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EQUIGENERIC AND EQUISINGULAR FAMILIES OF CURVES ON SURFACES

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Abstract: We investigate the following question: let C be an integral curve contained in a smooth complex algebraic surface X; is it possible to deform C in X into a nodal curve while preserving its geometric genus?

We affirmatively answer it in most cases when X is a Del Pezzo or Hirzebruch surface (this is due to Arbarello and Cornalba, Zariski, and Harris), and in some cases when X is a $K3$ surface. Partial results are given for all surfaces with numerically trivial canonical class. We also give various examples for which the answer is negative.

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Lascia lente le briglie del tuo ippogrifo, o Astolfo,
e sfrena il tuo volo dove più ferve l'opera dell'uomo.
Però non ingannarmi con false immagini
ma lascia che io veda la verità
e possa poi toccare il giusto.

– Banco del mutuo soccorso, freely inspired by *Orlando furioso*

Introduction

Historically, the study of families of nodal irreducible plane curves (the so-called *Severi varieties*, named after [**34**]) was motivated by the fact that every smooth projective curve is birational to such a plane curve, and that plane curves should be easier to study since they are divisors. One can of course consider similar families of curves in any smooth algebraic surface and, as it has turned out, their study is rewarding whether one is interested in surfaces or in curves.

Let X be a smooth algebraic surface, and ξ an element of its Néron–Severi group. For $\delta \in \mathbf{Z}_{\geq 0}$, we denote by $V^{\xi,\delta}$ the family of integral curves in X of class ξ, whose singular locus consists of exactly δ nodes (i.e. δ ordinary double points; we call such curves *nodal*, or δ-*nodal*). These families are quite convenient to work with, being fairly well-understood from a deformation-theoretic point of view. For instance, when the canonical class K_X is non-positive this enables one to show that they are smooth of the expected dimension in the usual cases (when K_X is positive however, they tend to behave more wildly, see, e.g., [**10, 11**]). Moreover, they have been given a functorial definition in [**40**] (see also [**33**, §4.7.2]).

Yet, there is no definitive reason why one should restrict one's attention to curves having this particular kind of singularities (even when X is the projective plane), and it seems much more natural from a modular point of view to consider the families V^{ξ}_{g}, $g \in \mathbf{Z}_{\geq 0}$, of integral curves in X of class ξ that have geometric genus g (i.e. the normalizations of which have genus g). We call these families *equigeneric*. These objects have however various drawbacks, for instance their definition only makes sense set-theoretically, and accordingly there is no such thing as a local equigeneric deformation functor (i.e. one that would describe equigeneric deformations over an Artinian base).

It is a fact that every irreducible equigeneric family V of curves in X contains a Zariski open subset, all members of which have the same kind of singularities (families enjoying the latter property are called *equisingular*), and these singularities determine via their deformation theory the codimension V is expected to have in the universal family of all class ξ curves in X. This expected codimension is the lowest possible when the general member of V is nodal (in such a case, the expected codimension equals the number of nodes, which itself equals the difference between the arithmetic and geometric genera of members of V), so that it makes sense to consider the following.

Problem (A). Let C be an integral curve in X. Is it possible to deform C in X into a nodal curve while preserving its geometric genus?

One may rephrase this as follows: let ξ be the class of C in $\mathrm{NS}(X)$, $p_a(\xi)$ the arithmetic genus of curves having class ξ, g the geometric genus of C, and $\delta = p_a(\xi) - g$; is V^{ξ}_{g} contained in the Zariski closure of $V^{\xi,\delta}$? Observe that whenever the answer is affirmative, the Severi varieties $V^{\xi,\delta}$ provide a consistent way of understanding the equigeneric families V^{ξ}_{g}.

In any event, it is a natural question to ask what kind of singularities does the general member of a given family V^{ξ}_{g} have (besides, this question is important for enumerative geometry, see [**6, 16, 24**]). Closely related to this is the problem of determining whether a given equisingular family has the expected dimension. The actual dimension is always greater or equal to the expected dimension, and whenever they differ the family is said to be *superabundant.*

In this text, we provide an answer to various instances of Problem (A). Some of these answers are not new, see below for details and proper attributions.

Theorem (B).

(B.1) (Arbarello–Cornalba [**1, 2**], Zariski [**43**]). *Let* $X = \mathbf{P}^2$ *and* $L = \mathcal{O}_{\mathbf{P}^2}(1) \in \operatorname{Pic} X = \operatorname{NS}(X)$. *For integers* $n \geq 1$ *and* $0 \leq g \leq p_a(nL)$, *the general element of every irreducible component of* V_g^{nL} *is a nodal curve.*

(B.2) (Harris [**20**]). *Let* X *be a degree* d *Hirzebruch surface. For every effective class* $L \in \operatorname{Pic} X = \operatorname{NS}(X)$ *and integer* $0 \leq g \leq p_a(L)$, *the general member of every irreducible component of* V_g^L *is a nodal curve.*

(B.3) (Harris [**20**]). *Let* X *be a degree* d *Del Pezzo surface, and* $K_X \in \operatorname{Pic} X = \operatorname{NS}(X)$ *its canonical class. For integers* $n \geq 1$ *and* $0 \leq g \leq p_a(-nK_X)$, *the general element of every irreducible component of* $V_g^{-nK_X}$ *is nodal unless* $dn \leq 3$ *(it is at any rate immersed unless* $d = n = 1$ *and* $g = 0$*).*

(B.4) *Let* X *be a very general algebraic* $K3$ *surface,* L *the positive generator of* $\operatorname{Pic} X = \operatorname{NS}(X)$, *and write* $L^2 = 2p - 2$. *For* $p/2 < g \leq p_a(L) = p$, *the general element of every irreducible component of* V_g^L *is nodal.*

For integers $k \geq 1$ *and* $0 < g \leq p_a(kL)$, *the general element of every irreducible component of* V_g^{kL} *is immersed; if its normalization is non-trigonal*[1], *then it is actually nodal.*

(B.5) *Let* X *be an Enriques surface, and* $L \in \operatorname{Pic} X = \operatorname{NS}(X)$ *an effective class. For* $3 \leq g \leq p_a(L)$, *if* $[C] \in V_g^L$ *has a non-hyperelliptic normalization* $\bar{C}$, *then the general element of every component of* V_g^L *containing* C *is immersed. If moreover* $\bar{C}$ *has Clifford index* ≥ 5, *then* C *is nodal.*

(B.6) *Let* X *be an Abelian surface and* $\xi \in \operatorname{NS}(X)$. *For* $2 < g \leq p_a(\xi)$, *the general element of every irreducible component of* V_g^ξ *is immersed; if its normalization is non-trigonal, then it is actually nodal.*

[1] When $k = 1$, [**14**] provides a sufficient condition for a general element of V_g^L to have a non-trigonal normalization (see Corollary 4.7).

(Here a curve is said to be *immersed* if the differential of its normalization morphism is everywhere injective.)

In all cases within Theorem (B), the corresponding Severi varieties $V^{\xi,p_a(\xi)-g}$ are smooth and of the expected dimension (if non empty; non-emptiness is also known, except for Enriques and Abelian[2] surfaces). In addition, their irreducibility has been proven in the following cases: when X is the projective plane [**20, 21**], when X is a Hirzebruch surface [**39**], and when X is a Del Pezzo surface, $g = 0$, and $(d, n) \neq (1, 1)$ [**38**]; when X is a $K3$ surface, only a particular case is known [**12**]. These irreducibility properties transfer to the corresponding equigeneric families when Problem (A) admits a positive answer.

For surfaces with trivial canonical class one can formulate the following conjecture, which Theorem (B) only partly solves.

Conjecture (C). *Let X be a $K3$ (resp. Abelian) surface, and $\xi \in \mathrm{NS}(X)$. For $g > 0$ (resp. $g > 2$) the general element of every irreducible component of V^{ξ}_{g} is nodal.*

Note however that Problem (A) does not always have a positive solution. This happens for instance when X is a $K3$ (resp. Abelian) surface and $g = 0$ (resp. $g = 2$); the latter case is however somewhat exceptional, since the corresponding equigeneric families are 0-dimensional (see Subsection 4.2 for further discussion).

We give other instances, hopefully less exceptional, of Problem (A) having a negative solution in Section 5. This comes with various examples, some of them new, of equigeneric and equisingular families having superabundant behaviour.

Surfaces of general type are missing from our analysis, as their Severi varieties are notably not well-behaved and, especially, not keen to be studied using the techniques of the present text. For information about this case one may consult [**10, 11**].

Problem (A) was first studied (and solved) for the projective plane in the (19)80s by Arbarello and Cornalba [**1, 2**] (see also [**3**, Chap. XXI, §§8–10] for a unified treatment in English), and Zariski [**43**], with different approaches. The latter considers curves in surfaces as divisors and studies the deformations of their equations (we call this the *Cartesian* point of view), while the former see them as images of maps from smooth curves (we call this the *parametric* point of view). Harris generalized this result using the Cartesian theory in [**20**], thus obtaining as

[2] After the present text was completed, Knutsen, Lelli-Chiesa, and Mongardi (`arXiv:1503.04465`) proved the non-emptiness of $V^{\xi,p_a(\xi)-g}$ for ξ the numerical class of a polarization of type $(1, n)$ on an Abelian surface, and $2 \leq g \leq p_a(\xi)$.

particular cases parts (B.2) and (B.3) of the theorem above. There is however a subtle flaw in this text [**20**, Prop. 2.1] which has been subsequently worked around using the parametric theory in [**21**]. Apparently it had not been spotted before; we analyze it in detail in Subsection 3.3.

Note also that [**8**, Lem. 3.1] states Conjecture (C) for $K3$ surfaces as a result, but the proof reproduces the incomplete argument of [**20**, Prop. 2.1]; unfortunately, in this case the parametric approach does not provide a full proof either. We also point out that the result of Conjecture (C) for $K3$ surfaces is used in [**7**, proof of Thm. 3.5]; the weaker part (B.4) of Theorem (B) should however be enough for this proof, see [**16, 6**].

Eventually let us mention that the recent [**24**] by Kleiman and Shende provides an answer to Problem (A) for rational surfaces under various conditions. They use the Cartesian approach, while in the Appendix Tyomkin reproves the same results using the parametric approach.

We need arguments from both the parametric and Cartesian approaches here. The core of the parametric theory in the present text is Theorem 2.5, which is essentially due to Arbarello–Cornalba, Harris, and Harris–Morrison. Except for its part (B.4), Theorem (B) is a more or less direct corollary of Theorem 2.5; parts (B.5) and (B.6), which to the best of our knowledge appear here for the first time[3], still require additional arguments from a different nature, admittedly not new either (see Subsection 4.2).

The parametric approach is more modern in spirit, and arguably more agile, but although it enables one to give a full solution to Problem (A) for minimal rational surfaces, it does not provide a fully satisfactory way of controlling equisingular deformations of curves; somehow, it requires too much positivity of $-K_X$ (see, e.g., Remark 2.6), which explains why Theorem (B) is not optimal in view of Conjecture (C). For $K3$ surfaces, part (B.4) of Theorem (B), which is our main original contribution to the subject, is beyond what is possible today with the mere parametric approach; we obtain it along the Cartesian approach, with the new tackle of formulating it in terms of generalized divisors on singular curves (see Subsection 3.4), and with the help of additional results from Brill–Noether theory. This is yet not a definitive answer either, and we believe finer arguments are required in order to fully understand the subtleties of the question.

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[3]Part (B.6) of Theorem (B) has later on been used by Knutsen, Lelli-Chiesa, and Mongardi (`arXiv:1503.04465`) to prove Conjecture (C) for X an Abelian surfaces and ξ the numerical class of a polarization of type $(1, n)$.

The organization of the paper is as follows. In Section 1, we define the abstract notions of equigeneric and equisingular families of curves and specify our setup. In Section 2 we recall the relevant facts from the parametric deformation theory, which culminate in the already mentioned Theorem 2.5. Section 3 is devoted to Cartesian deformation theory, which involves the so-called equisingular and adjoint ideals of an integral curve with planar singularities. In Section 4 we apply the results of the two former sections in order to prove Theorem (B), and in Section 5 we gather examples in which the situation is not the naively expected one.

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1. Equigeneric and equisingular families of curves

We work over the field $\mathbf{C}$ of complex numbers.

