where $k = 0, \ldots, 2^n - 1$ and $|k\rangle$ is the k-th element of the standard base of $\mathbb{C}^{2^n}$.



Remark

Remark that for couple of square matrices of the same dimension A and B, the matrix $L \otimes A + R \otimes B$ is classically referred to as the *Kronecker sum* (or *direct sum*) $A \oplus B$ of A and B.

As well as the Kronecker (tensor) product $\otimes$, the Kronecker sum of two unitary matrices is unitary.

Then by construction, the matrix B_{2^n} is product, tensor product and Kronecker sum of unitary matrices, and consequently, it is a unitary matrix.

As columns of a unitary matrix, the Bell states $|b_k\rangle$, with $k = 1, \ldots, 2^n$ form a complete orthonormal basis for $\mathbb{C}^{2^n}$.

Main Result

We show that the 2^n-dimensional Bell states are maximally entangled with respect to MW measure.

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We introduce the matrix

$$L_{2^n} := B_{2^n}^{\dagger} M_{2^n} B_{2^n}, \tag{4}$$

It is important, since provides a way to check if a state is entangled:

Lemma

If $|\langle\phi|L_{2^n}|\bar{\phi}\rangle| = 1$ and if $|\psi\rangle = B_{2^n}|\phi\rangle$ then $|\psi\rangle$ is maximally entangled with respect to the MW measure.

In particular, if $|\langle k|L_{2^n}|\bar{k}\rangle| = 1$, where $|k\rangle$ is the k-th element of the standard base, then the k-th Bell state is maximally entangled with respect to the MW measure.

Counter Example

There exist states ϕ which not satisfy $|\langle\phi|L_{2^n}|\bar{\phi}\rangle| = 1$ and such that $B_{2^n}|\phi\rangle$ is maximally entangled, an example of this phenomenon is given by the state $\phi = B_{2^n}^{-1}|GHZ_n\rangle$.

NUMBERS

Thue-Morse Sequence

Next result gives a closed formula for L_{2^n} and relates its diagonal elements to the *Thue-Morse sequence*,that is the binary sequence (τ_i) defined by the recursive relation

$$\tau_1 := 0$$
$$\tau_{2n} := 1 - \tau_n$$
$$\tau_{2n-1} := \tau_n$$

for all positive integers n. We notice that for all $n \geq 1$

$$\tau_{2^n+i} = 1 - \tau_i \quad \text{for all } i = 1, \ldots, 2^n. \tag{5}$$

Equality (5) characterises the Thue-Morse sequence via bitwise negation, indeed it states that every initial block of length 2^n, i.e, $\tau_1, \ldots, \tau_{2^n}$, is followed by a block of equal length that is its bitwise negation, i.e., $\tau_{2^n+1} = 1 - \tau_1, \ldots, \tau_{2^{n+1}} = 1 - \tau_{2^n}$.

Lemma

For all $n \geq 2$

$$L_{2^n} = -\underbrace{\sigma_z \otimes \cdots \otimes \sigma_z}_{n \text{ times}}. \tag{6}$$

Moreover L_{2^n} *is a diagonal matrix whose diagonal elements* $L_{2^n,i}$, $i = 1, \ldots, 2^n$, *satisfy*

$$L_{2^n,i} = 2\tau_i - 1, \qquad \text{for all } n = 1, \ldots, 2^n, \tag{7}$$

where (τ_i) *is the Thue-Morse sequence,*

Theorem

The 2^n*-dimensional Bell states are maximally entangled with respect to MW measure.*

Operational method

The matrix L_{2^n} provides an operative method for building maximally entangled states, indeed if $|\langle x^{\mathrm{t}} L_{2^n} x \rangle| = 1$ then $B_{2^n}|x\rangle$ is a maximally entangled state. Lemma 7 points out the intimate relation between L_{2^n} and the first 2^n terms of the (shifted) Thue-Morse sequence $1101\ 0010 \cdots$.

Indeed the i-th diagonal element of L_{2^n} is $2\tau_i - 1$, that is, the diagonal elements of L_{2n} are the finite sequence of digits $+1$ and -1 obtained by replacing with -1 every occurrence of 0 in the Thue-Morse sequence.

We proposed a family of unitary transformations generalising the cnot gate to an arbitrary number of qubits. We showed that a circuit composed by Walsh matrix and our general cnot gate yields a maximally entangled (with respect to MW measure) set of states, that we called *generalised Bell states*. In order to prove the validity of the method, we developed ad hoc entanglement criteria based on the definition of a suitable antilinear operator. The paper also contains a preliminary theoretical investigation of such operator, which turned out to be related with the celebrated Thue-Morse sequence.

Conclusion

- **generalisation of controlled unitary operations:** our results may suggest a way to further investigate the extension to other controlled unitary operations.
- **antiliear operators:** antilinear operators with zero expectation value on product states could represent a step towards an algebraic characterisation of the states with maximal MW measure.
- **generalised Thue-Morse sequences:** it could be interesting to better understand the intriguing relation between states with maximal MW measure and the Thue-Morse sequence.

Thanks!