Generalized golden ratios in ternary alphabets

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We explain the solution of this problem for three-letter alphabets.

Expansions

Given a finite *alphabet* $A = \{a_1 < \cdots < a_J\}$, $J \geq 2$, and a real *base* $q > 1$, by an **expansion** of a real number x we mean a sequence $c = (c_i) \in A^\infty$ satisfying the equality

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Example

If $A = \{0, 1\}$ and $q = 2$, then $U_{A,q}$ is the set of numbers $x \in [0, 1]$ except those of the form $x = m2^{-n}$ with two positive integers m, n, and $U'_{A,q}$ is the set of all sequences $(c_i) \in \{0, 1\}^\infty$, except those ending with 10^∞ or 01^∞.

Elementary characterization

Proposition

A sequence $c = (c_i) \in A^\infty$ belongs to $U'_{A,q}$ if and only the following conditions are satisfied:

$$\sum_{i=1}^{\infty} \frac{c_{n+i}}{q^i} < a_{j+1} - a_j \quad \textit{whenever} \quad c_n = a_j < a_J,$$

and (• • •)

$$\sum_{i=1}^{\infty} \frac{a_J - c_{n+i}}{q^i} < a_j - a_{j-1} \quad \textit{whenever} \quad c_n = a_j > a_1.$$

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- There exists a **critical base** p_A such that
  - there exist nontrivial unique expansions if $q > p_A$,
  - there are no nontrivial unique expansions if $q < p_A$.

Two-letter alphabets

Theorem

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Idea of the proof. We may assume by an affine transformation that $A = \{0, 1\}$. Then an expansion $(c_i) \in \{0, 1\}^\infty$ is unique $\Longleftrightarrow$

$$\sum_{i=1}^{\infty} \frac{c_{n+i}}{q^i} < 1 \quad \text{whenever} \quad c_n = 0,$$

and

$$\sum_{i=1}^{\infty} \frac{1 - c_{n+i}}{q^i} < 1 \quad \text{whenever} \quad c_n = 1.$$

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Every sequence satisfies these conditions if $q > 2$. The theorem follows by a similar but finer argument.

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For each fixed $m \geq 2$, we analyse the above characterization of unique expansions $(\bullet\bullet\bullet)$.
This yields an interesting property:

Lemma

If $(c_i) \neq 0^\infty$ is a unique expansion in a base $q \leq P_m := 1 + \sqrt{\frac{m}{m-1}}$, then (c_i) contains at most finitely many 0 digits.

Numerical tests

For each fixed $m = 2, 3, \ldots, 65536$ we were searching periodical nontrivial sequences $(c_i) \in \{0, 1, m\}^\infty$ satisfying the above given characterization $(\bullet\bullet\bullet)$ for as small bases $q > 1$ as possible. We have found essentially a unique minimal sequence in each case:

m	(c_i)
2	1^∞
3	$(m1)^\infty$
4	$(m1)^\infty$
5	$(mm1mm1m1)^\infty$
6	$(mm1)^\infty$
7	$(mm1)^\infty$
8	$(mm1)^\infty$
9	$(mmm1mm1)^\infty$

m	(c_i)
10	$(mmm1)^\infty$
11	$(mmm1)^\infty$
12	$(mmm1)^\infty$
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- $(m^{\mathbf{h}}1)^\infty$ with **h** $= [\log_2 m]$ for 65495 values;
- $(m^{\mathbf{h}}1)^\infty$ with **h** $= [\log_2 m] - 1$ for 33 values (**close to 2-powers**);
- seven **exceptional** values:

m	d
5	$(m^2 1 m^2 1 m 1)^\infty$
9	$(m^3 1 m^2 1)^\infty$
130	$(m^7 1 m^6 1)^\infty$
258	$(m^8 1 m^7 1)^\infty$
2051	$(m^{11} 1 m^{10} 1)^\infty$
4099	$(m^{12} 1 m^{11} 1)^\infty$
32772	$(m^{15} 1 m^{14} 1)^\infty$

Conjecture and proof

- It was natural to conjecture that p_m is the value such that the **minimal sequence** corresponding to m is univoque for $q > p_m$, but not univoque for $q < p_m$.

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- However, we had to solve the problem for all real values $m \geq 2$, and for this we had to understand the **general structure of the minimal sequences**, including the exceptional cases.
- We have observed that **no**ne of the minimal sequences contained **zero digits**.
- Next we have observed that all minimal sequences (c_i) satisfy the **lexicographic inequalities**

$$1c_2c_3\ldots \leq c_{n+1}c_{n+2}c_{n+3}\ldots \leq c_1c_2c_3\ldots$$

for all $n = 0, 1, \ldots,$ and we have conjectured that all these sequences played a role in our problem.

Main result

We consider expansions on the alphabets $A_m = \{0, 1, m\}$ with $m \geq 2$ in bases $q > 1$.

- For each $m \geq 2$ there exists a number p_m such that

$$q > p_m \Longrightarrow |U_{q,m}| > 2 \Longrightarrow q \geq p_m.$$

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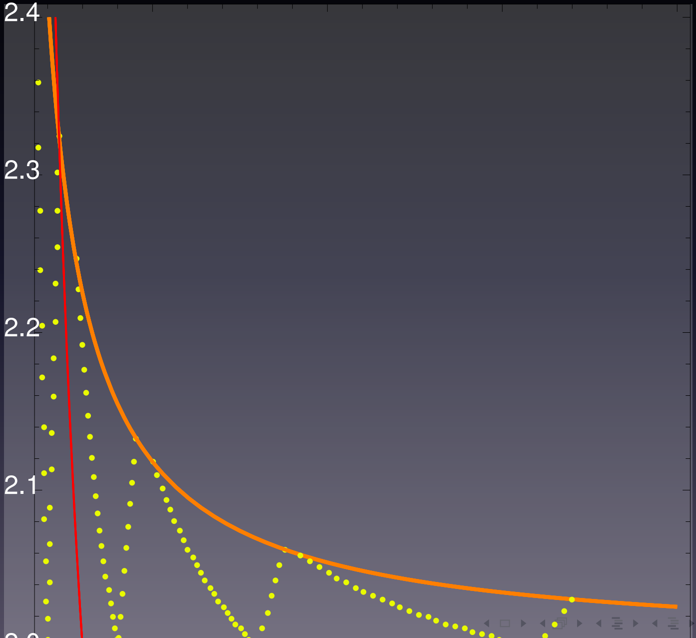
- Each connected component (m_d, M_d) of $[2, \infty) \setminus C$ has a point μ_d such that p decreases in (m_d, μ_d) and increases in (μ_d, M_d).

Intervals containing $m = 2^k$



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- The proof allows us to characterize those values of m for which $|U_{q,m}| > 2$ in the limiting case $q = p_m$.
- We do not know the Lebesgue measure and the Hausdorff dimension of the Cantor set $\{m \geq 2 \ : \ p_m = P_m\}$.