1.1. General definitions. While the definition of equigenericity is rather straightforward, that of equisingularity is much more subtle, and requires some care. The definition given here is taken from Teissier [**36, 37**], who slightly modified the one originally introduced by Zariski (see [**36**, §5.12.2] for a comment on this). The two versions are anyway equivalent in our setting (explicited in Subsection 1.2) by [**37**, II, Thm. 5.3.1]. We invite the interested reader to take a look at [**15**] as well.

Let $p\colon \mathcal{C} \to Y$ be a flat family of reduced curves, where Y is any separated scheme.

Definition 1.1. The family $p\colon \mathcal{C} \to Y$ is equigeneric if

(i) Y is reduced,
(ii) the locus of singular points of fibres is proper over Y, and

(iii) the sum of the δ-invariants of the singular points of the fibre C_y is a constant function on $y \in Y$.

When p is proper, condition (iii) above is equivalent to the geometric genus of the fibres being constant on Y.

Definition 1.2. The family $p\colon \mathcal{C} \to Y$ is equisingular if there exist

(i) disjoint sections $\sigma_1, \dots, \sigma_n$ of p, the union of whose images contains the locus of singular points of the fibres, and
(ii) a proper and birational morphism $\varepsilon\colon \bar{\mathcal{C}} \to \mathcal{C}$, such that
  (a) the composition $\bar{p} := p \circ \varepsilon\colon \bar{\mathcal{C}} \to Y$ is flat,
  (b) for every $y \in Y$, the induced morphism $\varepsilon_y\colon \bar{C}_y \to C_y$ is a resolution of singularities (here $\bar{C}_y$ and C_y are the respective fibres of $\bar{p}$ and p over y), and
  (c) for $i = 1, \dots, n$, the induced morphism $\bar{p}\colon \varepsilon^{-1}(\sigma_i(Y)) \to Y$ is locally (on $\varepsilon^{-1}(\sigma_i(Y))$) trivial.

In Definition 1.1, the reducedness assumption on the base is an illustration of the fact that equigenericity cannot be functorially defined, unlike equisingularity. The following result of Zariski, Teissier, Diaz–Harris provides a more intuitive interpretation of equisingularity. Two germs of isolated planar curve singularities $(C_1, 0) \subset (\mathbf{C}^2, 0)$ and $(C_2, 0) \subset (\mathbf{C}^2, 0)$ are said to be topologically equivalent if there exists a homeomorphism $(\mathbf{C}^2, 0) \to (\mathbf{C}^2, 0)$ mapping $(C_1, 0)$ to $(C_2, 0)$ (cf. [**19**, I.3.4]). The corresponding equivalence classes are called *topological types.*

Theorem 1.3 ([**37**, II, Thm. 5.3.1], [**15**, Prop. 3.32])**.** *Let $p\colon \mathcal{C} \to Y$ be a flat family of reduced curves on a smooth surface X, i.e. $\mathcal{C} \subset X \times Y$, and p is induced by the second projection. We assume that $\mathcal{C}$ and Y are reduced separated schemes of finite type. Let $\Sigma \subset \mathcal{C}$ be the locus of singular points of fibres of p. If Σ is proper over Y the following two conditions are equivalent:*

(i) *the family $p\colon \mathcal{C} \to Y$ is locally equisingular in the analytic topology;*
(ii) *for each topological type of isolated planar curve singularity, all fibres over closed points of Y have the same number of singularities of that topological type.*

One then has the following result, often used without any mention in the literature. It is an application of the generic smoothness theorem.

Proposition 1.4 ([**37**, II, 4.2])**.** *Let $p\colon \mathcal{C} \to Y$ be an equigeneric family of reduced curves. There exists a dense Zariski-open subset $U \subset Y$ such that the restriction $\mathcal{C} \times_Y U \to U$ is equisingular.*

The latter result implies the existence, for any flat family $p\colon \mathcal{C} \to Y$ of reduced curves on a smooth surface X, with Y reduced separated and of finite type, of an equisingular stratification of Y in the Zariski topology. Indeed, the geometric genus of the fibres being a lower semicontinuous function on Y (see e.g. [**15**, §2]), our family restricts to an equigeneric one over a Zariski-open subset of Y, to which we can apply Proposition 1.4.

Eventually we need the following result of Teissier, which shows that equigenericity can be interpreted in terms of the existence of a simultaneous resolution of singularities.

Theorem 1.5 ([**37**, I, Thm. 1.3.2])**.** *Let $p\colon \mathcal{C} \to Y$ be a flat family of reduced curves, where $\mathcal{C}$ and Y are reduced separated schemes of finite type. If Y is normal, then the following two conditions are equivalent:*

(i) *the family $p\colon \mathcal{C} \to Y$ is equigeneric;*
(ii) *there exists a proper and birational morphism $\varepsilon\colon \bar{\mathcal{C}} \to \mathcal{C}$ such that $\bar{p} = p \circ \varepsilon$ is flat and, for every $y \in Y$, the induced morphism $\bar{C}_y \to C_y$ is a resolution of singularities of the fibre $C_y = p^{-1}(y)$.*

In addition, whenever it exists, the simultaneous resolution ε is necessarily the normalization of $\mathcal{C}$.

1.2. Superficial setting. We now introduce our set-up for the remaining of this paper.

Unless explicit mention to the contrary, X shall design a nonsingular projective connected algebraic surface. Given an element $\xi \in \mathrm{NS}(X)$ of the Néron–Severi group of X we let

$$\mathrm{Pic}^{\xi}(X) := \{L \in \mathrm{Pic}(X) \mid L \text{ has class } \xi\}.$$

The Hilbert scheme of effective divisors of X having class ξ, which we denote by Curves^{ξ}_X, is fibered over $\mathrm{Pic}^{\xi}(X)$

$$\mathrm{Curves}^{\xi}_X \to \mathrm{Pic}^{\xi}(X)$$

with fibres linear systems. We write $p_a(\xi)$ for the common arithmetic genus of all members of Curves^{ξ}_X. In case $q(X) := h^1(X, \mathcal{O}_X) = 0$, i.e. X is *regular*, Curves^{ξ}_X is a disjoint union of finitely many linear systems $|L|$, with L varying in $\mathrm{Pic}^{\xi}(X)$.

For any given integer δ such that $0 \leq \delta \leq p_a(\xi)$ there is a well defined, possibly empty, locally closed *subscheme* $V^{\xi,\delta} \subset \mathrm{Curves}^{\xi}_X$, whose geometric points parametrize reduced and irreducible curves having exactly δ nodes and no other singularities. These subschemes are defined functorially in a well known way [**40**] and will be called *Severi varieties*.

More generally, given a reduced curve C representing $\xi \in \mathrm{NS}(X)$, there is a functorially defined subscheme $\mathrm{ES}(C) \subset \mathrm{Curves}_X^\xi$ whose geometric points parametrize those reduced curves that have the same number of singularities as C for every equivalence class of planar curve singularity [**41**]. The restriction to $\mathrm{ES}(C)$ of the universal family of curves over Curves_X^ξ is the largest equisingular family of curves on X that contains C.

We will also consider, for any given integer g such that $0 \le g \le p_a(\xi)$, the locally closed *subset* $V_g^\xi \subset \mathrm{Curves}_X^\xi$ whose geometric points parametrize reduced and irreducible curves C having geometric genus g, i.e. such that their normalization has genus g. When $\delta = p_a(\xi) - g$ we have $V^{\xi,\delta} \subset V_g^\xi$.

There is also, for each $L \in \mathrm{Pic}^\xi(X)$, a subscheme $V_L^\delta = V^{\xi,\delta} \cap |L|$ of $|L|$, and a locally closed subset $V_{L,g} = V_g^\xi \cap |L|$. These are the natural objects to consider when X is regular.

2. A parametric approach

2.1. The scheme of morphisms. We briefly recall some facts from the deformation theory of maps with fixed target, which will be needed later on. Our main reference for this matter is [**33**, §3.4]; [**3**, Chap. XXI, §§8–10] may also be useful. We consider a fixed nonsingular projective n-dimensional variety Y.

Remark 2.1. We use the definition of *modular family*, as given in [**23**, p. 171]. For every $g \ge 0$, there is a modular family $\pi_g \colon \mathcal{D}_g \to S_g$ of smooth projective connected curves of genus g by [**23**, Thm. 26.4 and Thm. 27.2], and S_g has dimension $3g - 3 + a_g$, with a_g the dimension of the automorphism group of any genus g curve. Then, setting $M_g(Y)$ to be the relative Hom scheme $\mathrm{Hom}(\mathcal{D}_g/S_g, Y \times S_g/S_g)$ and $\mathcal{D}_g(Y) := \mathcal{D}_g \times_{S_g} M_g(Y)$, there is a modular family of morphisms from nonsingular projective connected curves of genus $g \ge 0$ to Y in the form of the commutative diagram

$$\begin{tikzcd} \mathcal{D}_g(Y) \arrow[r, "\Phi_g"] \arrow[d] & Y \times M_g(Y) \arrow[dl] \\ M_g(Y) & \end{tikzcd} \tag{2.1}$$

which enjoys properties (a), (b), (c) of [**23**, Def. p. 171] (note that here we declare two morphisms to be isomorphic if they are equal). Note that the scheme $M_g(Y)$ and diagram (2.1) are unique only up to an étale base change; nevertheless, with an abuse of language we call $M_g(Y)$ *the scheme of morphisms* from curves of genus g to Y.

Let

$$\phi\colon D \to Y$$

be a morphism from a nonsingular connected projective curve D of genus g and $[\phi] \in M_g(Y)$ a point parametrizing it. There is an exact sequence [**33**, Prop. 4.4.7]

$$(2.2)\quad 0 \to \mathrm{H}^0(D, \phi^*T_Y) \to T_{[\phi]}M_g(Y) \to \mathrm{H}^1(D, T_D) \to \mathrm{H}^1(D, \phi^*T_Y),$$

and it follows from [**25**, I.2.17.1] that

$$(2.3)\quad -K_Y \cdot \phi_*D + (n-3)(1-g) + \dim(\mathrm{Aut}(D)) \leq \dim_{[\phi]} M_g(Y).$$

We denote by $\mathrm{Def}_{\phi/Y}$ the deformation functor of ϕ with fixed target Y, as introduced in [**33**, §3.4.2]. Recall that N_ϕ, the *normal sheaf of* ϕ, is the sheaf of $\mathcal{O}_D$-modules defined by the exact sequence on D

$$(2.4)\qquad 0 \to T_D \to \phi^*T_Y \to N_\phi \to 0.$$

It controls the functor $\mathrm{Def}_{\phi/Y}$: one has $\mathrm{Def}_{\phi/Y}(\mathbf{C}[\varepsilon]) = \mathrm{H}^0(D, N_\phi)$, and $\mathrm{H}^1(D, N_\phi)$ is an obstruction space for $\mathrm{Def}_{\phi/Y}$; in particular, if R_ϕ is the complete local algebra which prorepresents $\mathrm{Def}_{\phi/Y}$ [**33**, Thm. 3.4.8], we have

$$\chi(N_\phi) \leq \dim(R_\phi) \leq h^0(N_\phi).$$

Using the exact sequence (2.4), one computes

$$(2.5)\qquad \chi(N_\phi) = \chi(\omega_D \otimes \phi^*\omega_Y^{-1}) = -K_Y \cdot \phi_*D + (n-3)(1-g),$$

hence

$$(2.6)\quad -K_Y \cdot \phi_*D + (n-3)(1-g) \leq \dim(R_\phi) \leq h^0(N_\phi) = \chi(N_\phi) + h^1(N_\phi).$$

In particular, R_ϕ is smooth of dimension $-K_Y \cdot \phi_*D + (n-3)(1-g)$ if and only if $\mathrm{H}^1(N_\phi) = 0$.

