ALGEBRAIC CURVES AND THEIR MODULI

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Contents

This survey is modeled on the CIMPA introductory lectures I gave in Guanajuato in February 2016 about moduli of curves. Surveys on this topic abound and for this reason I decided to focus on few specific aspects giving some illustration of the relation between local and global properties of moduli. The central theme is the interplay between the various notions of "generality" for a curve of genus g. Of course these notes reflect my own view of the subject. For a recent comprehensive text on moduli of curves I refer to [5].

1. Parameters

All schemes will be defined over $\mathbb{C}$. The category of algebraic schemes will be denoted by (Schemes).

Algebraic varieties depend on parameters. This is clear if we define them by means of equations in some (affine or projective) space, because one can vary the coefficients of the equations. For example, by moving the coefficients of their equation we parametrize *nonsingular plane curves* of degree d in $\mathbb{P}^2$ by the points of (an open subset of) a $\mathbb{P}^N$, where $N = \frac{d(d+3)}{2}$.

Less obviously, consider a *nonsingular rational cubic curve* $C \subset \mathbb{P}^3$. Up to choice of coordinates it can be defined by the three quadric equations:

$$X_1X_3 - X_2^2 = X_0X_3 - X_1X_2 = X_0X_2 - X_1^2 = 0 \tag{1}$$

that I will write as

$$\mathrm{rk}\begin{pmatrix} X_0 & X_1 & X_2 \\ X_1 & X_2 & X_3 \end{pmatrix} \leq 1 \tag{2}$$

If we deform arbitrarily the coefficients of the equations (1) their intersection will consist of 8 distinct points. Bad choice!

Good choice: deform the entries of the matrix (2) to general linear forms in $X_0, \dots, X_3$. This will correspond to ask that the corresponding family of subvarieties is *flat*, and will guarantee that we obtain twisted cubics again.

2. From parameters to moduli

Parameters are a naive notion. Moduli are a more refined notion: they are *parameters of isomorphism classes* of objects.

Typical example: the difference between parametrizing plane conics and parametrizing plane cubics. Conics depend on 5 parameters but have no moduli, cubics depend on 9 parameters and have one modulus.

What does it mean that cubics have one modulus? This has been an important discovery in the XIX century. It consists of the following steps ([13], ch. 3):

- given an ordered 4-tuple of pairwise distinct points $P_i = [a_i, b_i] \in \mathbb{P}^1$, $i = 1, \dots, 4$, consider their *cross ratio*
$$\lambda = \frac{(a_1b_3 - a_3b_1)(a_2b_4 - a_4b_2)}{(a_1b_4 - a_4b_1)(a_2b_3 - a_3b_2)}$$
It is invariant under linear coordinate changes (direct computation) and takes all values $\neq 0, 1$.
As we permute the points their cross ratio takes the values
$$\lambda, 1-\lambda, \frac{1}{\lambda}, \frac{1}{1-\lambda}, \frac{\lambda}{1-\lambda}, \frac{1-\lambda}{\lambda}$$

and the expression

$$j(\lambda) := 2^8 \frac{(\lambda^2 - \lambda + 1)^3}{\lambda^2(\lambda - 1)^2}$$

takes the same value if and only if we replace λ by any of the above six expressions. Moreover each $j \in \mathbb{C}$ is of the form $j(\lambda)$ for some $\lambda \neq 0, 1$. Therefore we obtain a 1-1 correspondence between $\mathbb{A}^1_{\mathbb{C}}$ and unordered 4-tuples of distinct points of $\mathbb{P}^1$ up to coordinate changes.

- Given a nonsingular cubic $C \subset \mathbb{P}^2$ and $P \in C$ there are 4 distinct tangent lines to C passing through P besides the tangent line at P. View them as points of $\mathbb{P}(T_P\mathbb{P}^2) \cong \mathbb{P}^1$, and compute their $j(\lambda)$. Then it can be proved that $j(\lambda)$ is independent of P. Call it $j(C)$.

  For example if C is in *Hesse normal form*

  $$X_0^3 + X_1^3 + X_2^3 + 6\alpha X_0 X_1 X_2 = 0$$

  then $1 + 8\alpha^3 \neq 0$ and $j(C) = \frac{64(\alpha - \alpha^4)^3}{(1+8\alpha^3)^3}$
- For every $j \in \mathbb{C}$ there exists a nonsingular cubic C such that $j = j(C)$.
- (Salmon) Two nonsingular cubics C, C' are isomorphic if and only if $j(C) = j(C')$.

Therefore *in some sense* the set of isomorphism classes of plane cubics is identified with $\mathbb{A}^1_{\mathbb{C}}$. Or we might say that $\mathbb{A}^1_{\mathbb{C}}$ is the *moduli space* of plane cubics.

This is not yet a satisfactory definition, because the relation between $\mathbb{A}^1_{\mathbb{C}}$ and parametrized curves is not fully transparent. We will discuss why in §4.

It is difficult to distinguish which parameters are moduli.

For example consider the following linear pencil of plane quartics:

$$\lambda F_4(X_0, X_1, X_2) + \mu(X_0^4 + X_1^4 + X_2^4) = 0, \quad (\lambda, \mu) \in \mathbb{P}^1 \tag{3}$$

where $F_4(X_0, X_1, X_2)$ is a general quartic homogeneous polynomial. The two quartics $F_4(X_0, X_1, X_2) = 0$ and $X_0^4 + X_1^4 + X_2^4 = 0$ are not isomorphic because F_4 has 24 ordinary flexes and the other quartic has 12 hyperflexes (i.e. nonsingular points where the tangent lines meets the curve with multiplicity 4).

Can we infer, from the fact that it contains two non-isomorphic members, that the pencil depends continuously on one modulus? Of course our intuition suggests that this should be the case. But in moduli theory there are two pathologies that can appear, the existence of *isotrivial* *families* and the *jumping phenomenon* (both related to the presence of automorphisms). They are fatal to the existence of a good moduli space, and we cannot exclude just by intuition that they appear in this case. We will discuss later the subtle role of isotriviality, and we now give an example of jumping phenomenon.

Example 1. Consider the graded $\mathbb{C}[t]$-algebra $R = \mathbb{C}[t, X_0, X_1]$ and the graded R-module

$$M = \mathrm{coker}[\ R(-1) \xrightarrow{\begin{pmatrix} X_0 \\ X_1 \\ t \end{pmatrix}} R \oplus R \oplus R(-1)\ ]$$

Then $\mathrm{Proj}(R) = \mathbb{A}^1 \times \mathbb{P}^1$ and $\mathcal{F} := \widetilde{M}$ is a rank-two locally free sheaf on $\mathbb{A}^1 \times \mathbb{P}^1$. Then $\mathbb{P}(\mathcal{F}) \to \mathbb{A}^1 \times \mathbb{P}^1$ is a $\mathbb{P}^1$-bundle and, after composing with the projection, we get a proper smooth family:

$$f : \mathbb{P}(\mathcal{F}) \longrightarrow \mathbb{A}^1$$

whose fibres are rational ruled surfaces. Precisely, one checks that

$$f^{-1}(\alpha) = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}), \quad \alpha \neq 0$$
$$f^{-1}(0) = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(1))$$

Therefore all fibres over $\mathbb{A}^1 \setminus \{0\}$ are pairwise isomorphic, while $f^{-1}(0)$ belongs to a different isomorphism class. There is no continuous variation of isomorphism classes of the fibres because we have a jumping above 0. This excludes the possibility of having a space parametrizing isomorphism classes of ruled surfaces and with reasonable geometric properties. For details about this example we refer to [37], p. 53-54.

