DEFORMATION THEORY
Notes from a graduate course
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Contents

S:DTgoals

0.1. Goals and strategy of Deformation Theory

Deformation Theory (DT) is closely related with the problems of classification in Algebraic Geometry (AG). Given a class $\mathcal{M}$ of objects, for instance

$$\mathcal{M} = \{\text{isomorphism classes of projective nonsingular varieties of given invariants}\}$$

$$\mathcal{M} = \{\text{vector bundles of fixed rank and Chern classes on a variety } X\}$$

$$\mathcal{M} = \{\text{closed subschemes } X \subset \mathbb{P}^r \text{ with fixed Hilbert polynomial}\}$$

etc., the corresponding classification problem is: describe $\mathcal{M}$. The interest and difficulty of this problem are due to the existence of families. For each class $\mathcal{M}$ we will be able to define the appropriate notion of "family of objects of $\mathcal{M}$ parametrized by a scheme S", and this notion will be functorial. In other words we will be able to define a (contravariant) functor:

$$F_{\mathcal{M}} : \text{Schemes} \longrightarrow \text{Sets}$$

by sending

$$S \mapsto \{\text{families of objects of } \mathcal{M} \text{ parametrized by } S\}$$

The existence of this functor implies that $\mathcal{M}$ has some kind of structure, hopefully $\mathcal{M}$ will be the set of (closed) points a scheme M, which will be called the *(course) moduli scheme* of $\mathcal{M}$. Ideally $F_{\mathcal{M}}$ will even be representable, i.e. of the form $F_{\mathcal{M}}(-) = \mathrm{Hom}(-, M)$, where M parametrizes a *universal family* of objects of $\mathcal{M}$. All this is very optimistic, and in fact a scheme M equipped with a universal family almost never exists. In general the functor $F_{\mathcal{M}}$ will be endowed with a weaker structure, usually some kind of algebraic stack, and the goal is to describe such structure.[1]
A more limited goal is to study $\mathcal{M}$ locally, i.e. to study its structure at a fixed $m \in \mathcal{M}$. Even this goal can be technically very hard. The purpose of DT is to study $m \in \mathcal{M}$ using only infinitesimal methods. Suppose that we are working over a fixed algebraically closed field $\mathbf{k}$ and, to fix ideas, let's consider a specific case when $F_{\mathcal{M}}$ is representable.

THEOREM 0.1.1 (Grothendieck [**G**]). *Fix $r \geq 1$. Let $p(t)$ be a polynomial with rational coefficients such that $p(k) \in \mathbb{Z}$ for all integers k. There exists a projective scheme $H = H^r_{p(t)}$ and a closed subscheme $W \subset \mathbb{P}^r \times H$, flat over H such that the fibres over the closed points $h \in H$ are all the closed subschemes of $\mathbb{P}^r$ having $p(t)$ as Hilbert polynomial. Moreover the pair (H, W) is universal, in the following sense. If S is a scheme and $Z \subset \mathbb{P}^r \times S$ is a closed subscheme, flat over S, such that all fibres of Z over closed points of S have Hilbert polynomial $p(t)$, then there is a unique morphism $S \to H$ such that $Z = S \times_H W$.*

Suppose that we want to study H infinitesimally at the point h parametrizing a closed subscheme $C \subset \mathbb{P}^r$. We will then consider only flat families of the form $Z \subset \mathbb{P}^r \times \mathrm{Spec}(A)$, flat over $\mathrm{Spec}(A)$, where A is a local artinian $\mathbf{k}$-algebra with residue field $\mathbf{k}$, such that the closed fibre is C. Note that, set theoretically, $\mathrm{Spec}(A)$ consists of only one point and that, by the representability, families as above correspond in a 1-1 fashion to homomorphisms $\mathcal{O}_{H,h} \to A$. Studying such families will be already sufficient for the understanding of several important properties of the local ring $\mathcal{O}_{H,h}$ of H at h. This provides only an intermediate, but already useful, step

[1]More precisely, in each case the set of all families of objects im $\mathcal{M}$ will be a *category fibered in groupoids* (or a *groupoid fibration*). See [**B**] for definitions and details.