In analyzing the possibilities here, one has to keep in mind that N_ϕ can have torsion. In fact there is an exact sequence of sheaves of $\mathcal{O}_D$-modules

$$(2.7)\qquad 0 \to \mathcal{H}_\phi \to N_\phi \to \bar{N}_\phi \to 0,$$

where $\mathcal{H}_\phi$ is the torsion subsheaf of N_ϕ, and $\bar{N}_\phi$ is locally free. The torsion sheaf $\mathcal{H}_\phi$ is supported on the ramification divisor Z of ϕ, and it is zero if and only if $Z = 0$. Moreover, there is an exact sequence of locally free sheaves on D

$$0 \to T_D(Z) \to \phi^* T_Y \to \bar{N}_\phi \to 0. \tag{2.8}$$

The scheme $M_g(Y)$ and the functors $\mathrm{Def}_{\phi/Y}$ are related as follows. For each $[\phi] \in M_g(Y)$ we get by restriction a morphism from the prorepresentable functor $h_{\widehat{\mathcal{O}}_{M_g(Y),[\phi]}}$ to $\mathrm{Def}_{\phi/Y}$. Call ρ_ϕ this morphism. Its differential is described by the diagram:

$$\begin{tikzcd}
0 \arrow[r] & H^0(D,\phi^*T_Y) \arrow[r] \arrow[d] & T_{[\phi]}M \arrow[r] \arrow[d, "d\rho_\phi"] & H^1(D,T_D) \arrow[r] \arrow[d, equal] & H^1(D,\phi^*T_Y) \arrow[d, equal] \\
0 \arrow[r] & H^0(D,\phi^*T_Y)/H^0(D,T_D) \arrow[r] & H^0(D,N_\phi) \arrow[r] & H^1(D,T_D) \arrow[r] & H^1(D,\phi^*T_Y)
\end{tikzcd}$$

where the top row is the sequence (2.2) and the second row is deduced from the sequence (2.4). This diagram shows that $d\rho_\phi$ is surjective with kernel $H^0(D,T_D)$, whose dimension is equal to $\dim(\mathrm{Aut}(D))$. In particular, if $M_g(Y)$ is smooth at $[\phi]$, then $\mathrm{Def}_{\phi/Y}$ is smooth as well and $\dim(R_\phi) = \dim_{[\phi]}(M_g(Y)) - \dim(\mathrm{Aut}(D))$. This analysis is only relevant when $g = 0, 1$, because otherwise ρ_ϕ is an isomorphism.

2.2. Equigeneric families and schemes of morphisms. In view of the superficial situation set up in Subsection 1.2, we will often consider the case when ϕ is the morphism $\varphi\colon \bar{C} \to X$, where C is an integral curve in a smooth projective surface X, and φ is the composition of the normalization $\nu\colon \bar{C} \to C$ with the inclusion $C \subset X$; we may loosely refer to φ as the normalization of C. We then have

$$\bar{N}_\varphi \cong \varphi^*\omega_X^{-1} \otimes \omega_{\bar{C}}(-Z)$$

by the exact sequence (2.8). The embedded curve C is said to be *immersed* if the ramification divisor Z of φ is zero; in this case, we may also occasionally say that C has no (generalized) cusps.

The following result is based on a crucial observation by Arbarello and Cornalba [**2**, p. 26].

Lemma 2.2. *Let B be a semi-normal*[4] *connected scheme, $0 \in B$ a closed point, $\pi\colon \mathcal{D} \to B$ a flat family of smooth projective irreducible curves of genus g, and*

$$\begin{tikzcd}
\mathcal{D} \arrow[r, "\Phi"] \arrow[d, "\pi"'] & X \times B \arrow[dl, "\mathrm{pr}_2"] \\
B &
\end{tikzcd}$$

[4]We refer to [**25**, §I.7.2] for background on this notion.

a family of morphisms. We call D_0 the fibre of $\mathcal{D}$ over $0 \in B$, $\phi_0 \colon D_0 \to X$ the restriction of Φ, which we assume to be birational on its image, and ξ the class of $\phi_0(D_0)$ in $\mathrm{NS}(X)$*.*

(i) *The scheme-theoretic image $\Phi(\mathcal{D})$ is flat over B. This implies that there are two classifying morphisms p and q from B to $M_g(X)$ and* Curves_X^ξ *respectively, with differentials*

$$d\rho_{\phi_0} \circ dp_0 \colon T_{B,0} \to \mathrm{H}^0(D_0, N_{\phi_0}) \text{ and } dq_0 \colon T_{B,0} \to \mathrm{H}^0(\phi_0(D_0), N_{\phi_0(D_0)/X}).$$

(ii) *The inverse image by $d\rho_{\phi_0} \circ dp_0$ of the torsion* $\mathrm{H}^0(D_0, \mathcal{H}_{\phi_0}) \subset \mathrm{H}^0(D_0, N_{\phi_0})$ *is contained in the kernel of dq_0.*

Proof: The morphism $\varpi = \mathrm{pr}_2 \colon \Phi(\mathcal{D}) \to B$ is a well-defined family of codimension 1 algebraic cycles of X in the sense of [**25**, I.3.11]. Since B is semi-normal, it follows from [**25**, I.3.23.2] that ϖ is flat.

Given a non-zero section $\sigma \in H^0(D_0, N_{\phi_0})$, the first order deformation of ϕ_0 defined by σ can be described in the following way: consider an affine open cover $\{U_i\}_{i \in I}$ of C_0, and for each $i \in I$ consider a lifting $\theta_i \in \Gamma(U_i, \phi_0^* T_X)$ of the restriction $\sigma_{|U_i}$. Each θ_i defines a morphism

$$\psi_i \colon U_i \times \mathrm{Spec}(\mathbf{C}[\varepsilon]) \to X$$

extending $\phi_{0|U_i} \colon U_i \to X$. The morphisms ψ_i are then made compatible after gluing the trivial deformations $U_i \times \mathrm{Spec}(\mathbf{C}[\varepsilon])$ into the first order deformation of D_0 defined by the coboundary $\partial(\sigma) \in H^1(C_0, T_{C_0})$ of the exact sequence (2.4). In case $\sigma \in \mathrm{H}^0(D_0, \mathcal{H}_{\phi_0})$, everyone of the maps ψ_i is the trivial deformation of $\sigma_{|U_i}$ over an open subset. This implies that the corresponding first order deformation of ϕ_0 leaves the image fixed, hence the vanishing of $dq_0(\sigma)$. □

Lemma 2.3. *Let $m_0 \in M_g(X)$ be a general point of an irreducible component of $M_g(X)$, and $\phi_0 \colon D_0 \to X$ the corresponding morphism. Assume that ϕ_0 is birational onto its image $C_0 := \phi_0(D_0)$, and that $[C_0] \in V_g^\xi$. Then $[C_0]$ belongs to a unique irreducible component of V_g^ξ and*

$$\dim_{[C_0]} V_g^\xi = \dim R_{\phi_0} = \dim_{m_0} M_g(X) - \dim(\mathrm{Aut}\, D_0),$$

where R_{ϕ_0} is the complete local $\mathbf{C}$-algebra that prorepresents $\mathrm{Def}_{\phi_0/X}$*.*

Proof: Consider the reduced scheme $M_{\mathrm{red}} := M_g(X)_{\mathrm{red}}$, and let $\tilde{M}$ be its semi-normalization. Let

$$\begin{tikzcd} \mathcal{D}_{\tilde{M}} \arrow[r, "\Phi_{\tilde{M}}"] \arrow[d] & X \times \tilde{M} \arrow[dl] \\ \tilde{M} & \end{tikzcd}$$

be the pullback of the modular family (2.1). Then we have a diagram

$$\begin{tikzcd} \Phi_{\tilde M}(\mathcal{D}_{\tilde M}) \arrow[r, hook] \arrow[d, "\pi"'] & X \times \tilde M \arrow[dl] \\ \tilde M & \end{tikzcd}$$

where $\Phi_{\tilde M}(\mathcal{D}_{\tilde M})$ is the scheme-theoretic image. The morphism π is flat by Lemma 2.2, and therefore we have an induced functorial morphism $\Psi\colon \tilde M \to V_g^\xi$.

Suppose that $\Psi(m_1) = \Psi(m_0) = [C_0]$ for some $m_1 \in \tilde M$. Then m_0 and m_1 parametrize the same morphism up to an automorphism of the source D_0. By property (a) of modular families in [**23**], this implies that the fibres of Ψ have the same dimension as $\operatorname{Aut} D_0$, and therefore that

$$\dim_{m_0} M_g(X) - \dim(\operatorname{Aut} D_0) \le \dim_{[C]} V_g^\xi.$$

On the other hand, consider the normalization map $\bar V_g^\xi \to V_g^\xi$, and the pull-back to $\bar V_g^\xi$ of the universal family of curves over Curves_X^ξ. It has a simultaneous resolution of singularities $\bar{\mathcal{U}} \to \bar V_g^\xi$ by Theorem 1.5, which comes with a family of morphisms $N\colon \bar{\mathcal{U}} \to X \times \bar V_g^\xi$ over $\bar V_g^\xi$. By property (c) of the modular family $M_g(X)$ in [**23**], this implies that there exist an étale surjective $\eta\colon W \to \bar V_g^\xi$ and a morphism $w\colon W \to M_g(X)$ such that $\bar{\mathcal{U}}_W := \bar{\mathcal{U}} \times_{\bar V_g^\xi} W$ fits in the Cartesian diagram

$$\begin{tikzcd} \bar{\mathcal{U}}_W \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{D}_g \arrow[d] \\ W \arrow[r, "w"] & M_g(X) \end{tikzcd}$$

where the left vertical map is the pullback of N. The map $W \to M_g(X)$ is generically injective because the universal family of curves over V_g^ξ is nowhere isotrivial. Moreover, its image is transverse at every point m (corresponding to a morphism ϕ) to the subvariety of $M_g(X)$ parametrizing morphisms gotten by composing ϕ with an automorphism of its source. This implies that, for $c_0 \in \eta^{-1}([C_0])$

$$\dim_{[C_0]} V_g^\xi = \dim_{c_0} W \le \dim_{m_0} M_g(X) - \dim(\operatorname{Aut} D_0).$$

It is then clear that $[C_0]$ belongs to a unique irreducible component of V_g^ξ. $\square$

Remark 2.4. It can happen that $M_g(X)$ is non-reduced. For an example of such a situation, consider the pencil $|L|$ constructed in Example 5.1 below (we use the notations introduced therein), and let $C \subset X$ be a general element of $V_{L,9}$, which is open and dense in $|L|$. The curve C has one ordinary cusp s and no further singularity; we let $s' \in \bar{C}$ be the unique ramification point of the normalization $\varphi\colon \bar{C} \to X$.

One has $\chi(N_\varphi) = -8+8 = 0$ whereas $\dim_{[\varphi]} M_9(X) = 1$. The torsion part of N_φ is the skyscraper sheaf $\underline{\mathbf{C}}_{s'}$, and accordingly $h^0(\mathcal{H}_\varphi) = 1$. One has $h^0(\omega_{\bar{C}} \otimes \varphi^*\omega_X^{-1}) = 1$, and the unique divisor in $|\omega_{\bar{C}} \otimes \varphi^*\omega_X^{-1}|$ contains s' (with multiplicity 4), so that

$$h^0(\bar{N}_\varphi) = h^0\big(\omega_{\bar{C}} \otimes \varphi^*\omega_X^{-1}(-s')\big) = h^0\big(\omega_{\bar{C}} \otimes \varphi^*\omega_X^{-1}\big) = 1.$$

We then deduce from the exact sequence (2.7) that $h^0(N_\varphi) = 2$ and $h^1(N_\varphi) = 2$.

2.3. Conditions for the density of nodal (resp. immersed) curves. The following result is essentially contained in [**20, 21**]; the idea of condition (c) therein comes from [**1**].

Theorem 2.5. *Let $V \subset V_g^\xi$ be an irreducible component and let $[C] \in V$ be a general point, with normalization $\varphi\colon \bar{C} \to X$.*

(i) *Assume that the following two conditions are satisfied:*
   (a) $\omega_{\bar{C}} \otimes \varphi^*\omega_X^{-1}$ *is globally generated;*
   (b) $\dim(V) \geq h^0\big(\bar{C}, \omega_{\bar{C}} \otimes \varphi^*\omega_X^{-1}\big)$.

   Then C is immersed, i.e. all its singularities consist of (possibly non transverse) linear branches.

(ii) *Assume in addition that the following condition is satisfied:*
   (c) *the line bundle $\omega_{\bar{C}} \otimes \varphi^*\omega_X^{-1}$ separates any (possibly infinitely near) 3 points, i.e.*

   $$h^0\big(\bar{C}, \omega_{\bar{C}} \otimes \varphi^*\omega_X^{-1}(-A)\big) = h^0\big(\bar{C}, \omega_{\bar{C}} \otimes \varphi^*\omega_X^{-1}\big) - 3$$

   for every effective divisor A of degree 3 on $\bar{C}$.