As observed, we cannot a priori exclude that a similar jumping phenomenon takes place in the pencil (3). We will eventually do, but after some hard work has been done. It's now time to move to a more advanced point of view.

3. Families

A *family* of projective nonsingular irreducible curves of genus g parametrized by a scheme B is a projective smooth morphism:

$$f : \mathcal{C} \longrightarrow B$$

whose fibres are projective nonsingular curves of genus g. The fibre over a point $b \in B$ will be denoted by $f^{-1}(b)$ or by $\mathcal{C}(b)$. Recall that the *genus* of a nonsingular curve C is

$$g(C) := \dim(H^1(C, \mathcal{O}_C)) = \dim(H^0(C, \Omega^1_C))$$

More generally, if C is a possibly singular projective curve we define its *arithmetic genus* by

$$p_a(C) := 1 - \chi(\mathcal{O}_C)$$

Clearly $g(C) = p_a(C)$ if C is nonsingular and irreducible.

A *family of deformations of a given projective nonsingular curve* C parametrized by a pointed scheme (B, b) is a pullback diagram:

$$\begin{tikzcd} C \arrow[r, hook] \arrow[d] & \mathcal{C} \arrow[d, "f"] \\ \mathrm{Spec}(\mathbb{C}) \arrow[r, "b"] & B \end{tikzcd} \tag{4}$$

where f is a family of projective nonsingular curves of genus g. This means that, in addition to the family f, an isomorphism $C \cong \mathcal{C}(b)$ is given.

In the above definitions we can replace smooth curves by possibly singular ones, but then we must require that the family f is flat.

An *isomorphism between two families of curves* $f : \mathcal{C} \to B$ and $\varphi : \mathcal{D} \to B$ is just a B-isomorphism:

$$\begin{tikzcd} \mathcal{C} \arrow[rr, "\Phi"] \arrow[dr, "f"'] & & \mathcal{D} \arrow[dl, "\varphi"] \\ & B & \end{tikzcd}$$

The family $f : \mathcal{C} \to B$ is *trivial* if it is isomorphic to a *product family*

$$p : B \times C \to B$$

for some curve C.

An *isomorphism between two families of deformations of* C, say (4) and

$$\begin{tikzcd} C \arrow[r, hook] \arrow[d] & \mathcal{D} \arrow[d, "\varphi"] \\ \mathrm{Spec}(\mathbb{C}) \arrow[r, "b"] & B \end{tikzcd}$$

is an isomorphism between the families f and φ which commutes with the identifications of C with $\mathcal{C}(b)$ and with $\mathcal{D}(b)$. Similarly one defines the notion of *trivial deformation of* C.

A family of curves *embedded* in a projective variety X is a commutative diagram:

$$\begin{tikzcd} \mathcal{C} \arrow[r, hook, "j"] \arrow[d, "f"'] & B \times X \arrow[dl, "\pi"] \\ B & \end{tikzcd}$$

where f is a family of projective curves of genus g and π is the projection.

Most important case: $X = \mathbb{P}^r$. One may include the case $r = 1$ by replacing the inclusion j by a finite flat morphism. In this case for each closed point $b \in B$ the fibre $j(b) : \mathcal{C}(b) \to \mathbb{P}^1$ will be a ramified cover of degree independent of b.

If B is irreducible then $\dim(B)$ is defined to be the *number of parameters of the family f.*

In practice one often considers families including singular curves among their fibres. One then replaces the smoothness condition by flatness, which guarantees the constancy of the arithmetic genus of the fibres.

For example, the pencil of plane quartics considered before defines a family of curves embedded in $\mathbb{P}^2$:

$$\begin{tikzcd} \mathcal{C} \arrow[r, hook] \arrow[d] & \mathbb{P}^1 \times \mathbb{P}^2 \arrow[dl] \\ \mathbb{P}^1 & \end{tikzcd}$$

parametrized by $\mathbb{P}^1$, where $\mathcal{C}$ is defined by the bihomogeneous equation of the pencil. The general fibre of this pencil is nonsingular, but there are some singular fibres.

4. Moduli functors

We would like to construct, for each $g \geq 0$, an algebraic $\mathbb{C}$-scheme M_g, to be called "moduli space of curves", whose closed points are in 1–1 correspondence with the set of isomorphism classes of projective nonsingular irreducible curves (shortly "curves") of a given genus g.

We saw in §2 that in the case $g = 1$ the affine line $\mathbb{A}^1_{\mathbb{C}}$ does the job, even though we were not completely satisfied by this solution. One question remained unanswered:

Where should the structure of scheme of M_g come from?

We expect that such structure is *natural* in some sense, i.e. that it reflects in a precise way the nature of moduli as parameters. More clearly stated, we want a precise relation between M_g and families of curves of genus g. The functorial point of view comes to help at this point.

To every scheme X there is associated its *functor of points*

$$h_X : (\text{Schemes}) \to (\text{Sets}), \quad h_X(S) = \mathrm{Mor(S,X)}$$

and conversely X can be reconstructed from this functor. So we must look for a functor on the first place and it must be related with families of curves of genus g. Here is one.

Setting

$$\mathcal{M}_g(B) = \left\{ \begin{matrix} \text{families } \mathcal{C} \to B \\ \text{of curves of genus } g \end{matrix} \right\} \Big/ \cong$$

(where by $\cong$ we mean "isomorphism") we obtain a contravariant functor

$$\mathcal{M}_g : (\text{Schemes}) \to (\text{Sets})$$

called the *moduli functor* of nonsingular curves of genus g.

The optimistic expectation is that $\mathcal{M}_g$ is representable, i.e. that it is the functor of points of a scheme M_g. The representability implies that M_g comes equipped with a *universal family* $\pi : \mathcal{X} \to M_g$ of curves of genus g.

"Universal" means that every other family $f : \mathcal{C} \to B$ of curves of genus g is obtained by pulling back π via a unique morphism $\mu_f : B \to M_g$. This property implies that the closed points of M_g are in 1-1 correspondence with the set of isomorphism classes of curves of genus g (because such isomorphism classes are in turn in 1-1 correspondence with families of the form $C \to \mathrm{Spec}(\mathbb{C})$ up to isomorphism). Therefore the morphism μ_f necessarily maps a $\mathbb{C}$-rational point $b \in B$ to the isomorphism class $[\mathcal{C}(b)]$.

The pair (M_g, π) would then represent the functor $\mathcal{M}_g$. In other words it would imply the existence of an isomorphism of functors

$$\mathcal{M}_g \cong h_{M_g}$$

and it would be fair to call such M_g *the moduli space* (or moduli scheme) of curves of genus g. Actually its name would be *fine moduli space*.

The situation is not that simple though. A universal family of curves of genus g does not exist and this is due to the existence of non-trivial families $f : \mathcal{C} \to B$ whose geometric fibres are pairwise isomorphic. In fact, such a family, like any other, should be induced by pulling back the universal family:

$$\begin{array}{ccc} \mathcal{C} & \longrightarrow & \mathcal{X} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \pi} \\ B & \xrightarrow[\mu_f]{} & M_g \end{array}$$

But since all fibres of f are isomorphic μ_f is constant and therefore f would be trivial.

Example 2. The above phenomenon appears already in genus $g = 0$. There is only one isomorphism class of curves of genus zero, namely $[\mathbb{P}^1]$. So if $\mathcal{M}_0$ were representable the universal family would just be $\mathbb{P}^1 \to \mathrm{Spec}(\mathbb{C})$. This would imply, as shown above, that every family $f : \mathcal{C} \to B$ of curves of genus zero is trivial. But this contradicts the existence of non-trivial ruled surfaces $\mathcal{C} \to B$, i.e. ruled surfaces not isomorphic to $B \times \mathbb{P}^1 \to B$.