towards the full local understanding of H. On the other hand infinitesimal methods are much easier to control than general ones, and the resulting theory will be widely applicable in practise.
In the general situation of a class $\mathcal{M}$ of objects, DT only considers families in $\mathcal{M}$ parametrized by spectra of Artinian local **k**-algebras, **without assuming the existence of a "moduli space"** M of any kind. Under very mild assumptions on $\mathcal{M}$, DT is able to prove the existence of a complete local ring $\widehat{\mathcal{O}}_{M,m}$, which plays the role of a local moduli space, and to obtain useful informations on it. This will be discussed below.

Before starting with DT it will be useful to overview the use of infinitesimal methods in the study of a local ring, and see which kind of information it is possible to obtain using such methods.

CHAPTER 1

Infinitesimal methods

S:formsmooth

1.1. Formal smoothness

We fix an algebraically closed field $\mathbf{k}$. We will only consider $\mathbf{k}$-schemes. We denote by:

$\mathcal{A}$: the category of local artinian $\mathbf{k}$-algebras with residue field $\mathbf{k}$.
$\widehat{\mathcal{A}}$: the category of complete local noetherian $\mathbf{k}$-algebras with residue field $\mathbf{k}$.
$\mathcal{A}^*$: the category of local noetherian $\mathbf{k}$-algebras with residue field $\mathbf{k}$.

Morphisms are local $\mathbf{k}$-homomorphisms. $\mathcal{A}$ is a full subcategory of $\widehat{\mathcal{A}}$, which is a full subcategory of $\mathcal{A}^*$.
Let S be a ring. A surjective homomorphism $\pi : A \to \overline{A}$ of S-algebras with kernel I is called a *square-zero extension* of $\overline{A}$ by I if $I^2 = 0$; in this case I has a structure of $\overline{A}$-module. The notion of isomorphism of square-zero extensions is given in the obvious way. The set of isomorphism classes of square-zero extensions of $\overline{A}$ by I is denoted by $\mathrm{Ex}_S(\overline{A}, I)$. Exactly as in the case of module extensions, it is possible to give $\mathrm{Ex}_S(\overline{A}, I)$ a structure of $\overline{A}$-module.
A surjection $\pi : A \to \overline{A}$ in $\mathcal{A}$ is called a *semi-small extension* if $\mathfrak{m}_A I = 0$, and it is *small* if $\dim_{\mathbf{k}}(I) = 1$. By definition, the kernel I of a semi-small extension is a $\mathbf{k} = A/\mathfrak{m}_A$-vector space. This holds in particular for a small extension. Every semi-small extension is square-zero.

The *ring of dual numbers* is $\mathbf{k}[t]/(t^2)$, usually written as $\mathbf{k}[\epsilon]$, where $\epsilon = t \bmod (t^2)$, and $\epsilon^2 = 0$. Clearly

$$0 \longrightarrow \mathbf{k}\epsilon \longrightarrow \mathbf{k}[\epsilon] \longrightarrow \mathbf{k} \longrightarrow 0$$

is a small extension.

More generally, given a finite dimensional $\mathbf{k}$-vector space V, we can consider the $\mathbf{k}$-vector space $\mathbf{k} \oplus V$, and define a ring structure by

$$(a, v)(b, w) = (ab, aw + bv)$$

We denote this ring by $\mathbf{k}[V]$. Clearly $\mathbf{k}[V]$ is a local artinian $\mathbf{k}$-algebra with square-zero maximal ideal V. The square zero extension $\pi : \mathbf{k}[V] \to \mathbf{k}$ has a section $\sigma : \mathbf{k} \to \mathbf{k}[V]$, given by

$$\sigma(a) = (a, 0)$$

π is called the *trivial extension* with kernel V.