   Then C is nodal. Equivalently $V \subset \overline{V^{\xi,\delta}}$, with $\delta = p_a(\xi) - g$.

Proof: For simplicity we give the proof in the case $g \geq 2$. Assume by contradiction that the curve C has (generalized) cusps. This is equivalent to the fact that $Z \neq 0$, where $Z \subset \bar{C}$ is the zero divisor of the differential of φ. By generality, $[C]$ is a smooth point of V, so we may (and will) assume without loss of generality that V is smooth. As in the proof of Lemma 2.3, it follows from Theorem 1.5 that there is a simultaneous resolution of singularities

$$\begin{tikzcd} \bar{\mathcal{C}} \arrow[r, "\Phi"] \arrow[dr, "\bar{\pi}"'] & \mathcal{C} \arrow[d, "\pi"] \\ & V \end{tikzcd}$$

of the universal family of curves over V. This is a deformation of the morphism φ, so we have a characteristic morphism $p\colon V \to M_g(X)$. The differential

$$dp_{[\varphi]}\colon T_{[C]}V \to \mathrm{H}^0(\bar{C}, N_\varphi)$$

is injective because to every tangent vector $\theta \in T_{[C]}V$ corresponds a non-trivial deformation of C. On the other hand, it follows from Lemma 2.2 that the intersection $\operatorname{Im} dp_{[\varphi]} \cap \mathrm{H}^0(\bar{C}, \mathcal{H}_\varphi)$ is trivial. Eventually, we thus have

$$\dim V = \dim T_{[C]}V \leq h^0(\bar{C}, \bar{N}_\varphi) = h^0\big(\omega_{\bar{C}} \otimes \varphi^*\omega_X^{-1}(-Z)\big).$$

By assumption (a), this implies $\dim V < h^0\big(\omega_{\bar{C}} \otimes \varphi^*\omega_X^{-1}\big)$, a contradiction. This proves (i).

Assume next that (c) is also satisfied and, by contradiction, that C is not nodal. We shall show along the lines of [**1**, pp. 97–98] that it is then possible to deform C into curves with milder singularities, which contradicts the generality of C in V and thus proves (ii). First note that since C is immersed by (i), one has $N_\varphi = \bar{N}_\varphi$, so that condition (b) implies the smoothness of the scheme of morphisms $M_g(X)$ at a point $[\varphi]$ parametrizing φ, the tangent space at this point being

$$\mathrm{H}^0(N_\varphi) = \mathrm{H}^0\big(\omega_{\bar{C}} \otimes \varphi^*\omega_X^{-1}\big).$$

The assumption that C is not nodal means that there is a point $x \in C$ at which C has either (1) (at least) 3 local branches meeting transversely, or (2) (at least) 2 local branches tangent to each other. In case (1), there are three pairwise distinct points $p, q, r \in \bar{C}$ such that $\varphi(p) = \varphi(q) = \varphi(r) = x$. It follows from condition (c) that there exists a section $\sigma \in \mathrm{H}^0(N_\varphi)$ such that $\sigma(p) = \sigma(q) = 0$ and $\sigma(r) \neq 0$. Such a section corresponds to a first-order deformation (hence, by smoothness, to an actual deformation) of φ leaving both $\varphi(p)$ and $\varphi(q)$ fixed while moving $\varphi(r)$: it therefore turns the triple point constituted at x by the 3 local branches of C under consideration into 3 nodes. In particular it is not equisingular, a contradiction.

In case (2), there are 2 distinct points $p, q \in \bar{C}$, such that $\varphi(p) = \varphi(q)$, and $\operatorname{im}(d\varphi_p) = \operatorname{im}(d\varphi_q)$, and it follows from condition (c) that there exists a section $\sigma \in \mathrm{H}^0(N_\varphi)$ such that $\sigma(p) = 0$ and $\sigma(q) \notin \operatorname{im}(d\varphi_p)$. The corresponding deformation of C leaves $\varphi(p)$ fixed and moves $\varphi(q)$ in a direction transverse to the common tangent to the 2 local branches of C under consideration (if the 2 branches of C are simply tangent, the tacnode they constitute at x is turned into 2 nodes). It is therefore not equisingular either, a contradiction also in this case. □

In many cases the conditions considered in Theorem 2.5 are not satisfied: this tends to happen when ω_X^{-1} is not positive enough.

- Clearly enough, (a) does not hold in general. Critical occurrences of this phenomenon are to be observed for rational curves on $K3$ surfaces (Remark 4.10) and for anticanonical rational curves on a degree 1 Del Pezzo surface (Remark 4.3). In these two situations, the conclusion of Theorem 2.5 is not true in general.
- There can also be actual obstructions to deform the normalization of the general member of V and then (b) does not hold, see Remark 2.4 and Example 5.1. The conclusion of Theorem 2.5 is not true for this example.

Remark 2.6. Condition (c) of Theorem 2.5, albeit non-redundant (see Warning 3.8), is too strong, as the following example shows. Let (X, L) be a very general primitively polarized $K3$ surface, with $L^2 = 12$. It follows from Proposition 4.8 that the general element C of every irreducible component of $V_{L,4}$ is nodal. On the other hand, having genus 4 the curve $\bar{C}$ is trigonal, i.e. there exists an effective divisor of degree 3 on $\bar{C}$ such that $h^0\big(\mathcal{O}_{\bar{C}}(A)\big) = 2$, whence

$$h^0\big(\omega_{\bar{C}}(-A)\big) = 2 > h^0\big(\omega_{\bar{C}}\big) - 3 = 1,$$

and condition (c) does not hold.

A finer analysis is required in order to get the right condition. The approach described in Section 3 might provide a possibility for doing so.

The following result provides a convenient way to apply Theorem 2.5.

Corollary 2.7. *Assume that $V \subset V_g^\xi$ is an irreducible component and let $[C] \in V$ be general. If $\omega_{\bar{C}} \otimes \varphi^*\omega_X^{-1}$ is non-special and base-point-free then C has no cusps. If moreover*

$$\deg(\omega_{\bar{C}} \otimes \varphi^*\omega_X^{-1}) \geq 2g + 2 \tag{2.9}$$

then C is nodal.

Proof: Condition (a) of Theorem 2.5 is satisfied by hypothesis. The non-speciality of $\omega_{\bar{C}} \otimes \varphi^*\omega_X^{-1}$ implies that

$$\chi(N_\varphi) \geq h^0(\omega_{\bar{C}} \otimes \varphi^*\omega_X^{-1})$$

and therefore also condition (b) is satisfied, thanks to (2.6). The last assertion is clear because (2.9) implies that condition (c) is also satisfied. $\square$

3. A Cartesian approach

The situation and notations are as set-up in Subsection 1.2.

3.1. Ideals defining tangent spaces. Let C be a reduced curve in the surface X. We consider the sequence of sheaves of ideals of $\mathcal{O}_C$

$$J \subseteq I \subseteq A \subseteq \mathcal{O}_C,$$

where:

(1) J is the Jacobian ideal: it is locally generated by the partial derivatives of a local equation of C;
(2) I is the equisingular ideal [**41**]: it does not have any non-deformation-theoretic interpretation;
(3) A is the adjoint ideal: it is the conductor $\mathcal{C}_\nu := \mathrm{Hom}_{\mathcal{O}_C}(\nu_*\mathcal{O}_{\bar{C}}, \mathcal{O}_C)$ of the normalization $\nu\colon \bar{C} \to C$ of C.

Being ν birational, $\mathcal{C}_\nu$ is the annihilator ideal $\mathrm{Ann}_{\mathcal{O}_C}\big(\nu_*\mathcal{O}_{\bar{C}}/\mathcal{O}_C\big)$. It follows that $A \subset \mathcal{O}_C$ is also a sheaf of ideals of $\nu_*\mathcal{O}_{\bar{C}}$, which implies that there exists an effective divisor $\bar{\Delta}$ on $\bar{C}$ such that

$$A \cong \nu_*\big(\mathcal{O}_{\bar{C}}(-\bar{\Delta})\big). \tag{3.1}$$

Moreover, we have

$$\omega_{\bar{C}} = \nu^*(\omega_C) \otimes \mathcal{O}_{\bar{C}}(-\bar{\Delta}). \tag{3.2}$$

Lemma 3.1. *For $i = 0, 1$, one has*

$$\mathrm{H}^i\big(C, A \otimes \mathcal{O}_C(C)\big) \cong \mathrm{H}^i\big(\bar{C}, \omega_{\bar{C}} \otimes \varphi^*\omega_X^{-1}\big),$$

where $\varphi\colon \bar{C} \to X$ is the composition of the normalization map ν with the inclusion $C \subset X$.

Proof: By (3.1) and the projection formula, we have

$$\begin{aligned}\mathrm{H}^0\big(C, \mathcal{O}_C(C) \otimes A\big) &= \mathrm{H}^0\big(C, \mathcal{O}_C(C) \otimes \nu_*\big(\mathcal{O}_{\bar{C}}(-\bar{\Delta})\big)\big)\\ &= \mathrm{H}^0\big(\bar{C}, \nu^*\mathcal{O}_C(C) \otimes \mathcal{O}_{\bar{C}}(-\bar{\Delta})\big).\end{aligned}$$

By (3.2) and the adjunction formula $\omega_C = \mathcal{O}_C(C) \otimes \omega_X$, we have

$$\nu^*\mathcal{O}_C(C) \otimes \mathcal{O}_{\bar{C}}(-\bar{\Delta}) = \nu^*\omega_C \otimes \varphi^*\omega_X^{-1} \otimes \mathcal{O}_{\bar{C}}(-\bar{\Delta}) = \omega_{\bar{C}} \otimes \varphi^*\omega_X^{-1},$$

and the statement follows in the case $i = 0$. For the second identity, observe that $R^1\nu_*(\omega_{\bar{C}} \otimes \varphi^*\omega_X^{-1}) = 0$, hence

$$\mathrm{H}^1\big(C, \mathcal{O}_C(C) \otimes A\big) = \mathrm{H}^1\big(C, \nu_*(\omega_{\bar{C}} \otimes \varphi^*\omega_X^{-1})\big) = \mathrm{H}^1\big(\bar{C}, \omega_{\bar{C}} \otimes \varphi^*\omega_X^{-1}\big)$$

by Leray's spectral sequence. $\square$

Let $\xi \in \mathrm{NS}(X)$ be the class of C. From the functorial identification of $T_{[C]}\mathrm{Curves}_X^\xi$ with $\mathrm{H}^0\big(C, \mathcal{O}_C(C)\big)$ we may deduce the sequence of inclusions

$$(3.3)\quad \mathrm{H}^0\big(C, J\otimes\mathcal{O}_C(C)\big) \subseteq \mathrm{H}^0\big(C, I\otimes\mathcal{O}_C(C)\big) \subseteq \mathrm{H}^0\big(C, A\otimes\mathcal{O}_C(C)\big) \subseteq T_{[C]}\mathrm{Curves}_X^\xi,$$

which has the following deformation-theoretic interpretation.

Proposition 3.2 ([**15**, Prop. 4.19])**.** (i) $\mathrm{H}^0\big(C, J\otimes\mathcal{O}_C(C)\big)$ *is the tangent space at* $[C]$ *to the subscheme of* Curves_X^ξ *of formally locally trivial deformations of* C*.*

(ii) $\mathrm{H}^0\big(C, I\otimes\mathcal{O}_C(C)\big)$ *is the tangent space at* $[C]$ *to* $\mathrm{ES}(C)$*. In particular,*

$$\dim_{[C]} \mathrm{ES}(C) \leq h^0\big(C, I\otimes\mathcal{O}_C(C)\big).$$

(iii) $\mathrm{H}^0(C, A\otimes\mathcal{O}_C(C))$ *contains the reduced tangent cone to* $V^\xi_{g(C)}$ *at* $[C]$*. In particular,*

$$\dim_{[C]} V^\xi_{g(C)} \leq h^0\big(C, A\otimes\mathcal{O}_C(C)\big) = h^0\big(\bar{C}, \omega_{\bar{C}}\otimes\varphi^*\omega_X^{-1}\big).$$

As is the case for $\mathrm{ES}(C)$, the subscheme of Curves_X^ξ of formally locally trivial deformations of C is functorially defined [**40**]; in contrast, V^ξ_g is only set-theoretically defined.