Example 3. Consider the pencil of nonsingular plane cubics $\mathcal{C} \to \mathbb{A}^1$ given in affine coordinates by:

$$Y^2 = X^3 + t, \quad t \in \mathbb{A}^1 \setminus \{0\} \tag{5}$$

The j-invariant of $\mathcal{C}(t)$ is constant and equal to zero. Therefore all $\mathcal{C}(t)$ are pairwise isomorphic. But it is impossible to give an isomorphism of this family with the constant family $Y^2 = X^3 + 1$ without introducing the irrationality $t^{1/6}$. Therefore (5) is non-trivial.

A systematic way of producing non-trivial families of curves of higher genus whose geometric fibres are pairwise isomorphic is by means of the notion of isotriviality.

Definition 4. Let $f : \mathcal{Z} \to S$ be a flat family of algebraic schemes. Then f is called *isotrivial* if there is a finite surjective and etale morphism $S' \to S$ such that the induced family $f_{S'} : S' \times_S \mathcal{Z} \to S'$ is trivial. If $S' \times_S \mathcal{Z} \cong S' \times Z$ we say that f is *isotrivial with fibre* Z.

Example 5 (An example of isotrivial family)**.** Consider the family

$$f : \mathrm{Spec}(\mathbb{C}[Z, t, t^{-1}]/(Z^2 - t)) \longrightarrow \mathrm{Spec}(\mathbb{C}[t, t^{-1}]) = \mathbb{A}^1 \setminus \{0\}$$

For each $0 \neq a \in \mathbb{A}^1$ the fibre $f^{-1}(a) = \mathrm{Spec}(\mathbb{C}[Z]/(Z^2 - a))$ consists of two distinct reduced points: hence all fibres of f are isomorphic to $X = \mathrm{Spec}(\mathbb{C}[Z]/(Z^2 - 1))$. As in Example 3 one checks that the family is not trivial. Consider the etale morphism:

$$\beta : \mathrm{Spec}(\mathbb{C}[u, u^{-1}]) \longrightarrow \mathrm{Spec}(\mathbb{C}[t, t^{-1}]), \quad t \mapsto u^2$$

Pulling back f by β we obtain the family:

$$\mathrm{Spec}(\mathbb{C}[Z,u,u^{-1}]/(Z^2-u^2)) \longrightarrow \mathrm{Spec}(\mathbb{C}[u,u^{-1}])$$

and this family is trivial. Therefore f is isotrivial but not trivial.

The existence of isotrivial families is regulated by the following simple result.

Proposition 6. *The following conditions are equivalent on a quasi-projective scheme X:*

- *There exists a non-trivial isotrivial family with fibre X.*
- *The group $Aut(X)$ contains a non-trivial finite subgroup.*

For the proof we refer to [44], Th. 2.6.15. For each $g \geq 2$ there exist curves of genus g with non-trivial automorphisms, and all such curves have a finite group of automorphisms; therefore the proposition applies to them and implies the existence of non-trivial isotrivial families of curves of any genus $g \geq 2$.

The conclusion is that *a universal family of curves of genus g does not exist,* for all $g \geq 0$. Equivalently, *the moduli functor $\mathcal{M}_g$ is not representable* for all g.

No panic: despite these discouraging phenomena we still are on the right track because any reasonable structure on the set of genus g curves must be somehow compatible with the moduli functor. All we have to do is to weaken the condition that there is a universal family. There are several ways to do this. The one we choose is via the notion of *coarse moduli space.*

5. The coarse moduli space of curves

The following definition is due to Mumford [39].

Definition 7. *The* coarse moduli space of curves of genus g *is an algebraic scheme M_g such that:*

- *There is a morphism of functors $\mathcal{M}_g \to h_{M_g}$ which induces a bijection*

$$\mathcal{M}_g(\mathbb{C}) \cong M_g(\mathbb{C}) = \{\mathbb{C}\text{-}\textit{rational points of } M_g\}$$

- *If N is another scheme such that there is a morphism of functors $\mathcal{M}_g \to h_N$ then there is a unique morphism $M_g \to N$ such that the following diagram commutes:*

$$\begin{tikzcd} \mathcal{M}_g \arrow[r] \arrow[rr, bend left] & h_{M_g} \arrow[r] & h_N \end{tikzcd}$$

The definition implies that:

- The closed points of M_g are in 1–1 correspondence with the isomorphism classes of (nonsingular projective) curves of genus g.
- for every family $f : \mathcal{C} \to B$ of curves of genus g the set theoretic map
$$B(\mathbb{C}) \ni b \mapsto [f^{-1}(b)] \in M_g$$
defines a morphism $\mu_f : B \to M_g$. (this is the *universal property* of M_g).

It is easy to prove that, if it exists, M_g is unique up to isomorphism. In that case we say that $\mathcal{M}_g$ is *coarsely represented* by M_g.

The case $g = 0$ is trivial: $M_0 = \mathrm{Spec}(\mathbb{C})$ because $[\mathbb{P}^1]$ is the unique isomorphism class of curves of genus 0.

The case of genus 1, despite having served us as a useful heuristic introductory example, requires a special treatment because curves of genus 1 have a continuous group of automorphisms. It turns out to be more natural to consider families of *pointed* curves of genus 1. The functor of such curves then admits a coarse moduli space, which is isomorphic to $\mathbb{A}^1_{\mathbb{C}}$. The details of its construction are worked out in [28]. The general case is covered by the following deep result.

Theorem 8. *Let $g \geq 2$. Then:*
(i) *(Mumford* [39]*) M_g exists and is a quasi-projective normal algebraic scheme of dimension* $3g - 3$.
(ii) *(Deligne-Mumford* [11], *Fulton* [18]*) M_g is irreducible.*

The construction of M_g is obtained by means of Geometric Invariant Theory, which will not be considered in these lectures. As explained in the introduction of [11] the irreducibility of M_g had already been proved classically, via a topological analysis of families of branched coverings of $\mathbb{P}^1$. But an algebro-geometric proof was still lacking.

Moduli are local parameters on M_g around a given point $[C]$ and the *number of moduli* on which an abstract curve C depends is the dimension of M_g at $[C]$.

Now it is clear, at least theoretically, *how to distinguish moduli among parameters.* A family of curves $f : \mathcal{C} \to B$, with B an irreducible algebraic scheme, depends on $\dim(B)$ parameters and on $\dim(\mu_f(B))$ moduli.

For example in the *product family*
$$p : B \times C \longrightarrow B$$

all closed fibres are isomorphic to C and therefore $\mu_f(B) = \{[C]\}$: thus the number of moduli of this family is zero. This happens more generally if the family is isotrivial.

On the opposite side, an *effectively parametrized family* is one which depends on $\dim(B)$ moduli and such that μ_f is finite. This means that the fibre $\mathcal{C}(b)$ over any closed point $b \in B$ is isomorphic to only finitely many others.

Because of the universal property, every 1-parameter family of curves which contains two non-isomorphic fibres is effectively parametrized. In particular we can now deduce that the pencil of plane quartics (3) considered in §2 depends on one modulus.

A family $f : \mathcal{C} \to B$ is said to have *general moduli* if $\mu_f : B \to M_g$ is dominant. If a curve C is given as the general member of a family having general moduli, we say *C has general moduli* or that C is a *general curve of genus g.* This definition can be sometimes misleading because it presupposes that a family containing C has been given before we can say that it is a general curve. Nevertheless it is a classical and ubiquitous terminology.

Variants: moduli of pointed curves

Given $g \geq 0$ and $n \geq 1$ a useful variant of M_g is the coarse moduli space $M_{g,n}$ of *n-pointed curves of genus g.*

It parametrizes pairs $(C; p_1, \dots, p_n)$ consisting of a curve C of genus g and an ordered n-tuple $(p_1, \dots, p_n)$ of distinct points of C.