L:small

LEMMA 1.1.1. *Every surjection $\pi : A \to \overline{A}$ in $\mathcal{A}$ can be obtained as a composition of small extensions, hence, in particular, as a composition of semi-small extensions.*

PROOF. By induction on $d := \dim_k(A)$. If $d = 1$ there is nothing to prove. Assume $d \geq 2$. Let $I = \ker(\pi)$, and let n be the highest integer such that $\mathfrak{m}_A^n I \neq$

(0), and let $0 \neq t \in \mathfrak{m}_A^n I$. Then $(t) = k \cdot t$ is a 1-dimensional k-vector space, $A \to A' := A/(t)$ is a small extension and $\overline{A} = A'/(I/(t))$ Therefore we have a factorization:

$$\begin{tikzcd} A \arrow[rr] \arrow[dr, "\pi"'] & & A' \arrow[dl, "\pi'"] \\ & \overline{A} & \end{tikzcd}$$

with π' surjective and $\dim_k(A') < \dim_k(A)$. Now we conclude by induction. $\square$

Let $(R, \mathfrak{m})$ be a noetherian local $\mathbf{k}$-algebra with residue field $\mathbf{k}$. Typically R will be the local ring of an algebraic scheme, or the completion of such a ring. To R there is associated the representable functor

$$\mathrm{Hom}_{\mathcal{A}^*}(R, -) : \mathcal{A}^* \longrightarrow \mathrm{Sets},$$

We can consider the restriction of this functor to $\mathcal{A}$, to be denoted by

$$h_R : \mathcal{A} \longrightarrow \mathrm{Sets}, \quad h_R(A) = \mathrm{Hom}_{\mathcal{A}^*}(R, A)$$

This is an example of *functor of Artin rings*, i.e. of a covariant functor $F : \mathcal{A} \to \mathrm{Sets}$. Those of the form h_R as above are called *prorepresentable*. They have the special property that $h_R(\mathbf{k})$ consists of one point, namely the unique map $R \to R/\mathfrak{m} = \mathbf{k}$, sending $r \mapsto \overline{r}$, where $\overline{r} = r \bmod \mathfrak{m}_R$. If we consider any k-algebra R, not necessarily local, we can still consider the corresponding functor of Artin rings $h_R : \mathcal{A} \longrightarrow \mathrm{Sets}$: in this case $h_R(\mathbf{k})$ will consist of the $\mathbf{k}$-rational points of $\mathrm{Spec}(R)$.
Returning to the local case, note that if $\widehat{R}$ is the $\mathfrak{m}$-adic completion of R, the two functors h_R and $h_{\widehat{R}}$ coincide. In particular the functor h_R will not be able to distinguish R from any other local ring S having the same completion.
An *infinitesimal notion* on objects and morphisms of $\mathcal{A}^*$ is one which can be defined in terms of the corresponding functors h_R and natural transformations between them. For example, the *Zariski tangent space* $\mathbf{t}_R := \left(\mathfrak{m}/\mathfrak{m}^2\right)^\vee$ of R is an infinitesimal object associated to R because

$$\mathbf{t}_R = h_R(\mathbf{k}[\epsilon])$$

Ex:tgspace

EXAMPLE 1.1.2. It is an easy exercise to check that the vector space structure on $\mathbf{t}_R$ can be deduced in a functorial way from properties of the functor $F = h_R$. Consider the homomorphism

$$(\alpha, \beta) : \mathbf{k}[\epsilon] \times_{\mathbf{k}} \mathbf{k}[\epsilon] \to \mathbf{k}[\epsilon]$$

given by $(c + a\epsilon, c + b\epsilon) \mapsto c + (\alpha a + \beta b)\epsilon$. The canonical map

$$D : F(\mathbf{k}[\epsilon] \times_{\mathbf{k}} \mathbf{k}[\epsilon]) \longrightarrow F(\mathbf{k}[\epsilon]) \times F(\mathbf{k}[\epsilon])$$

sends $\phi : R \to \mathbf{k}[\epsilon] \times_{\mathbf{k}} \mathbf{k}[\epsilon]$, $v \mapsto \overline{v} + d_1(v) + d_2(v)$, to (ϕ_1, ϕ_2), where $\phi_i(v) = \overline{v} + d_i(v)$. Therefore D is bijective. Then the vector space structure on $F(\mathbf{k}[\epsilon])$ is given by the composition

$$F(\mathbf{k}[\epsilon]) \times F(\mathbf{k}[\epsilon]) \xrightarrow{D^{-1}} F(\mathbf{k}[\epsilon] \times_{\mathbf{k}} \mathbf{k}[\epsilon]) \xrightarrow{F(\alpha,\beta)} F(\mathbf{k}[\epsilon])$$