Proof: (i) is based on results of Artin and Schlessinger respectively; since we will not use this, we refer to [**15**] for the precise references. (ii) follows from [**37, 41**], as explained in [**28**, Prop. 3.3.1]. (iii) stems from [**36**] (the last equality comes from Lemma 3.1). □

The next result is conceptually important: it explains why one would envisage an affirmative answer to Problem (A) in the first place.

Proposition 3.3 ([**43**])**.** *Let* (C,p) *be a reduced planar curve germ, and consider the local ideals* $I_p \subseteq A_p \subseteq \mathcal{O}_{C,p}$*. Then* $I_p = A_p$ *if and only if* p *is a node.*

This also occurs as [**28**, Thm. 3.3.2] and [**15**, Lem. 6.3], where enlightening proofs are provided.

3.2. Effective computations. Next, we collect some results enabling one to compute in practice the ideals A and I which will be needed in the sequel.

Lemma 3.4 ([**35**, II.6–7])**.** *Let $C \subset X$ be a reduced curve in a smooth surface. Consider a finite chain of birational morphisms*

$$X_{s+1} \xrightarrow{\varepsilon_s} X_s \to \cdots \to X_2 \xrightarrow{\varepsilon_1} X_1 = X$$

such that each ε_r is the blow-up of a single closed point $q_r \in X_r$, with exceptional divisor E_r ($1 \le r \le s$). Let furthermore

- $\varepsilon_{s,r} = \varepsilon_r \circ \cdots \circ \varepsilon_s \colon X_{s+1} \to X_r$,
- *C_r be the proper transform of C in X_r ($C_1 = C$), and*
- *m_r be the multiplicity of C_r at $q_r \in X_r$.*

If the proper transform of C in X_{s+1} is smooth, then the adjoint ideal of C is

$$A_C = (\varepsilon_{s,1})_* \mathcal{O}_{X_s}\Big(-(m_1-1)\varepsilon_{s,2}^* E_1 - \cdots \\ \cdots - (m_{s-1}-1)\varepsilon_{s,s}^* E_{s-1} - (m_s - 1)E_s\Big) \otimes_{\mathcal{O}_X} \mathcal{O}_C.$$

As far as the equisingular ideal is concerned, we shall only need two special instances of [**19**, Prop. 2.17], and refer to loc. cit. §II.2.2 for further information.

Recall that a polynomial $f = \sum_{(\alpha,\beta)\in\mathbf{N}^2} a_{\alpha,\beta} x^\alpha y^\beta$ is said to be *quasi-homogeneous* if there exist positive integers w_1, w_2, d such that

$$\forall\, (\alpha,\beta) \in \mathbf{N}^2, \quad a_{\alpha,\beta} \neq 0 \implies w_1\alpha + w_2\beta = d.$$

In such a case, $(w_1, w_2; d)$ is called the *type* of f. An isolated planar curve singularity $(C, 0)$ is said to be *quasihomogeneous* if it is analytically equivalent to the singularity at the origin of a plane affine curve defined by a quasihomogeneous polynomial f, i.e. if the complete local ring $\hat{\mathcal{O}}_{C,0}$ is isomorphic to $\mathbf{C}[[x,y]]/\langle f\rangle$.

Lemma 3.5 ([**19**, Prop. 2.17])**.** *Let $f \in \mathbf{C}[x,y]$ be a quasihomogeneous polynomial of type $(w_1, w_2; d)$, and consider the affine plane curve C defined by f. If C has an isolated singularity at the origin 0, then the local equisingular ideal of C at 0 is*

$$I = J + \langle x^\alpha y^\beta \mid w_1\alpha + w_2\beta \ge d\rangle,$$

in the local ring $\mathcal{O}_{C,0}$ (where J denotes as usual the Jacobian ideal $\langle \partial_x f, \partial_y f\rangle$).

Recall that simple curve singularities are those defined by one of the following equations:

$$\begin{aligned} A_\mu : &\quad y^2 - x^{\mu+1} = 0 \quad (\mu \geq 1), \\ D_\mu : &\quad x(y^2 - x^{\mu-2}) = 0 \quad (\mu \geq 4), \\ E_6 : &\quad y^3 - x^4 = 0, \\ E_7 : &\quad y(y^2 - x^3) = 0, \\ E_8 : &\quad y^3 - x^5 = 0. \end{aligned}$$

Simple singularities are quasihomogeneous, and one obtains as a corollary of Lemma 3.5 that *the equisingular ideal I of a simple singularity equals its Jacobian ideal J.* This means that simple singularities do not admit non topologically trivial equisingular deformations.

Example 3.6 (Double points)**.** Any double point of a curve is a simple singularity of type A_μ, $\mu \geq 1$. At such a point p, the adjoint and equisingular ideals are respectively

$$A = \langle y, x^{\lfloor \frac{\mu+1}{2} \rfloor} \rangle \quad \text{and} \quad I = \langle y, x^\mu \rangle$$

in the local ring of the curve at p.

Example 3.7 (Ordinary m-uple points)**.** Let m be a positive integer. An ordinary m-uple point of a curve is analytically equivalent to the origin in an affine plane curve defined by an equation

$$f(x,y) = f_m(x,y) + \tilde{f}(x,y) = 0, \tag{3.4}$$

where f_m is a degree m homogeneous polynomial defining a smooth subscheme of $\mathbf{P}^1$, and $\tilde{f}$ is a sum of monomials of degree $> m$; such a polynomial f is said to be *semi-homogeneous.* In particular, [**19**, Prop. 2.17] applies to this situation, and the adjoint and equisingular ideals at the origin of the curve defined by(3.4) are respectively

$$A = \langle x^\alpha y^\beta \mid \alpha + \beta \geq m-1 \rangle \quad \text{and} \quad I = \langle \partial_x f, \partial_y f \rangle + \langle x^\alpha y^\beta \mid \alpha + \beta \geq m \rangle$$

in the local ring of the curve (A is computed with Lemma 3.4).

3.3. Pull-back to the normalization. In this subsection we discuss the possibility of proving Theorem 2.5 by "lifting" the sequence of tangent spaces (3.3) on the normalization of a general member of a maximal irreducible equigeneric family. First of all, we would like to point out a fallacy: we explain below why a certain line of argumentation does not enable one to remove assumption (c) in (ii). This incomplete argumentation is used in the proofs of [**20**, Prop. 2.1] (last paragraph of p. 448) and of [**8**, Lem. 3.1] (last paragraph of the proof). As indicated in [**20**], it is nevertheless possible to prove [**20**, Prop. 2.1] using the parametric approach, see, e.g., [**21**, pp. 105–117]. As for [**8**, Lemma 3.1] however, we do not know of any valid proof.

Warning 3.8. As in Theorem 2.5, consider an irreducible component V of V_g^ξ, and $[C]$ a general member of V, and assume that conditions (a) and (b) of Theorem 2.5 hold. Suppose moreover that C is not nodal; this implies by Proposition 3.3 that $I_C \subsetneq A_C$.

Being $C \in V$ general, one has $T_{[C]}V \subset T_{[C]}\,\mathrm{ES}(C)$ by Proposition 1.4. Therefore

$$\begin{aligned}(3.5)\qquad \dim V \leq \dim T_{[C]}V \leq \dim\bigl(T_{[C]}\,\mathrm{ES}(C)\bigr) &= h^0\bigl(C, I_C \otimes \mathcal{O}_C(C)\bigr)\\ \leq h^0\bigl(\bar{C}, \nu^*\bigl(I_C \otimes \mathcal{O}_C(C)\bigr)\bigr) &= h^0\bigl(\bar{C}, I' \otimes \omega_{\bar{C}} \otimes \varphi^*\omega_X^{-1}\bigr),\end{aligned}$$

where I' is the ideal of $\mathcal{O}_{\bar{C}}$ determined by the relation $\nu^* I_C = I' \otimes \nu^* A_C$ (as usual, $\nu\colon \bar{C} \to C$ is the normalization of C and φ its composition with the inclusion $C \subset X$).

Now: *although $\omega_{\bar{C}} \otimes \varphi^*\omega_X^{-1}$ is globally generated by our hypothesis (a)* and $I_C \subsetneq A_C$ because C is not nodal, in general it does not follow from the sequence of inequalities (3.5) that $\dim V < h^0\bigl(\bar{C}, \omega_{\bar{C}} \otimes \varphi^*\omega_X^{-1}\bigr)$, i.e. there is a priori no contradiction with assumption (b).

The reason for this is that $\nu^* I_C$ and $\nu^* A_C$ may be equal even if I_C and A_C are not (see Examples 3.9 and 3.10 below). In such a situation, I' is trivial, and (3.5) only gives $\dim V \leq h^0(\bar{C}, \omega_{\bar{C}} \otimes \varphi^*\omega_X^{-1})$. Example 4.14 displays a situation when both (a) and (b) hold, but the general member $C \in V$ is not nodal (i.e. conditions (a) and (b) hold but the conclusion of (ii) doesn't): in this example one has $\mathrm{H}^0\bigl(I_C \otimes \mathcal{O}_C(C)\bigr) = \mathrm{H}^0\bigl(A_C \otimes \mathcal{O}_C(C)\bigr)$ although $\nu^*\bigl(A_C \otimes \mathcal{O}_C(C)\bigr)$ is globally generated and $I_C \subsetneq A_C$. Therefore, condition (c) of (ii) is not redundant.

With this respect, it is important to keep in mind that base-point-freeness of the linear system $\nu^*\bigl|A \otimes \mathcal{O}_C(C)\bigr|$ on $\bar{C}$ does not imply base-point-freeness of the linear system (of generalized divisors, see Remark 3.13 below) $\bigl|A \otimes \mathcal{O}_C(C)\bigr|$ on C. And indeed, it is almost always the case that $\bigl|A \otimes \mathcal{O}_C(C)\bigr|$ has base points (see Remark 3.16).

Also, note that the linear subsystem $\nu^*\bigl|I_C \otimes \mathcal{O}_C(C)\bigr|$ of $\bigl|\nu^*\bigl(I_C \otimes \mathcal{O}_C(C)\bigr)\bigr|$ is in general not complete (see Example 3.9), in contrast with the fact that $\nu^*\bigl|A \otimes \mathcal{O}_C(C)\bigr| = \bigl|\nu^*\bigl(A \otimes \mathcal{O}_C(C)\bigr)\bigr|$ by independence of the adjoint conditions (Lemma 3.1).

Example 3.9 ([**30**])**.** Let $C \subset \mathbf{P}^2$ be a degree n curve with one ordinary tacnode (i.e. a singularity of type A_3) at a point p and smooth otherwise. At p, there are local holomorphic coordinates (x, y) such that C has equation $y^2 = x^4$. Then

$$A_{C,p} = \langle y, x^2 \rangle \quad \text{and} \quad I_{C,p} = \langle y, x^3 \rangle,$$

(see Example 3.6) whence the linear system $\left|A_C \otimes \mathcal{O}_C(C)\right|$ (resp. $\left|I_C \otimes \mathcal{O}_C(C)\right|$) on C is cut out by the system of degree n curves tangent at p to the two local branches of C there (resp. having third order contact at p with the reduced tangent cone to C there).

Now, a third order contact with the reduced tangent cone at p does not imply anything beyond simple tangency with each of the two local branches of C there. In coordinates, this translates into the fact that

$$\nu^* A_{C,p_i} = \nu^* I_{C,p_i} = \langle t_i^2 \rangle$$

at the two preimages p_i, $i = 1, 2$, of p, t_i being a local holomorphic coordinate of $\bar{C}$ at p_i. Nevertheless the linear system $\nu^*\left|I_C \otimes \mathcal{O}_C(C)\right|$ has codimension 1 in $\left|\nu^*\big(I_C \otimes \mathcal{O}_C(C)\big)\right| = \left|\nu^*\big(A_C \otimes \mathcal{O}_C(C)\big)\right|$ and is free from base point.

Example 3.10. We consider an ordinary m-uple planar curve singularity $(C, 0)$ as in Example 3.7. Without loss of generality, we assume that the line $x = 0$ is not contained in the tangent cone of C at 0. Then x is a local parameter for each local branch, and it follows from the computations of $A_{C,0}$ and $I_{C,0}$ in Example 3.7 that

$$\nu^* A_{C,0} = \nu^* I_{C,0} = \langle x^{m-1} \rangle,$$

where ν is the normalization of C.