The corresponding moduli functor is

$$\mathcal{M}_{g,n}(B) = \{(f : \mathcal{C} \to B, \sigma_1, \dots, \sigma_m)\} \,/\text{isomorphism}$$

where $\sigma_1, \dots, \sigma_n : B \to \mathcal{C}$ are disjoint sections of $f : \mathcal{C} \to B$, and the notion of isomorphism is the obvious one.

6. The dimension of M_g

Riemann was able to count the number of moduli of curves of genus g, i.e. $\dim(M_g)$, by exhibiting a family of curves of genus g with general moduli in the following way. Assume $g \geq 4$.

Consider the family of all ramified covers of $\mathbb{P}^1$ of genus g and of a fixed degree n such that

$$\frac{g+2}{2} \leq n \leq g-1$$

We can represent it as

$$
\begin{tikzcd}
\mathcal{C} \arrow[r, hook, "j"] \arrow[d, "f"'] & B \times \mathbb{P}^1 \arrow[dl, "\pi"] \\
B &
\end{tikzcd}
$$

where B is a certain irreducible scheme. Riemann existence theorem implies that, associating to a cover the (ordered) set of its branch points we obtain a *finite* morphism $r : B \to (\mathbb{P}^1)^{2(n+g-1)}$. Therefore $\dim(B) = 2(n+g-1)$.

Consider $\mu_f : B \to M_g$. We have the following facts:

- each curve of genus g can be expressed as a ramified cover of $\mathbb{P}^1$ defined by a line bundle of degree n provided $n \geq \frac{g+2}{2}$. Therefore μ_f is surjective.
- Composing a cover $f : C \to \mathbb{P}^1$ with a non-trivial automorphism $\alpha : \mathbb{P}^1 \to \mathbb{P}^1$ we obtain another cover $\alpha \cdot f : C \to \mathbb{P}^1$ defined on the same curve by the same line bundle L.
- In the range $\frac{g+2}{2} \leq n \leq g-1$ the line bundles L of degree n with two sections on a given curve C depend on $2n-2-g$ parameters.

Therefore the general fibre of μ_f has dimension

$$\dim(\mathrm{PGL}(2)) + 2n - 2 - g = 2n + 1 - g$$

Then we conclude that:

$$
\begin{aligned}
\dim(M_g) &= \dim(\mathrm{Im}(\mu_{\mathrm{f}})) \\
&= \dim(B) - (2n+1-g) \\
&= 2(n+g-1) - (2n+1-g) \\
&= 3g-3
\end{aligned}
$$

This computation depends on several implicit assumptions but is essentially correct.

If $g = 2, 3$ one can take $n = 2, 3$ resp. and get the same result by a similar computation.

The previous computation is an example of *parameter counting*, a method that can be applied in several situations and is useful in computing the dimension of various loci in M_g. In such computations the universal property of M_g is used. There is a better way to perform them and it is by means of *deformation theory* (see §9).

7. Stable curves

The moduli space M_g is quasi-projective but it is not projective. The reason is that curves varying in a family may become singular. In that case one speaks of a *degenerating family* of curves.

It is therefore natural to consider the functor $\widetilde{\mathcal{M}}_g$, of which $\mathcal{M}_g$ is a subfunctor, defined as follows:

$$\widetilde{\mathcal{M}}_g(B) := \qquad \left\{ \begin{array}{c} \text{isom. classes of flat families} \\ \text{of curves of arithmetic genus } g \end{array} \right\}$$

and ask: is it possible to embed M_g into a projective scheme $\widetilde{M}_g$ which is a coarse moduli scheme for the functor $\widetilde{\mathcal{M}}_g$?

The answer is NO, as the following example shows.

Example 9. Consider a nonsingular curve C and a parameter nonsingular curve B. Let $\beta : S \to B \times C$ be the blow-up at a point $x \in B \times C$. We get a flat family:

$$f : S \xrightarrow{\beta} B \times C \xrightarrow{p} B$$

whose fibres over $B \setminus p(x)$ are isomorphic to C while $f^{-1}(p(x))$ is a reducible curve. This is again an example of jumping phenomenon and it implies that the functor $\widetilde{\mathcal{M}}_g$ cannot be coarsely represented.

But there is a nice solution if we modify the question by *allowing only certain singular curves.*

Definition 10. *A* stable curve *of genus $g \geq 2$ is a connected reduced curve of arithmetic genus g having at most nodes (i.e. ordinary double points) as singularities and such that every nonsingular rational component meets the rest of the curve in ≥ 3 points.*

Define the *moduli functor of stable curves* of genus $g \geq 2$ as follows:

$$\overline{\mathcal{M}}_g(B) := \qquad \left\{ \begin{array}{c} \text{isom. classes of flat families} \\ \text{of stable curves of genus } g \end{array} \right\}$$

We obviously have injective natural transformations of functors:

$$\mathcal{M}_g(B) \subseteq \overline{\mathcal{M}}_g(B) \subseteq \widetilde{\mathcal{M}}_g(B)$$

Theorem 11 (Deligne-Mumford)**.** *There is a projective scheme $\overline{M}_g$ containing M_g and coarsely representing the functor $\overline{\mathcal{M}}_g$. The complement $\overline{M}_g \setminus M_g$ is a divisor with normal crossings.*

Remark 12. M_g is not projective but not affine either. It is known that a priori it may contain projective subvarieties having up to dimension

$g-2$ (Diaz [12]) but the precise bound for their dimension is not known in all genera.

Other variants of M_g are the moduli spaces of *stable pointed curves.* We will not introduce them since they will not appear in our discussion.

8. The local structure of a scheme

Denote by $\widehat{\mathfrak{Loc}}$ the category of local noetherian complete $\mathbb{C}$-algebras with residue field $\mathbb{C}$, and local homomorphisms. Let

$$\mathbb{C}[\epsilon] := \mathbb{C}[t]/(t^2)$$

and $D := \mathrm{Spec}(\mathbb{C}[\epsilon])$. If $(R, \mathfrak{m})$ is in $\widehat{\mathfrak{Loc}}$ then

$$t_R = \left(\mathfrak{m}/\mathfrak{m}^2\right)^\vee = \mathrm{Hom}(R, \mathbb{C}[\epsilon])$$

is the (Zariski) tangent space of R.

The strictly local structure of a scheme X around a point $x \in X(\mathbb{C})$ is encoded by the *complete* local ring $\widehat{\mathcal{O}}_{X,x}$, which is by definition the following object of $\widehat{\mathfrak{Loc}}$:

$$\widehat{\mathcal{O}}_{X,x} = \lim_{\longleftarrow} \mathcal{O}_{X,x}/\mathfrak{m}^n_{X,x}$$

This ring tells us in particular about the dimension and the singularity of X at x. For example, X is nonsingular of dimension d at x if and only if $\widehat{\mathcal{O}}_{X,x}$ is isomorphic to the formal power series ring $\mathbb{C}[[X_1, \dots, X_d]]$ where $d = \dim(t_R)$.