Note that we have used the property $F(\mathbf{k}) = \{\text{one point}\}$.
More generally, assume given a functor of Artin rings F satisfying $F(\mathbf{k}) = \{\text{one point}\}$ and such that the map D is bijective. Then in the same fashion we can define a

$\mathbf{k}$-vector space structure on $\mathbf{t}_F := F(\mathbf{k}[\epsilon])$. This will be called the *tangent space of* F, and we will say that F has a tangent space.

The most important infinitesimal notion is given by the following:

DEFINITION 1.1.3. *A ring homomorphism $f : S \to R$ is called* formally smooth *(resp.* formally étale*) if for every commutative diagram:*

E:smoothdia1 (1.1)
$$\begin{tikzcd} R \arrow[r, "\overline{\varphi}"] & \overline{A} \\ S \arrow[u, "f"] \arrow[r] & A \arrow[u, "\pi"'] \end{tikzcd}$$

where $\pi : A \to \overline{A}$ is a surjection in $\mathcal{A}$, there exists $\varphi : R \to A$ (resp. a unique $\varphi : R \to A$) such that the resulting diagram

E:smoothdia2 (1.2)
$$\begin{tikzcd} R \arrow[r, "\overline{\varphi}"] \arrow[dr, "\varphi"'] & \overline{A} \\ S \arrow[u, "f"] \arrow[r] & A \arrow[u, "\pi"'] \end{tikzcd}$$

is commutative. If $S = \mathbf{k}$ and $f : \mathbf{k} \to R$ is formally smooth then we say that R is a formally smooth $\mathbf{k}$-algebra.
f is called smooth *(resp.* étale*) if it is formally smooth (resp. formally étale) and essentially of finite type (i.e. R is a localization of an S-algebra of finite type).*

R:sqzero REMARK 1.1.4. Thanks to Lemma 1.1.1, in order to check if $f : S \to R$ is formally smooth (resp. étale) one can limit to check the condition of the definition on square-zero extensions $A \to \overline{A}$ in $\mathcal{A}$, or even on small extensions.

R:smooth REMARK 1.1.5. It can be proved that if $f : S \to R$ is formally smooth then φ as in (1.2) exists for every diagram (1.1) where π is *any surjection of rings with* $\ker(\pi)$ *nilpotent* (see [**Se**], theorem C9).

We can rephrase the previous definition as follows.

P:formsmooth PROPOSITION 1.1.6. *A morphism $f : S \to R$ in $\mathcal{A}^*$ is formally smooth (resp. formally étale) if and only if for every surjection $\pi : A \to \overline{A}$ in $\mathcal{A}$ the natural map:*
$$h_R(A) \longrightarrow h_R(\overline{A}) \times_{h_S(\overline{A})} h_S(A)$$
is surjective (resp. bijective). In particular R is formally smooth if and only if $h_R(A) \to h_R(\overline{A})$ is surjective for all π surjective.

PROOF. Observing that elements of $h_R(\overline{A}) \times_{h_S(\overline{A})} h_S(A)$ are precisely commutative diagrams (1.1), the proposition is obvious. □

C:formsmooth COROLLARY 1.1.7. *Suppose that $f : S \to R$ is a formally smooth (resp formally étale) homomorphism of local rings. Then, for every A in $\mathcal{A}$ the map*
$$h_R(A) \longrightarrow h_S(A)$$
is surjective (resp. bijective). In particular the differential $df : \mathbf{t}_R \to \mathbf{t}_S$ is surjective (resp. an isomorphism).