It might nevertheless be possible to use the argument given in Warning 3.8 to give another proof of (i), i.e. that (a) and (b) of Theorem 2.5 imply that the general member of V is immersed. We have indeed not found any example of a non immersed planar curve singularity such that the pull-backs by the normalization of I and A are equal. The next statement is a first step in this direction.

Remark 3.11. Let $(C, 0)$ be a simple curve singularity, and ν its normalization. Then $\nu^* A_{C,0} \neq \nu^* I_{C,0}$ if and only if $(C, 0)$ is not immersed.

Proof: This is a basic computation. We treat the case of E_8, and leave the remaining ones to the reader. The normalization ν of the E_8 singularity factors as the sequence of blow-ups $\varepsilon_1 \circ \varepsilon_2$ corresponding to the morphisms of $\mathbb{C}$-algebras

$$\begin{array}{ccccc} \dfrac{\mathbf{C}[u_1,v_2]}{\langle u_1 - v_2^2\rangle} & \xleftarrow{\quad\varepsilon_2^*\quad} & \dfrac{\mathbf{C}[x,u_1]}{\langle x^2 - u_1^3\rangle} & \xleftarrow{\quad\varepsilon_1^*\quad} & \dfrac{\mathbf{C}[x,y]}{\langle y^3 - x^5\rangle} \\[2ex] (v_2\, u_1, u_1) & \longleftarrow\!\dashv & (x,u_1)\ ;\ (x, u_1\, x) & \longleftarrow\!\dashv & (x,y) \end{array}$$

and it follows from Lemma 3.4 that its adjoint is

$$\begin{aligned} A = (\varepsilon_1)_*\langle x,u_1\rangle \cdot \langle x,y\rangle^2 &= \langle x^3, x^2y, xy^2, u_1x^2, u_1xy, u_1y^2\rangle \\ &= \langle x^3, x^2y, xy^2, yx, y^2, x^4\rangle \\ &= \langle x^3, xy, y^2\rangle. \end{aligned}$$

On the other hand, its equisingular ideal is $I = J = \langle x^4, y^2\rangle$ by Lemma 3.5. Eventually, one has

$$\begin{aligned} \nu^*A = \varepsilon_2^*\varepsilon_1^*\langle x^3, xy, y^2\rangle = \varepsilon_2^*\langle x^3, u_1x^2, u_1^2x^2\rangle &= \varepsilon_2^*\langle x^3, u_1x^2\rangle \\ &= \langle v_2^3u_1^3, v_2^2u_1^3\rangle = \langle v_2^2u_1^3\rangle = \langle v_2^8\rangle \end{aligned}$$

and

$$\nu^*I = \varepsilon_2^*\varepsilon_1^*\langle x^4, y^2\rangle = \varepsilon_2^*\langle x^4, u_1^2x^2\rangle = \langle v_2^4u_1^4, v_2^2u_1^4\rangle = \langle v_2^2u_1^4\rangle = \langle v_2^{10}\rangle,$$

so that $\nu^*A \neq \nu^*I$, and indeed the E_8 singularity is non-immersed. $\square$

Remark 3.12. In any event, the tendency is that one loses information during the pull-back, even in the case of non-immersed singularities. For instance, in the case of an A_{2n} singularity one has $\dim_{\mathbf{C}} \nu^*A/\nu^*I = 1$ whereas $\dim_{\mathbf{C}} A/I = n$.

3.4. Generalized divisors. As explained in the previous subsection, one loses information as one pulls back the strict inclusion $I \subsetneq A$ to the normalization. In other words, in order to exploit the full strength of this inequality, it is required to work directly on the singular curve under consideration. Here, we describe a possibility for doing so, namely by using the theory of generalized divisors on curves with Gorenstein singularities (a condition obviously fulfilled by divisors on smooth surfaces), as developed by Hartshorne [**22**]. A meaningful application will be given for $K3$ surfaces in Subsection 4.2.

Recall from [**22**, §1] that generalized divisors on an integral curve C with Gorenstein singularities are defined as being *fractional ideals* of C, i.e. as those nonzero subsheaves of $\mathcal{K}_C$ (the constant sheaf of the function field of C) that are coherent $\mathcal{O}_C$-modules; note that fractional ideals of C are rank 1 torsion-free coherent $\mathcal{O}_C$-modules. As a particular case, nonzero coherent sheaves of ideals of $\mathcal{O}_C$ are generalized divisors; these correspond to 0-dimensional subschemes of C, and are called *effective* generalized divisors.

The addition of a generalized divisor and a Cartier divisor is well-defined (and is a generalized divisor), but there is no reasonable way to define the addition of two generalized divisors. There is an inverse mapping $D \mapsto -D$, which at the level of fractional ideals reads $\mathcal{I} \mapsto \mathcal{I}^{-1} := \{f \in \mathcal{K}_C \mid f \cdot \mathcal{I} \subset \mathcal{O}_C\}$. Hartshorne moreover defines a *degree* function on the set of generalized divisors, which in the case of a 0-dimensional subscheme Z coincides with the length of $\mathcal{O}_Z$. He then shows that both the Riemann–Roch formula and Serre duality hold in this context.

Remark 3.13. Let Z be a generalized divisor on C, and $\mathcal{O}_C(Z)$ the inverse of the fractional ideal corresponding to Z. The projective space of lines in $\mathrm{H}^0\big(\mathcal{O}_C(Z)\big)$ is in bijection with the set $|Z|$ of effective generalized divisors linearly equivalent to Z. A point $p \in C$ is a base point of $|Z|$ if $p \in \operatorname{Supp} Z'$ for every $Z' \in |Z|$. One has to be careful that $\mathcal{O}_C(Z)$ may be generated by global sections even though $|Z|$ has base points, and that it is in general not possible to associate to $|Z|$ a base-point-free linear system by subtracting its base locus, the latter being a generalized divisor, see [**22**, pp. 378–379 and Ex. (1.6.1)].

Let C be an integral curve in a smooth surface X, and ξ its class in $\mathrm{NS}(X)$. The adjoint and equisingular ideals A and I of C define two effective generalized divisors on C, which we shall denote respectively by Δ and E. As a reformulation of Proposition 3.3, we have:

$$C \text{ not nodal} \quad \Longleftrightarrow \quad \deg E > \deg \Delta.$$

Now, to argue along the lines of Warning 3.8, one has to estimate

$$h^0\big(C, N_{C/X}\otimes A\big) - h^0\big(C, N_{C/X}\otimes I\big) = h^0\big(C, \mathcal{O}_C(C-\Delta)\big) - h^0\big(C, \mathcal{O}_C(C-E)\big).$$

Lemma 3.14. *If in the situation above C is not nodal, then*

$$h^0\big(C, N_{C/X} \otimes A\big) - h^0\big(C, N_{C/X} \otimes I\big) > h^0\big(C, \omega_X \otimes \mathcal{O}_C(\Delta)\big) - h^0\big(C, \omega_X \otimes \mathcal{O}_C(E)\big).$$

Proof: By the Riemann–Roch formula together with Serre duality and the adjunction formula, we have

$$\big[h^0\big(C, N_{C/X} \otimes A\big) - h^0\big(C, \omega_X \otimes \mathcal{O}_C(\Delta)\big)\big] - \big[h^0\big(C, N_{C/X} \otimes I\big) - h^0\big(C, \omega_X \otimes \mathcal{O}_C(E)\big)\big] = -\deg \Delta + \deg E. \qquad \square$$

Remark 3.15. As a sideremark (which will nevertheless be useful in our application to $K3$ surfaces), note that $\deg \Delta = p_a(C) - g(C)$, the so-called δ-invariant of the curve C. Moreover, it follows from Serre duality and Lemma 3.1 that

$$h^1\bigl(C, \mathcal{O}_C(\Delta)\bigr) = h^0\bigl(C, \omega_C(-\Delta)\bigr) = g(C).$$

The Riemann–Roch formula then tells us that $h^0\bigl(\mathcal{O}_C(\Delta)\bigr) = 1$, i.e. Δ is a rigid (generalized) divisor.

Remark 3.16. The linear system $|N_{C/X} \otimes A|$ has almost always base points. To see why, consider the typical case when C has an ordinary m-uple point p and no further singularity. Then it follows from Example 3.7 that $|N_{C/X} \otimes A|$ consists of those effective generalized divisors linearly equivalent to $N_{C/X} - (m-1)p$. Now, every effective divisor linearly equivalent to $N_{C/X}$ and containing p has to contain it with multiplicity $\geq m$, so that $|N_{C/X} \otimes A|$ has p as a base point.

4. Applications

Historically, the first instance of Problem (A) to be studied was that of curves in the projective plane, by Zariski on the one hand, and by Arbarello and Cornalba on the other. In this situation, the parametric approach of Section 2 can be efficiently applied.

Usually, inequality (b) of Theorem 2.5 is obtained from the estimate

$$\dim\bigl(\mathcal{G}^2_d\bigr) \geq 3d + g - 9 \tag{4.1}$$

proved in [**1**], where $\mathcal{G}^2_d$ is the moduli space of pairs (C, V) consisting of a genus g (smooth projective) curve C and of a g^2_d on C (i.e., V is a degree d linear system of dimension 2 on C), together with the fact that the group of projective transformations of the plane has dimension 8. As a sideremark, note that

$$3d + g - 9 = \dim \mathcal{M}_g + \rho(2, d, g),$$

where $\mathcal{M}_g$ is the moduli space of genus g curves, and $\rho(r, d, g) = g - (r+1)(g + r - d)$ is the Brill–Noether number (see [**4**, p. 159]).

In Subsection 4.1 below, we deduce inequality (b) of Theorem 2.5 in a more abstract nonsensical way from (2.6), which actually shows that we have equality in (b) of Theorem 2.5, hence also in (4.1), even when $\rho < 0$.

4.1. Applications to rational surfaces. We now collect various applications of Corollary 2.7 that settle Problem (A) for common rational surfaces. The paper [**24**] contains results going in the same direction.

We make repeated use of the elementary fact that any line bundle of degree $\geq 2g$ on a smooth genus g curve is non-special and globally generated.

Corollary 4.1 ([**1, 43**]). *The general element of the Severi variety $V_{d,g}$ of integral plane curves of degree d and genus g is a nodal curve.*

Proof: This is trivial for $d = 1$, and if $d \geq 2$, one has for $[C] \in V_{d,g}$
$$\deg(\omega_{\bar{C}} \otimes \varphi^* \omega_X^{-1}) = 2g - 2 + 3d \geq 2g + 2,$$
whence Corollary 2.7 applies. □

Corollary 4.2. *Let X be a Del Pezzo surface of degree d, i.e. $-K_X$ is ample and $K_X^2 = d$. Then for every $n \geq 1$, the general element C of any irreducible component of $V_{-nK_X,g}$ is nodal, unless $dn \leq 3$. In any event, C is immersed unless $d = n = 1$ and $g = 0$.*

Proof: For $[C] \in V_{-nK_X,g}$ we have
$$\deg(\omega_{\bar{C}} \otimes \varphi^* \omega_X^{-1}) = 2g - 2 + nd,$$
which is $\geq 2g + 2$ if $nd \geq 4$ and $\geq 2g$ if $nd \geq 2$, so that Corollary 2.7 applies. When $d = n = 1$, we are considering the pencil $|-K_X|$, the general member of which is a smooth irreducible curve of genus 1. □

Remark 4.3. Observe that the case of $V_{-K_X,0}$ when X is a Del Pezzo surface of degree 1 is a true exception, as the following example shows.

Let $D \subset \mathbf{P}^2$ be an irreducible cuspidal cubic, and let X be the blow-up of $\mathbf{P}^2$ at eight of the nine points of intersection of D with a general cubic. The proper transform C of D is isolated in $V_{-K_X,0}$, and is not nodal. In fact $\bar{C} = \mathbf{P}^1$, and
$$h^0\big(\bar{C}, \omega_{\bar{C}} \otimes \varphi^* \omega_X^{-1}\big) = h^0\big(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(-1)\big) = 0 = \dim\big(V_{\omega_X^{-1},0}\big),$$
but $\omega_{\bar{C}} \otimes \varphi^* \omega_X^{-1} = \mathcal{O}_{\mathbf{P}^1}(-1)$ is not globally generated, so that Theorem 2.5 does not apply.