If $\mu : B \to M$ is a morphism of reduced and irreducible schemes and $b \in B(\mathbb{C})$ is a nonsingular point then much of the local behaviour of μ at b is encoded by the local homomorphism:

$$\widehat{\mu}^\sharp : \widehat{\mathcal{O}}_{M,\mu(b)} \to \widehat{\mathcal{O}}_{B,b}$$

induced by μ. For example if

$$\mathrm{Spec}(\widehat{\mathcal{O}}_{B,b}) \to \mathrm{Spec}(\widehat{\mathcal{O}}_{M,\mu(b)})$$

is dominant then μ is dominant. The smoothness of μ at b is also encoded by the local homomorphism $\widehat{\mu}^\sharp$ because it is equivalent to the surjectivity of the differential:

$$d\mu_b : t_{\widehat{\mathcal{O}}_{B,b}} = T_bB \to T_{\mu(b)}M = t_{\widehat{\mathcal{O}}_{M,\mu(b)}}$$

To a ring R in $\widehat{\mathfrak{Loc}}$ one may associate a covariant functor defined on the category $\mathfrak{A}rt$ of local artinian $\mathbb{C}$-algebras with residue field $\mathbb{C}$:

$$\widehat{h}_R : \mathfrak{A}rt \longrightarrow (\mathrm{Sets})$$

defined by:

$$\widehat{h}_R(A) = \mathrm{Hom}_{\mathbb{C}-alg}(R, A)$$

for all objects A of $\mathfrak{A}rt$. Covariant functors $F : \mathfrak{A}rt \longrightarrow (\mathrm{Sets})$ are called *functors of Artin rings* and one of the type $\widehat{h}_R$ is said to be *prorepresented* by R. Note that $\widehat{h}_R$ is the restriction to $\mathfrak{A}rt$ of the representable functor

$$h_R : \widehat{\mathfrak{L}oc} \longrightarrow (\mathrm{Sets})\,, \quad h_R(S) = \mathrm{Hom}_{\mathbb{C}-alg}(R, S)$$

on the larger category $\widehat{\mathfrak{L}oc}$.

A morphism $\varphi : R \to S$ in $\widehat{\mathfrak{L}oc}$ induces a natural transformation $\widehat{h}_S \to \widehat{h}_R$ which, among other information, encodes the differential

$$d\varphi : t_S = \widehat{h}_S(\mathbb{C}[\epsilon]) \to \widehat{h}_R(\mathbb{C}[\epsilon]) = t_R$$

For an arbitrary functor of Artin rings F one can consider $t_F := F(\mathbb{C}[\epsilon])$. Under some conditions t_F ha a structure of $\mathbb{C}$-vector space and we are authorized to call it the *tangent space to the functor* F.

It is interesting that a ring R in $\widehat{\mathfrak{L}oc}$ can be recovered if we know the functor $\widehat{h}_R$ ([44], Prop. 2.3.1). Therefore the prorepresentable functors play a role with respect to local properties of schemes analogous to (the role of) functors of points in characterizing schemes globally. One can start from a functor of Artin rings and try to find conditions for its prorepresentability. This problem arises naturally for example in the formalization of deformation theory due to A. Grothendieck. He introduced the functors of Artin rings and gave a characterization of the prorepresentable ones (see [22, 23]). His results have been later improved by M. Schlessinger [42].

9. The local structure of M_g

We want to apply the remarks made in the previous section to the study of the local structure of M_g at a given point $[C]$. For doing this some of the technicalities of deformation theory are needed. Since they are not appropriate for a survey article of this kind we will only briefly outline the main steps.

Using the universal property it is natural to consider families of the form $f : \mathcal{C} \to \mathrm{Spec}(A)$, where $(A, \mathfrak{m})$ is in $\mathfrak{A}rt$, such that there exists an isomorphism $C \cong f^{-1}([\mathfrak{m}])$. We call them *infinitesimal families at* $[C]$ parametrized by A (or by $\mathrm{Spec}(A)$).

We can now define a functor of Artin rings:[1]

$$\mathrm{Inf}_{[C]} : \mathfrak{A}rt \longrightarrow (\mathrm{Sets})$$

[1]This functor is called the *crude local functor* in [28], §18.

by setting:

$$\mathrm{Inf}_{[C]}(A) := \begin{pmatrix}\text{infinitesimal families at } [C] \\ \text{parametrized by } A\end{pmatrix} \Big/ \cong$$

Since M_g represents the moduli functor $\mathcal{M}_g$ only coarsely, we cannot expect that $\mathrm{Inf}_{[C]}$ is prorepresentable. All we can get from the universal property of M_g is a natural transformation:

$$\mu_C : \mathrm{Inf}_{[C]} \longrightarrow \widehat{h}_{\widehat{\mathcal{O}}}$$

where $\widehat{\mathcal{O}} = \widehat{\mathcal{O}}_{M_g,[C]}$. But this is not very useful. On the other hand an infinitesimal family at $[C]$

$$f : \mathcal{C} \to \mathrm{Spec}(A)$$

is very close to being a deformation of C; we only need to further specify an isomorphism $C \cong f^{-1}([\mathfrak{m}])$. Once we do this we call the resulting deformation:

$$\begin{tikzcd} C \arrow[r] \arrow[d] & \mathcal{C} \arrow[d, "f"] \\ \mathrm{Spec}(\mathbb{C}) \arrow[r] & \mathrm{Spec}(A) \end{tikzcd}$$

an *infinitesimal deformation* of C parametrized by A and we define a functor of Artin rings Def_C by:

$$\mathrm{Def}_C(A) := \begin{pmatrix}\text{infinitesimal deformations of } C \\ \text{parametrized by } A\end{pmatrix} \Big/ \cong$$

This functor is more interesting. For example its tangent space is easy to describe.

Definition 13. *A* first order deformation *of C is a family of deformations of C parametrized by $\mathbb{C}[\epsilon]$:*

$$\begin{tikzcd} C \arrow[r, hook] \arrow[d] & \mathcal{C} \arrow[d, "f"] \\ \mathrm{Spec}(\mathbb{C}) \arrow[r, "(\epsilon)"] & D \end{tikzcd} \tag{6}$$

Therefore $t_{\mathrm{Def}_C} = \mathrm{Def}_C(\mathbb{C}[\epsilon])$ is the set of isomorphism classes of first order deformations of C.

Proposition 14. *There is a natural identification:*

$$\kappa : \mathrm{Def}_C(\mathbb{C}[\epsilon]) \cong H^1(C, T_C)$$

The cohomology class $\kappa(f)$ associated to a first order deformation (6) *is called the* Kodaira-Spencer class *of f (KS class).*

For a proof of Proposition 14 we refer to [44], Prop. 1.2.9, where it is proved, more generally, for the deformation functor of a nonsingular projective variety of any dimension. The KS class can be used to introduce an important map associated to any family of deformations of C:

$$\begin{tikzcd} C \arrow[r, hook] \arrow[d] & \mathcal{C} \arrow[d, "f"] \\ \mathrm{Spec}(\mathbb{C}) \arrow[r, "b"] & B \end{tikzcd}$$

called *Kodaira-Spencer map* of f. It is the map:

$$\kappa : T_b B \longrightarrow H^1(C, T_C)$$

which associates to a tangent vector $v : D \to B$ at b the KS class $\kappa(f_v)$ of the first order deformation of C:

$$\begin{tikzcd} C \arrow[r, hook] \arrow[d] & \mathcal{C}_D \arrow[r] \arrow[d, "f_v"] & \mathcal{C} \arrow[d, "f"] \\ \mathrm{Spec}(\mathbb{C}) \arrow[r, "(\epsilon)"] & D \arrow[r, "v"] & B \end{tikzcd}$$

obtained by pulling back f to D. The KS map is linear. As the name suggests, it has been introduced by Kodaira and Spencer in [34], and will play an important role in what follows.