PROOF. Just apply Proposition 1.1.6 to $\pi : A \to \mathbf{k}$. □

Proposition 1.1.6 characterizes formal smoothness of $S \to R$ only in terms of the morphism of functors $h_R \to h_S$. We can generalize as follows:

D:formsmfunctor

DEFINITION 1.1.8. *Let $\Phi : F \to G$ be a morphism (a natural transformation) of functors of Artin rings such that $F(\mathbf{k})$ and $G(\mathbf{k})$ consist of one element. Then Φ is called* smooth *if for every surjection $A \to \overline{A}$ in $\mathcal{A}$ the natural map:*

$$F(A) \longrightarrow F(\overline{A}) \times_{G(\overline{A})} G(A)$$

is surjective.

F is called smooth *if the morphism to the constant functor*

$$G(A) = \{\textit{one element}\} \quad \textit{for all } A$$

is smooth.

The following is an obvious consequence of the definitions:

PROPOSITION 1.1.9. *If $F \to G$ is a smooth morphism of functors of Artin rings such that $F(\mathbf{k})$ and $G(\mathbf{k})$ consist of one element, then $F(A) \to G(A)$ is surjective for all A in $\mathcal{A}$.*
F is smooth if and only if $F(A) \to F(\overline{A})$ is surjective for every surjection $A \to \overline{A}$ in $\mathcal{A}$.

L:relabssmooth

LEMMA 1.1.10.
(i) *Let $f : S \to R$ be a formally smooth homomorphism in $\mathcal{A}^*$. Then R is a formally smooth $\mathbf{k}$-algebra if and only if S is a formally smooth $\mathbf{k}$-algebra.*
(ii) *Let $F \to G$ be a smooth morphism of functors of Artin rings. Then F is smooth if and only if G is smooth.*

PROOF. It is an easy exercise. □

The following important result is well known:

T:smooth1

THEOREM 1.1.11. *Let R be a noetherian local $\mathbf{k}$-algebra with residue field $\mathbf{k}$. The following conditions are equivalent:*

(1) *R is a regular local ring.*
(2) *$\widehat{R} \cong \mathbf{k}[[X_1, \dots, X_d]]$, where $d = \dim(R)$.*
(3) *R is a formally smooth $\mathbf{k}$-algebra.*

For the proof see e.g. [**Se**], Thm. C4, p. 296.
More generally, we have the following:

T:smooth2

THEOREM 1.1.12. *A homomorphism $f : S \to R$ in $\mathcal{A}^*$ is formally smooth if and only if there is an isomorphism $\widehat{S}[[X_1, \dots, X_d]] \cong \widehat{R}$, where $d = \dim(R) - \dim(S)$.*

PROOF. See [**Se**], Prop. C6, p. 297. □

Recall also the following classical:

T:cohen

THEOREM 1.1.13 (Cohen structure theorem). *Let R be a complete local noetherian $\mathbf{k}$-algebra with residue field $\mathbf{k}$. Then $R \cong \mathbf{k}[[X_1, \dots, X_d]]/I$, where $d = \dim(R)$ and $I \subset (X_1, \dots, X_d)^2$.*

See [**E**], §7.4.

1.2. Liftings and derivations

S:liftings

Consider the situation of diagram (1.2), where π is just a surjective homomorphism of rings. The homomorphism $\varphi : R \to A$ is called a *lifting* of $\overline{\varphi}$ as a homomorphism of S-algebras. In the particular case $S = \mathbf{k}$, a lifting of $\overline{\varphi}$ is simply a $\varphi : R \to A$ making the following diagram commutative:

E:ext1

$$\begin{tikzcd} R \arrow[r, "\overline{\varphi}"] \arrow[dr, "\varphi"'] & \overline{A} \\ & A \arrow[u, "\pi"'] \end{tikzcd} \tag{1.3}$$

Notation. If $S \to R$ is a ring homomorphism and M an R-module, we denote by $\mathrm{Der}_S(R, M)$ the R-module of S-derivations of R in M.