It is remarkable that, when unlike the situation above X is $\mathbf{P}^2$ blown-up at eight general enough points, all members of $V_{-K_X,0}$ are nodal curves.

Corollary 4.4. *Let $X := \mathbf{F}_n = \mathbf{P}\big(\mathcal{O}_{\mathbf{P}^1} \oplus \mathcal{O}_{\mathbf{P}^1}(n)\big)$ be a Hirzebruch surface ($n \geq 0$). For every effective $L \in \operatorname{Pic} X$ and $0 \leq g \leq p_a(L)$, the general member of every irreducible component of $V_{L,g}$ is a nodal curve.*

Proof: Let E and F be the respective linear equivalence classes of the exceptional section and a fibre of the ruling, and $H = E + nF$. It is enough to consider the case $L = dH + kF$, $d, k \geq 0$, since every effective divisor on X not containing the exceptional section belongs to such an $|L|$. Consider an integral curve $C \in |L|$ of genus g. One has

$$\deg\bigl(\omega_{\bar{C}} \otimes \varphi^* \omega_X^{-1}\bigr) = 2g - 2 - K_X \cdot C = 2g - 2 + dn + 2d + 2k,$$

which is $\geq 2g + 2$ (so that Corollary 2.7 safely applies), unless either $d = 0$ and $k = 1$ or $d = 1$, $k = 0$, and $n \leq 1$. An elementary case by case analysis shows that the latter cases are all trivial. □

4.2. Applications to surfaces with numerically trivial canonical bundle. We now deal with the case when $K_X \equiv 0$. In this situation Corollary 2.7 does not apply directly and further arguments are required.

***K*3 surfaces.** Let (X, L) be a polarized $K3$ surface, with $L^2 = 2p - 2$, $p \geq 2$, and let $0 \leq g \leq p$. Then X is regular, and p equals both the dimension of $|L|$ and the arithmetic genus of a member of this linear system. Moreover, it follows from Lemma 2.3 and (2.6) that

$$g - 1 \leqslant \dim V_{L,g} \leqslant g. \tag{4.2}$$

In this case, the existence of deformations of projective $K3$ surfaces into non algebraic ones enables one to refine the former dimension estimate, still by using the techniques of Subsection 2.1. This is well-known to the experts. We shall nevertheless prove it here for the sake of completeness, along the lines of [**25**, Exercise II.1.13.1] and [**29**, Corollary 4].

Proposition 4.5. *Every irreducible component V of $V_{L,g}$ has dimension g.*

Proof: Using Lemma 2.3 and inequality (4.2), it suffices to prove that for every irreducible component M of $M_g(X)$ and general $[\phi\colon D \to X] \in M$, one has

$$\dim_{[\phi]} M \geq g + \dim(\operatorname{Aut} D). \tag{4.3}$$

We consider $\mathcal{X} \to \Delta$ an analytic deformation of X parametrized by the disc, such that the fibre over $t \neq 0$ does not contain any algebraic curve. Then we let $\pi_g\colon \mathcal{D}_g \to S_g$ be a modular family of smooth projective curves of genus g as in Remark 2.1, and

$$M'_g(\mathcal{X}) := \operatorname{Hom}(\mathcal{D}_g \times \Delta / S_g \times \Delta, \mathcal{X} \times S_g / S_g \times \Delta).$$

By [**25**, Theorem II.1.7], we have

$$\begin{aligned}\dim_{[\phi]} M'_g(\mathcal{X}) &\geq \chi(\varphi^*(T_X)) + \dim(S \times \Delta)\\ &= \big[-\deg(\varphi^* K_X) + 2\chi(\mathcal{O}_{\bar{C}})\big] + \big[3g - 2 + \dim(\operatorname{Aut} D)\big]\\ &= g + \dim(\operatorname{Aut} D).\end{aligned}$$

By construction and functoriality, an étale cover of $M'_g(\mathcal{X})$ maps finite-to-one into $M_g(X)$, so the inequality above implies the required (4.3). $\square$

Note that Proposition 4.5 does not imply that the varieties $V_{L,g}$ are non-empty. If the pair (X, L) is general, this is true for $0 \leq g \leq p_a$, as a consequence of the main theorem in [**8**].

Proposition 4.6. *For $g > 0$, the general element C of every irreducible component of $V_{L,g}$ is immersed. If moreover C has a non-trigonal normalization, then it is nodal.*

Proof: We have $\omega_{\bar{C}} \otimes \varphi^* \omega_X^{-1} = \omega_{\bar{C}}$. This line bundle is globally generated since $g \geq 1$, and $h^0\big(\omega_{\bar{C}} \otimes \varphi^* \omega_X^{-1}\big) = g = \dim(V)$ by Proposition 4.5. Therefore conditions (a) and (b) of Theorem 2.5 are satisfied and the first part follows. If the normalization $\bar{C}$ is not trigonal, then condition (c) of Theorem 2.5 is also satisfied and C is nodal. $\square$

Corollary 4.7. *Let (X, L) be a very general primitively polarized $K3$ surface (i.e. L is indivisible in* $\operatorname{Pic} X$*) with $L^2 = 2p - 2$, $p \geq 2$, and $0 < g \leq p$. If*

$$g + \left\lfloor \frac{g}{4} \right\rfloor \left(g - 2\left\lfloor \frac{g}{4} \right\rfloor - 2\right) > p, \tag{4.4}$$

then the general element of every irreducible component of $V_{L,g}$ is nodal.

Proof: By [**14**, Thm 3.1], inequality (4.4) ensures that for every $C \in |L|$, the normalization of C does not carry any g^1_3. $\square$

Proposition 4.8. *Let (X, L) be a very general primitively polarized $K3$ surface, with $L^2 = 2p - 2$. If $g > \frac{p}{2}$, then the general element of every irreducible component of $V_{L,g}$ is nodal.*

Before we prove this, recall that the *Clifford index* of an integral projective curve C of arithmetic genus $p \geq 2$ is

$$\operatorname{Cliff}(C) := \min\Bigg\{ \left\lfloor \frac{p-1}{2} \right\rfloor,\\ \min_{\{A \in \overline{\operatorname{Pic}}\, C \,|\, r(A) \geq 1 \text{ and } r(K_C - A) \geq 1\}} \big(\deg A - 2r(A)\big) \Bigg\},$$

where $\overline{\operatorname{Pic}}\,C$ is the set of rank 1 torsion-free sheaves on C, and $r(M)$ stands for $h^0(M)-1$ for any $M \in \overline{\operatorname{Pic}}\,C$. The bigger $\operatorname{Cliff}(C)$ is, the more general C is with respect to Brill–Noether theory.

Proof of Proposition 4.8: We apply the strategy described in Warning 3.8, and circumvent the issue therein underlined by using the theory of generalized divisors on singular curves, as recalled in Subsection 3.4 (we freely use the notations introduced in that subsection): let V be an irreducible component of $V_{L,g}$, $[C]$ a general member of V, and assume by contradiction that C is not nodal. We have

$$\dim V \leq \dim \operatorname{ES}(C) = h^0\big(C, I_C \otimes \mathcal{O}_C(C)\big),$$

and we shall show that

$$h^0\big(C, I_C \otimes \mathcal{O}_C(C)\big) < h^0\big(C, A_C \otimes \mathcal{O}_C(C)\big) = g, \tag{4.5}$$

thus contradicting Proposition 4.5 and ending the proof (the right-hand-side equality in (4.5) comes from Lemma 3.1).

If $h^1\big(C, \mathcal{O}_C(E)\big) < 2$, then

$$h^0\big(I \otimes \mathcal{O}_C(C)\big) = h^0\big(\omega_C(-E)\big) = h^1\big(\mathcal{O}_C(E)\big) \leq 1 < g,$$

and (4.5) holds. If on the other hand $h^0\big(C, \mathcal{O}_C(E)\big) < 2$, then (4.5) still holds, since Lemma 3.14 together with Remark 3.15 yield

$$h^0\big(C, N_{C/X} \otimes A)\big) - h^0\big(C, N_{C/X} \otimes I\big) > 1 - h^0\big(C, \mathcal{O}_C(E)\big).$$

For the remaining of the proof, we therefore assume that both $h^0\big(C, \mathcal{O}_C(E)\big)$ and $h^1\big(C, \mathcal{O}_C(E)\big)$ are ≥ 2.

Now, being (X, L) a very general primitively polarized $K3$ surface, and $C \in |L|$ an integral curve of geometric genus $g \geq 2$, it follows from [**5**] together with [**17**] that the Clifford index of C is that of a general smooth curve of genus p, i.e. $\operatorname{Cliff}(C) = \lfloor \frac{p-1}{2} \rfloor$. This implies

$$p + 1 - \big[h^0\big(\mathcal{O}_C(E)\big) + h^0\big(\omega_C(-E)\big)\big] = \deg E - 2r(E) \geq \left\lfloor \frac{p-1}{2} \right\rfloor,$$

hence

$$h^0\big(\omega_C(-E)\big) \leq \frac{p}{2} + 2 - h^0\big(\mathcal{O}_C(E)\big) \leq \frac{p}{2},$$

so that (4.5) again holds. □

Remark 4.9. In a private correspondence concerning a previous version of this paper, X. Chen has shown (using methods completely different from ours) that the statement of Proposition 4.8 holds more generally without the limitation $g > \frac{p}{2}$.

Remark 4.10. The case $g = 0$ in Proposition 4.6 is a true exception. For example, there exist irreducible rational plane quartic curves with one cusp and two nodes. Pick a general such curve: then there is a nonsingular quartic surface in $\mathbf{P}^3$ containing it as a hyperplane section.

On the other hand, it seems fairly reasonable to formulate the following conjecture, which predicts that the case $g = 0$ holds for very general (X, L). It is of particular interest in the context of enumerative geometry, in that it provides a good understanding of the various formulae counting rational curves on $K3$ surfaces (see [**16, 6**]).

Conjecture 4.11. *Let (X, L) be a very general polarized $K3$ surface. Then all rational curves in $|L|$ are nodal.*

This has been proved by Chen [**9**] in the particular case of indivisible L, using a degeneration argument.

Enriques surfaces.

Theorem 4.12. *Let X be an Enriques surface and L an invertible sheaf on X. If $g \geq 3$, and $[C] \in V_{L,g}$ has a non-hyperelliptic normalization $\bar{C}$, then the general element of every component of $V_{L,g}$ containing C has no cusps. If moreover* $\operatorname{Cliff}(\bar{C}) \geq 5$ *then C is nodal.*

Proof: The sheaf $M := \omega_{\bar{C}} \otimes \varphi^* \omega_X^{-1}$ has degree $2g - 2$, and it is Prym-canonical: in particular, it is non-special. On a non-hyperelliptic curve, every Prym-canonical sheaf is globally generated (see, e.g., [**26**, Lemma (2.1)]). Therefore, the first part follows from Corollary 2.7. If $\operatorname{Cliff}(\bar{C}) \geq 5$ then $\varphi_M(\bar{C}) \subset \mathbf{P}^{g-2}$ has no trisecants, by [**26**, Prop. (2.2)], and therefore condition (c) of Theorem 2.5 is also satisfied. □

Abelian surfaces. Let (X, ξ) be a polarized Abelian surface, and let $p = p_a(\xi)$. For each $[C] \in \mathrm{Curves}_X^\xi$ we have $\dim |C| = p - 2$, so that Curves_X^ξ is a $\mathbf{P}^{p-2}$-fibration over the dual Abelian surface $\widehat{X}$. A general Abelian surface does not contain any curve of geometric genus ≤ 1. On the other hand, the arguments for Propositions 4.5 and 4.6 apply mutatis mutandis to this situation, so one has:

Proposition 4.13. *Let $2 \leq g \leq p_a$, and V an irreducible component of V_g^ξ. Then $\dim V = g$, and the general $[C] \in V$ corresponds to a curve with only immersed singularities. If moreover C has non-trigonal normalization, then it is nodal.*

Note however that, unlike the case of $K3$ surfaces, we do not know in general whether the varieties V_g^ξ are non-empty for $2 \leq g \leq p_a$.

For genus 2 curves, more is known [**27**, Prop. 2.2]: if (X, L) is an Abelian surface of type (d_1, d_2), then any genus 2 curve in $|L|$ has at most ordinary singularities of multiplicity $\leq \frac{1}{2}(1 + \sqrt{8d_1d_2 - 7})$. We have the following enlightening and apparently well-known example which, among other things, shows that this bound is sharp.