Since two isomorphic infinitesimal deformations of C are supported on isomorphic infinitesimal families at $[C]$, we get a natural "forgetful" transformation of functors

$$\mathrm{Def}_C \longrightarrow \mathrm{Inf}_{[C]}$$

obtained by forgetting the isomorphism $C \cong f^{-1}([\mathfrak{m}])$. By composition with the transformation μ_C we obtain a natural transformation:

$$\Phi : \mathrm{Def}_C \longrightarrow \widehat{h}_{\widehat{\mathcal{O}}}$$

The precise relation between these functors is given by the following:

Theorem 15. (i) Def_C *is prorepresentable. Precisely, if* $g \geq 2$ Def_C *is prorepresented by* $R = \mathbb{C}[[X_1, \dots, X_{3g-3}]]$*, if* $g = 1$ *by* $\mathbb{C}[[X]]$ *and if* $g = 0$ *by* $\mathbb{C}$*.*

(ii) Φ *corresponds to a local homomorphism* $\widehat{\mathcal{O}} \to R$ *such that the induced morphism* $\widetilde{\Phi} : \mathrm{Spec}(R) \to \mathrm{Spec}(\widehat{\mathcal{O}})$ *is dominant with finite fibres.*

(iii) *The following conditions are equivalent:*
- C *has no non-trivial automorphisms.*
- $\widetilde{\Phi}$ *is an isomorphism.*

– *M_g is nonsingular at $[C]$.*

Proof. See [44] and [28]. □

As far as the study of families of curves is concerned, it is not so important to dig into the structure of the local ring $\widehat{\mathcal{O}}$ anymore. It suffices to record that the GIT construction shows that, for $g \geq 2$, M_g is locally the quotient of a nonsingular variety of dimension $3g-3$ by the finite group $\mathrm{Aut}(C)$. Theorem 15 is the formal analogue of this fact.

10. Families with general moduli

Suppose given a family of curves of genus g:

$$f : \mathcal{C} \to B$$

parametrized by an algebraic variety B. Let $b \in B$ be a closed nonsingular point and $C = f^{-1}(b)$. We would like to have a criterion to decide whether $\mu_f : B \to M_g$ is dominant, i.e. if f has general moduli, using only local information about f at b. If the differential:

$$d\mu_{f,b} : T_b B \longrightarrow T_{[C]} M_g$$

is surjective then μ_f is smooth at b and therefore the family f has general moduli. This criterion can possibly work only if M_g is nonsingular at $[C]$. A more efficient result is the following.

Proposition 16. *Let $f : \mathcal{C} \to B$ be a family of curves of genus g parametrized by an algebraic variety B, let $b \in B$ be a closed nonsingular point and $C = f^{-1}(b)$. If the KS map:*

$$\kappa : T_b B \longrightarrow H^1(C, T_C)$$

is surjective then f is a family with general moduli.

Proof. Let $S = \widehat{\mathcal{O}}_{B,b}$. The family f defines a natural transformation of functors of Artin rings:

$$\widehat{h}_S \longrightarrow \mathrm{Def}_C$$

by associating to every A in $\mathfrak{A}rt$ and $(\alpha : S \to A) \in \widehat{h}_S(A)$ the infinitesimal deformation:

$$\begin{array}{ccc} C & \longrightarrow & \mathrm{Spec}(A) \times_B \mathcal{C} \\ \downarrow & & \downarrow \\ \mathrm{Spec}(\mathbb{C}) & \longrightarrow & \mathrm{Spec}(A) \end{array}$$

obtained by pulling f via α. This natural transformation corresponds to a homomorphism: $\Psi : R \to S$ where R is the ring prorepresenting Def_C (Theorem 15(i)). We have a commutative diagram:

$$\begin{tikzcd} \mathrm{Spec}(S) \arrow[rr, "\widehat{\mu}"] \arrow[dr, "\widetilde{\Psi}"'] & & \mathrm{Spec}(\widehat{\mathcal{O}}) \\ & \mathrm{Spec}(R) \arrow[ur, "\widetilde{\Phi}"'] & \end{tikzcd}$$

where $\widehat{\mu}$ is induced by the functorial morphism $\mu_f : B \to M_g$. Since $\kappa = d\Psi$ is surjective $\widetilde{\Psi}$ is smooth, and by Theorem 15(ii) $\widetilde{\Phi}$ is dominant. Thus $\widehat{\mu}$ is dominant and therefore also μ_f is dominant. $\square$

Note that the criterion of Proposition 16 applies regardless of whether $\mathrm{Aut}(C)$ is trivial or not.

11. The Hilbert scheme

The next step is to have a good description of families of curves in a given projective space, which are the most important families appearing in nature. This is done by introducing new objects, the Hilbert schemes. For details on this section we refer to [5, 44].

We will introduce first the functors that are represented by the Hilbert schemes. Fix a polynomial $p(t) \in \mathbb{Q}[t]$ and define a contravariant functor:

$$Hilb^r_{p(t)} : (\text{Schemes}) \longrightarrow (\text{Sets})$$

setting

$$Hilb^r_{p(t)}(B) = \left\{ \begin{array}{c} \text{families of closed subschemes of } \mathbb{P}^r \\ \text{param. by } B \text{ and with Hilbert polyn. } p(t) \end{array} \right\}$$

This is the *Hilbert functor* for the polynomial $p(t)$.

Theorem 17 (Grothendieck [24])**.** *For every $r \geq 2$ and $p(t)$ there is a projective scheme $\mathrm{Hilb}^r_{p(t)}$ and a family*

(7)
$$\begin{tikzcd} \mathcal{X} \arrow[r, hook] \arrow[d, "f"'] & \mathrm{Hilb}^r_{p(t)} \times \mathbb{P}^r \arrow[dl] \\ \mathrm{Hilb}^r_{p(t)} & \end{tikzcd}$$

which is universal for the functor $Hilb^r_{p(t)}$. In particular $Hilb^r_{p(t)}$ is representable.

$\mathrm{Hilb}^r_{p(t)}$ is called *Hilbert scheme* of $\mathbb{P}^r$ relative to the Hilbert polynomial $p(t)$.

It is a very complicated object, highly reducible and singular, in general non-reduced, but connected. Its local properties at a point $[X \subset \mathbb{P}^r]$ depend only on the geometry of the embedding $X \subset \mathbb{P}^r$, as the following shows.

Theorem 18 (Grothendieck [24]). *Let $X \subset \mathbb{P}^r$ be a local complete intersection with Hilbert polynomial $p(t)$. Let $\mathcal{I} \subset \mathcal{O}_{\mathbb{P}^r}$ be its ideal sheaf and $N = N_{X/\mathbb{P}^r} := \mathcal{H}om(\mathcal{I}/\mathcal{I}^2, \mathcal{O}_X)$ its normal bundle. Then:*

- *$H^0(X, N)$ is the Zariski tangent space to $\mathrm{Hilb}^r_{p(t)}$ at $[X]$.*
- *$h^0(X, N) - h^1(X, N) \leq \dim_{[X]}(\mathrm{Hilb}^r_{p(t)}) \leq h^0(X, N)$.*
- *If $H^1(X, N) = 0$ then $\mathrm{Hilb}^r_{p(t)}$ is nonsingular of dimension $h^0(X, N)$ at $[X]$.*

If $C \subset \mathbb{P}^r$ is a nonsingular curve of degree d and genus g then the Hilbert polynomial of C is $p(t) = dt + 1 - g$ and we write $\mathrm{Hilb}^r_{d,g}$ instead of $\mathrm{Hilb}^r_{p(t)}$. Then:

$$h^0(C, N) - h^1(C, N) = \chi(C, N) = (r+1)d + (r-3)(1-g)$$

For example, if C is a nonsingular plane curve of degree d its genus is $g = \binom{d-1}{2}$ and $N = \mathcal{O}_C(d)$. Then $H^1(C, N) = 0$ and

$$h^0(C, N) = 3d + g - 1 = \frac{d(d+3)}{2} = \binom{d+2}{2} - 1$$

If $C \subset \mathbb{P}^3$ is a nonsingular curve of degree d and genus g then $\chi(N) = 4d$ does not depend on g. $\mathrm{Hilb}^3_{d,g}$ can be singular and/or of dimension $> 4d$.