Example 4.14. Let X be the Jacobian of a general genus 2 curve Σ, and choose an isomorphism $X \simeq \operatorname{Pic}^1 \Sigma$; it yields an identification $\Sigma \simeq \Theta_X$. Denote by $\{\Theta_X\}$ the corresponding polarization on X. The curve Σ has six Weierstrass points $w_1, \ldots, w_6$, and the divisors $2w_i$ on Σ are all linearly equivalent. It follows that the image of $\Sigma \subset X$ by multiplication by 2 is an irreducible genus 2 curve C which belongs to the linear system $\left|2^2 \cdot \Theta_X\right|$, and has a 6-fold point, the latter being ordinary by [**27**, Prop. 2.2] quoted above.

The curve C and its translates are parametrized by an irreducible (two-dimensional) component V of $V_2^{\{4\Theta_X\}}$. Since $\omega_\Sigma \otimes \varphi^*\omega_X^{-1} = \omega_\Sigma$ is globally generated and $\dim V = 2 = h^0(\omega_\Sigma)$, conditions (a) and (b) of Theorem 2.5 are satisfied. On the other hand, condition (c) of Theorem 2.5 is clearly not fullfilled and C is not nodal, showing that this condition is not redundant.

We emphasize that this is an explicit illustration of Warning 3.8. We have here (letting as usual ν denote the normalization of C)

$$\nu^*\big|N_{C/X} \otimes A_C\big| = \big|\nu^*(N_{C/X} \otimes A_C)\big| = |\omega_\Sigma|$$

which is a base-point-free linear system on $\bar{C}$, whereas

$$\big|N_{C/X} \otimes A_C\big| = \big|N_{C/X} \otimes I_C\big|$$

even though $I_C \subsetneq A_C$. Observe also that $\nu^*I_C = \nu^*A_C$ by Example 3.10.

5. A museum of noteworthy behaviours

5.1. Maximal equigeneric families with non-nodal general member. The examples in this subsection are mainly intended to show that the assumption that $\omega_{\bar{C}} \otimes \varphi^*\omega_X^{-1}$ is globally generated in Theorem 2.5 is necessary. The same goal was achieved by the examples provided in Remarks 4.3 and 4.10, but the ones presented here are hopefully less peculiar (e.g., the involved equigeneric families are in general not 0-dimensional).

Example 5.1 (A complete positive dimensional ample linear system on a rational surface, all members of which have a cuspidal double point)**.** The surface will be a plane blown-up at distinct points, which will allow us the use of a Cayley–Bacharach type of argument. Let $C_1, C_2 \subset \mathbf{P}^2$ be two irreducible sextics having an ordinary cusp at the same point $s_0 \in \mathbf{P}^2$, with the same principal tangent line, no other singularity, and meeting transversely elsewhere. Their local intersection number at s_0 is $(C_1 \cdot C_2)_{s_0} = 6$, so we can consider 26 pairwise distinct transverse intersection points $p_1, \ldots, p_{26} \in C_1 \cap C_2 \setminus \{s_0\}$. Let $\pi \colon X \to \mathbf{P}^2$ be the blow-up at $p_1, \ldots, p_{26}$, and let $L := 6H - \sum_{1 \leq i \leq 26} E_i$, where $H = \pi^* \mathcal{O}_{\mathbf{P}^2}(1)$, and the E_i's are the exceptional curves of π. Then, since $\dim |\mathcal{O}_{\mathbf{P}^2}(6)| = 27$, $|L|$ is a pencil generated by the proper transforms of C_1 and C_2, hence consists entirely of curves singular at the point $s = \pi^{-1}(s_0)$ and with a non ordinary singularity there. The general $C \in |L|$ is irreducible of genus nine, and $V_{L,9}$ is therefore an open subset of $|L|$, not containing any nodal curve.

For general $C \in |L|$, one computes $h^0\big(\omega_{\bar{C}} \otimes \varphi^* \omega_X^{-1}\big) = 1$, which shows that the line bundle $\omega_{\bar{C}} \otimes \varphi^* \omega_X^{-1}$ on $\bar{C}$ is not globally generated (we let, as usual, $\bar{C} \to C$ be the normalization of C, and φ its composition with the inclusion $C \subset X$). Thus condition (a) of Theorem 2.5 does not hold, while condition (b) is verified. As a sideremark, note that $(-K_X \cdot L) < 0$ and L is ample (see also Remark 2.4 above about this example).

This example can be generalized to curves with an arbitrary number of arbitrarily nasty singularities: simply note that the dimension of $|\mathcal{O}_{\mathbf{P}^2}(d)|$ grows as $d^2/2$ when d tends to infinity, and is therefore smaller by as much as we want than the intersection number of two degree d plane curves for d big enough.

The forthcoming examples all are degree n cyclic coverings $\pi \colon X \to \mathbf{P}^2$, branched over a smooth curve $B \subset \mathbf{P}^2$ of degree d. They are smooth and regular. Let $L = \pi^* \mathcal{O}_{\mathbf{P}^2}(1)$. One has

$$\mathrm{H}^0\big(X, kL\big) = \pi^* \, \mathrm{H}^0\big(\mathbf{P}^2, \mathcal{O}_{\mathbf{P}^2}(k)\big)$$

if and only if $k < \frac{d}{n}$.

Example 5.2 (A complete ample linear system with a codimension 1 equigeneric stratum, the general member of which has an A_{n-1}-double point)**.** As a local computation shows, the inverse image in X of a plane curve simply tangent to B is a curve with an A_{n-1}-double point at the preimage of the tangency point.

It follows that for $1 \leq k < \frac{d}{n}$ there is a codimension 1 locus in $|kL|$ that parametrizes curves with an A_{n-1}-double point, although the general member of $|kL|$ is a smooth curve. This is an irreducible component of $V_{kL, p_a(kL) - \lfloor n/2 \rfloor}$. It is superabundant, since one expects in general that codimension c equigeneric strata are components of $V_{kL, p_a(kL) - c}$.

Example 5.3 (A complete ample linear system containing two codimension 1 equigeneric strata, that respectively parametrize curves of genera g_1 and g_2, $g_1 \neq g_2$)**.** The inverse image in X of a plane curve having a node outside of B is a curve having n distinct nodes. Consequently, there is for every $3 \leq k < \frac{d}{n}$ a codimension 1 locus in $|kL|$ that parametrizes integral curves with n distinct nodes. This is an irreducible component of $V_{kL,p_a(kL)-n}$, and it is superabundant.

As a conclusion, notice that the discriminant locus in $|kL|$ is reducible, and has two of its irreducible components contained in $V_{kL,p_a(kL)-[n/2]}$ and $V_{kL,p_a(kL)-n}$ respectively.

Example 5.4 (A further example of 0-dimensional equigeneric locus)**.** Assume there exists a line which meets B at some point s with multiplicity 4. Then its inverse image in X is a curve with a singularity of type $y^n = x^4$. The corresponding point of $|L|$ is a component of $V_{L,p_a(L)-\delta}$, where δ is the δ-invariant of the singularity $y^n = x^4$, which may be computed using Lemma 3.4.

When $n = 2$, this singularity is a tacnode and $\delta = 2$. In this case, one gets a component of $V_{L,p_a(L)-2}$ that is not superabundant and parametrizes non-nodal curves.

It should be clear by now, how these two examples (5.1 and the series 5.2–5.4) can be generalized to produce an infinite series of examples.

5.2. Singular maximal equisingular families. Let X be a smooth projective surface, $\xi \in \mathrm{NS}(X)$, and C an integral curve of genus g and class ξ. We wish to illustrate in this subsection the fact that the local structures at $[C]$ of both V_g^ξ and $\mathrm{ES}(C)$ are not as nice as one would expect them to be by looking at their counterparts in the deformation theory of a single planar curve singularity. In fact, the situation is already messy in the simplest case $X = \mathbf{P}^2$.

Let $p_1, \ldots, p_\delta$ be the singular points of C, and let $\hat{C}_i$ be the germ of C at p_i for each $i = 1, \ldots, \delta$, and

$$\begin{array}{ccc} \hat{C}_i & \hookrightarrow & \hat{\mathcal{C}}_i \\ \downarrow & & \downarrow \\ \mathrm{Spec}(\mathbf{C}) & \longrightarrow & B_i \end{array}$$

be the étale semiuniversal deformation of $\hat{C}_i$ (see [**15**] for a precise account on this). By their universal properties, there exists an étale neighborhood $W \to \mathrm{Curves}_X^\xi$ of $[C]$ such that there is a restriction morphism

$$r \colon W \to \textstyle\prod_i B_i.$$

The general philosophy we want to underline can be summed up as follows.

Remark 5.5. In general, the restriction map r is not smooth.

Note that both domain and codomain of r are smooth. In particular, the smoothness of r is equivalent to the surjectivity of its differential.

The equigeneric and equisingular loci inside each one of the deformation spaces B_i are known to be well-behaved (we refer to [**15**] for details). Among others, let us mention that the equisingular locus is smooth, and that the general point in the equigeneric locus corresponds to a deformation of p_i in a union of nodes. Now, the smoothness of r would transport these good properties to V_g^ξ and $\mathrm{ES}(C)$. In particular, it would imply the two following facts:

(1) the general point of every irreducible component V of V_g^ξ corresponds to a nodal curve;
(2) $\mathrm{ES}(C)$ is smooth, and of the expected codimension in Curves_X^ξ.

Now Remark 5.5 follows from the fact that neither (1) nor (2) is true in general. For (1), this was discussed previously in Subsection 5.1. On the other hand, property (2) can be contradicted in several ways: we refer to [**18**] for a discussion of these problems and for a survey of what is known. Here we solely mention a few examples which are relevant to our point of view.

If C has n nodes, κ ordinary cusps, and no further singularity, then $\mathrm{ES}(C)$ is the locus of curves with n nodes and κ cusps, and has expected codimension $n+2\kappa$ in Curves_X^ξ. Here, we let $X = \mathbf{P}^2$, and adopt the usual notation $V_{d,n,\kappa}$ for the scheme of irreducible plane curves of degree d, with n nodes, κ cusps, and no further singularity.

Example 5.6 (B. Segre [**31**], see also [**42**, p. 220])**.** For $m \geq 3$, there exists an irreducible component of $V_{6m,0,6m^2}$, which is nonsingular and has dimension strictly larger than the expected one.

Example 5.7 (Wahl [**40**])**.** The scheme $V_{104,3636,900}$ has a non-reduced component of dimension $174 > 128 = \frac{104\cdot 107}{2} - 3636 - 2\cdot 900$.

Example 5.8. There also exists an equisingular stratum $V_{d,n,\kappa}$ having a reducible connected component.

The construction of the latter, which we shall now outline, follows the same lines as that of Wahl [**40**], and is based on the example of [**32**] (for a thorough description of which we refer to [**23**, §13 Exercises]).

Start from a nonsingular curve A of type $(2,3)$ on a nonsingular quadric $Q \subset \mathbf{P}^3$, and let $F, G \subset \mathbf{P}^3$ be respectively a general quartic and a general sextic containing A. Then $F \cap G = A \cup \gamma$ where γ is a nonsingular curve of degree 18 and genus 39. As shown in [**32**], the curve γ is obstructed. Precisely, $[\gamma]$ is in the closure of two components of $\mathrm{Hilb}^{\mathbf{P}^3}$, each consisting generically of projectively normal, hence unobstructed, curves.

Now consider an irreducible surface $S \subset \mathbf{P}^3$ of degree $N \gg 0$, having γ as an ordinary double curve, and let $C \subset \mathbf{P}^2$ be the branch curve of a generic projection of S on $\mathbf{P}^2$, $d := \deg(C)$. By [**13**], C is irreducible, and has n nodes and κ-cusps as its only singularities. It then follows from the results of [**40**], that $\mathrm{Hilb}^{\mathbf{P}^3}$ at $[\gamma]$ is smoothly related with $V_{d,n,\kappa}$ at $[C]$. Therefore $V_{d,n,\kappa}$ is analytically reducible at $[C]$.

In fact, one can show more precisely that $V_{d,n,\kappa}$ is reducible at $[C]$, by taking generic projections of irreducible surfaces S' of degree N having ordinary singularities along curves γ' which are in a neighbourhood of $[\gamma] \in \mathrm{Hilb}^{\mathbf{P}^3}$.

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