If $C \subset \mathbb{P}^r$ is nonsingular of degree d and genus g the KS map of the universal family (7) at the point $[C]$ is a map:

$$\kappa_C : T_{[C]}\mathrm{Hilb}^r_{d,g} = H^0(C, N_{C/\mathbb{P}^r}) \longrightarrow H^1(C, T_C) \tag{8}$$

Proposition 19. *κ_C is the coboundary map of the* normal sequence

$$0 \longrightarrow T_C \longrightarrow T_{\mathbb{P}^r|C} \longrightarrow N_{C/\mathbb{P}^r} \longrightarrow 0$$

Proof. An easy diagram chasing. □

It is of primary importance to decide for which d, g, r there is an irreducible component of the Hilbert scheme $\mathrm{Hilb}^r_{d,g}$ such that the universal family restricted to it has general moduli. For the investigation of this condition we introduce a new object.

Definition 20. *Let C be a curve and L an invertible sheaf on C. The multiplication map:*

$$\mu_0(L) : H^0(L) \otimes H^0(\omega_C L^{-1}) \longrightarrow H^0(\omega_C)$$

is the Petri map *relative to L.*

We have:

Proposition 21. *Let $C \subset \mathbb{P}^r$ be of degree d and genus g, $L = \mathcal{O}_C(1)$, and assume that $\mu_0(L)$ is injective. Then $[C] \in \mathrm{Hilb}^r_{d,g}$ is a nonsingular point and the map (8) is surjective. Therefore the universal family (7) has general moduli around $[C]$ (shortly, the curves parametrized by $\mathrm{Hilb}^r_{d,g}$ nearby $[C]$ have general moduli).*

Proof. We can reduce to the case $r+1 = h^0(L)$. Then we use the *restricted Euler sequence*

$$0 \longrightarrow \mathcal{O}_C \longrightarrow H^0(L)^\vee \otimes L \longrightarrow T_{\mathbb{P}^r|C} \longrightarrow 0$$

to show that the injectivity of $\mu_0(L)$ is equivalent to $H^1(C, T_{\mathbb{P}^r|C}) = 0$. Then we use Proposition 16, Theorem 18 and Proposition 19 to conclude. $\square$

If L is an invertible sheaf on C such that $\deg(L) = d$ and $h^0(L) = r+1$ then the *expected corank* of $\mu_0(L)$ is

$$\rho(g, r, n) := g - (r+1)(g - d + r)$$

It is called the *Brill-Noether number* relative to g, r, d. Of course for $\mu_0(L)$ to be injective it is necessary that $\rho(g, r, n) \geq 0$. The condition $\rho(g, r, d) \geq 0$ is equivalent to $d \geq \frac{1}{2}g + 1$ if $r = 1$, and to $d \geq \frac{2}{3}g + 2$ if $r = 2$, etc.

On the other hand $\mu_0(L)$ can have a non-zero kernel even if $\rho(g, r, n) \geq 0$, as shown by plenty of examples. Nevertheless the following important result holds.

Theorem 22. *Let $r \geq 3$ and $d, g \geq 0$ such that $\rho(g, r, d) \geq 0$. Then $\mathrm{Hilb}^r_{d,g}$ has a unique irreducible component $\mathbf{I}^r_{d,g}$ whose general point parametrizes a nonsingular irreducible curve $C \subset \mathbb{P}^r$ such that $h^0(\mathcal{O}_C(1)) = r+1$ and $\mu_0(\mathcal{O}_C(1))$ is injective. In particular an open non-empty subset of $\mathbf{I}^r_{d,g}$ parametrizes a family of curves of genus g with general moduli.*

If $r = 2$ and $d, g \geq 0$ are such that $d \geq \frac{2}{3}g + 2$ then there is an irreducible locally closed $\mathcal{V}_{d,g} \subset |H^0(\mathbb{P}^2, \mathcal{O}(d))|$ parametrizing irreducible plane curves of degree d and geometric genus g such that:

(a) *Every* $\Gamma \subset \mathbb{P}^2$ *parametrized by a point of* $\mathcal{V}_{d,g}$ *has only nodes as singularities and the sheaf* $L = \nu^*\mathcal{O}_\Gamma(1)$ *on the normalization* $\nu : C \to \Gamma$ *satisfies* $h^0(L) = 3$ *and* $\mu_0(L)$ *injective.*
(b) *The restriction to* $\mathcal{V}_{d,g}$ *of the universal family can be simultaneously desingularized and the resulting family of nonsingular curves of genus g has general moduli.*
(c) *There is a unique* $\mathcal{V}_{d,g}$ *maximal with respect to properties (a) and (b).*

This theorem has been forseen since the beginning of curve theory, starting from Brill-Noether [6] and Severi [46]. It is due to the concentrated efforts of several mathematicians during the 1970's and 80's: Kleiman-Laksov [33], Kempf [32], Arbarello-Cornalba [4], Eisenbud-Harris [14], Gieseker [20], Fulton-Lazarsfeld [19], Harris [26].

12. Petri general curves

The Petri map is a central object in curve theory.

Definition 23. *A curve C is called* Petri general *if the Petri map* $\mu_0(L)$ *is injective for all invertible sheaves* $L \in Pic(C)$.

Note that if $\deg(L) < 0$ or $\deg(L) > 2g-2$ then $\mu_0(L)$ is clearly injective. Moreover $\mu_0(L) = \mu_0(\omega_C L^{-1})$. Therefore the condition that C is Petri general has to be checked only on invertible sheaves such that $0 \leq \deg(L) \leq g-1$.

A Petri general curve has the property that any embedding $C \subset \mathbb{P}^r$, $r \geq 3$, by a complete linear system corresponds to a nonsingular point of one of the components $\mathbf{I}^r_{d,g}$ of the Hilbert scheme described by Theorem 22.

The definition of Petri general curve is completely intrinsic, i.e. it does not make use of families. In a footnote to [41] K. Petri, a student of M. Noether, stated as a fact what has been subsequently considered as

Petri's conjecture: For every g a general curve of genus g is Petri general.

This conjecture asserts that for a curve C the condition of being Petri general does not impose any closed condition on its moduli. According to the conjecture Petri general curves should be the most natural ones available in nature. But in fact this is not the case.

For example nonsingular plane curves of degree $d \geq 5$ are not Petri general.

In fact if $C \subset \mathbb{P}^2$ is of degree d then $\omega_C = \mathcal{O}(d-3)$ and therefore

$$H^0(\omega_C L^{-2}) = H^1(\mathcal{O}_C(2))^\vee \neq 0$$

if $d \geq 5$. Then use a simple remark which shows that $H^0(\omega_C L^{-2}) \subset \ker(\mu_0(L))$ for any L on any curve. A similar argument holds for complete intersections of multidegree $(d_1, \ldots, d_{r-1})$ such that $\sum d_j \geq r+3$.

It is very difficult to produce explicit examples of Petri general curves (see the final section for more about this). So the challenge of Petri's conjecture, if true, is to modify our intuitive idea of a general curve.

The conjecture is in fact true. It has been proved for the first time by Gieseker [20], and subsequently it has been given simpler proofs by Eisenbud and Harris [14] and by Lazarsfeld [35]. Special cases of the conjecture had been proved before by Arbarello and Cornalba [4].

13. Unirationality

To describe explicitly a general curve of genus g is the most elusive part of the theory. To write down equations of a general curve with coefficients depending on parameters requires solving a mixture of abstract and concrete problems that are very difficult to concile.

For instance, it is very difficult to give an explicit description/construction of the family parametrized by $\mathbf{I}^r_{d,g}$ described in Theorem 22, even if we know that it exists. The attempts to construct such explicit families have a long history and have motivated a large amount of work on M_g. The classical geometers succeeded for the first values of g and observed that for the families they got the parameter variety is rational or unirational. For example, a canonical curve of genus 3 is just a nonsingular plane quartic: it moves in the linear system $|H^0(\mathbb{P}^2, \mathcal{O}(4))| \cong \mathbb{P}^{14}$. A similar remark can be made for genus 4 and 5 since canonical curves of genus 4 and 5 are complete intersections in $\mathbb{P}^3$ (resp. $\mathbb{P}^4$).

One can prove, because of Theorem 22, that the general curve C of genus 6 can be realized as a plane sextic curve with 4 double points (see the table below). The linear system of adjoints to C of degree 3 maps $\mathbb{P}^2$ birationally to a Del Pezzo surface $S \subset \mathbb{P}^5$ containing the canonical model of C. Now $C \subset \mathbb{P}^5$ is the complete intersection of S with a quadric, therefore it varies in the linear system $|\mathcal{O}_S(2)|$, which is 15-dimensional. Since any two general 4-tuples of points in $\mathbb{P}^2$ are projectively equivalent, all the Del Pezzo surfaces costructed in this way are isomorphic. Therefore the general curve C of genus 6 is parametrized by a point varying in a $(15 = 3 \cdot 6 - 3)$-dimensional linear system on a fixed surface S, and M_6 is thus unirational. A more delicate argument, due to Shepherd-Barron [47], shows that M_6 is even rational.

One may ask whether an analogous statement is true for higher values of g, namely whether it is possible to produce a family of projective curves with general moduli parametrized by a rational variety, say an open subset of a projective space. For such a family $f : \mathcal{C} \to B$ the functorial morphism $\mu_f : B \to M_g$, being dominant, proves that M_g is unirational.

To my knowledge M. Noether was the first to ask such a question. He extended the above constructions to the more difficult case of genus 7 proving the unirationality of M_7 by explicit describing the canonical curves [40].

Subsequently Severi [45] extended the result up to genus 10. The proof given by Severi is quite simple and can be easily understood by looking at the following table which lists degree and number of nodes of plane curves of minimal degree and non-negative Brill-Noether number.

genus	degree	δ	$\frac{d(d+3)}{2} - 3\delta$	ρ
0	1	0	2	0
1	3	0	9	1
2	4	1	11	2
3	4	0	14	0
4	5	2	14	1
5	6	5	12	2
6	6	4	15	0
7	7	8	11	1
8	8	13	5	2
9	8	12	8	0
10	9	18	0	1
11	10	25	-10	2
12	10	24	-7	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

`Families of plane nodal curves with general moduli`

In outline it goes as follows. The table shows that if $g \leq 10$ it is possible to realize a general curve of genus g as a plane curve of degree d with δ singular points in such a way that

$$3\delta \leq \frac{d(d+3)}{2}$$

This implies, modulo a careful argument, that we can assign the singular points of such a curve in general position. Then the parameter space of the corresponding family $f : \mathcal{C} \to \mathcal{V}_{d,g}$ is fibered over an open subset of $(\mathbb{P}^2)^{(\delta)}$ with fibres linear systems, and therefore $\mathcal{V}_{d,g}$ is rational.

Based on such limited evidence Severi conjectured the unirationality of M_g for all g [45]. This conjecture resisted for 67 years, even though it was certainly considered with great interest (see e.g. [38]), until it was finally disproved by Harris and Mumford in 1982 [27]. One year earlier the unirationality of M_{12} had been proved [43] and other results followed promply ([36, 8, 15, 9, 29, 30, 31, 47]. Other results have been proved in more recent years [16, 7, 17, 48]. Now we have a quite precise information about the Kodaira dimension of M_g for almost all g, summarized in the table below. The problem is not yet closed though because as of today (june 2016) we have no idea about the Kodaira dimension of M_g for $17 \leq g \leq 21$. This is a challenge for younger generations!

For a detailed discussion of the vast topic touched in this section I refer the reader to the survey article of Verra [49].

genus	K-dim	credit
$1, 2, 3, 4, 6$	rational	Weierstrass-Salmon (1), Igusa (2) Katsylo (3,4,5), Shepherd-Barron (6)
11	uniruled	Mori-Mukai
≤ 14	unirational	Noether (≤ 7), Severi (≤ 10), Sernesi (12), Chang-Ran $(11, 13)$, Verra (14)
15	rat.lly connected	Chang-Ran ($\kappa = -\infty$), Bruno-Verra
16	uniruled	Chang-Ran ($\kappa = -\infty$), Farkas
$22, \geq 24$	gen. type	Farkas (22), Harris-Mumford (odd), Eisenbud-Harris (even)
23	≥ 2	Farkas

State of the art about the Kodaira dimension κ of M_g

14. Construction of Petri general curves

The existence of Petri general curves has been proved originally [20] by degeneration, thus in a non-effective way. The following has been a breakthrough:

Theorem 24 (Lazarsfeld [35])**.** *If S is a K3 surface with* $\mathrm{Pic}(S) = \mathbb{Z}[H]$ *then a general curve $C \in |H|$ is Petri general.*

This result is still non-effective, but it brings very clearly on the forefront the fact that *Petri general curves are not necessarily appearing in families with general moduli.* In fact the specific classes of Petri general curves described by Theorem 24 (we will call them *K3-curves*) are an illustration of this fact. Let's count parameters.

- Pairs (S, H) depend on 19 moduli.
- The linear system $|H|$ on a given (S, H) has dimension $g = \frac{1}{2}(H \cdot H) + 1$.

Therefore K3-curves depend on $\leq g + 19$ moduli. If $g \geq 12$ then $3g - 3 > g + 19$ and therefore a K3-curve of genus g is not a general curve if $g \geq 12$, even though it is Petri general.

K3-curves of genus $g \geq 12$ have been characterized recently among the Petri general ones by means of a cohomological condition by Arbarello, Bruno and Sernesi [3]. The condition is that the so-called *Wahl map*

$$\bigwedge^2 H^0(\omega_C) \longrightarrow H^0(\omega^3) \tag{9}$$

is not surjective. Precisely, the map (9) is known to be surjective on general curves [10] and therefore its non-surjectivity defines a closed locus $W \subset M_g$. It has also been known since some time [50] that the locus of K3-curves is contained in W. In [3] it is proved that W intersects the open set of Petri general curves precisely along the closure of the locus of K3-curves.

This argument leaves open the possibility that there exist Petri general curves in W that are limits of K3-curves without being K3-curves, namely Petri general curves not contained in a K3 surface but only on a (singular) limit of K3 surfaces. This question has been considered recently by Arbarello, Bruno, Farkas and Saccà in [2]. In their work the authors give examples of Petri general curves C of every genus $g \geq 1$ on certain rational surfaces that are limits of K3 surfaces. These surfaces are obtained by contracting the exceptional elliptic curve on the blow-up of $\mathbb{P}^2$ at 10 points $p_1, \ldots, p_{10} \in \mathbb{P}^2$ conveniently chosen on a cubic. The curves C, called *Du Val curves*, are the images of the proper transforms of curves of degree $3g$ passing through the points $p_1, \ldots, p_9, p_{10}$ with multiplicity $(g, \ldots, g, g-1, 1)$. In [1] Arbarello and Bruno have finally shown that there exist Du Val curves which are not K3-curves even though they are limits of K3-curves, thereby proving that the result of [3] is sharp.

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