C:action1

LEMMA 1.2.1. *Assume that we are given a commutative diagram*

$$\begin{tikzcd} R \arrow[r, "\overline{\varphi}"] & \overline{A} \\ S \arrow[u, "f"] \arrow[r] & A \arrow[u, "\pi"'] \end{tikzcd}$$

where f is a ring homomorphism and π is a surjective homomorphism of rings with kernel I such that $I^2 = (0)$. Then for any two liftings $\varphi, \psi : R \to A$ of $\overline{\varphi}$ the map $\psi - \varphi : R \to I$ in an S-derivation; conversely, given any lifting φ of $\overline{\varphi}$ and $d \in \mathrm{Der}_S(R, I)$, the map $\psi = \varphi + d$ is a lifting of $\overline{\varphi}$.
Therefore the set of liftings of $\overline{\varphi}$, if not empty, is a torsor under $\mathrm{Der}_S(R, I)$ (i.e. $\mathrm{Der}_S(R, I)$ acts simply and transitively on it).

PROOF. In the statement the structure of R-module on I is the one given via $\overline{\varphi}$. Let $\varphi, \psi : R \to A$ be liftings of $\overline{\varphi}$, and let $d := \psi - \varphi$. Then $d : R \to I \subset A$, since $\pi \cdot \varphi = \pi \cdot \psi = \overline{\varphi}$. Moreover d is an S-linear map and:

$$\begin{aligned} d(xy) &= \psi(xy) - \varphi(xy) = \psi(x)\psi(y) - \varphi(x)\varphi(y) \\ &= \psi(x)\psi(y) + [\psi(x)\varphi(y) - \psi(x)\varphi(y)] - \varphi(x)\varphi(y) \\ &= [\psi(x)\psi(y) - \psi(x)\varphi(y)] + [\psi(x)\varphi(y) - \varphi(x)\varphi(y)] \\ &= \psi(x)d(y) + d(x)\varphi(y) \\ &= \overline{\varphi}(x)d(y) + d(x)\overline{\varphi}(y) \end{aligned}$$

Therefore $d \in Der_S(R, I)$. Conversely, given $d \in Der_S(R, I)$, the map

$$\psi := \varphi + d : R \to A$$

is a lifting of $\overline{\varphi}$, because $I^2 = 0$; and $\psi = \varphi$ if and only if $d = 0$. □

Consider a local $\mathbf{k}$-algebra R in $\mathcal{A}^*$, and a small extension

$$0 \to \mathbf{k}t \to A \xrightarrow{\pi} \overline{A} \to 0$$

Lemma 1.2.1 tells us that two homomorphisms $\varphi, \psi : R \to A$ have the same image in $\mathrm{Hom}(R, \overline{A})$ if and only if there exists $d \in \mathrm{Der}_{\mathbf{k}}(R, \mathbf{k}) = \mathbf{t}_R$ (which is uniquely determined) such that

$$\psi(r) = \varphi(r) + d(r)t \quad \text{for all } r \in R$$

This defines an action $\tau : \mathbf{t}_R \times h_R(A) \to h_R(A)$, given by $\tau(d,\varphi) = \varphi + d$, which preserves the non-empty fibres of

$$h_R(A) \longrightarrow h_R(\overline{A})$$

and makes them torsors. We can give an equivalent definition of τ using the functor $F = h_R$ as follows. Let

$$b : \mathbf{k}[\epsilon] \times_{\mathbf{k}} A \longrightarrow A, \quad (x + y\epsilon, a) \mapsto a + yt$$

Then

$$\tau : F(\mathbf{k}[\epsilon]) \times F(A) \xrightarrow{D^{-1}} F(\mathbf{k}[\epsilon] \times_{\mathbf{k}} A) \xrightarrow{F(b)} F(A)$$

where $D : F(\mathbf{k}[\epsilon] \times_{\mathbf{k}} A) \to F(\mathbf{k}[\epsilon]) \times F(A)$ is the natural bijection (see Example 1.1.2). For this reformulation of τ we only used the fact that $F(\mathbf{k})$ contains only one element and the bijectivity of D.
Warning: this functorial description does not prove that the action is free and transitive on the fibres of $F(A) \to F(\overline{A})$ (see §3.2).

1.3. Deformations of abstract schemes

Denote by $\star = \mathrm{Spec}(\mathbf{k})$, and by $T := \mathrm{Spec}(\mathbf{k}[\epsilon]$. Let X_0 be an algebraic $\mathbf{k}$-scheme and Δ a scheme. A *deformation* (or a *family of deformations*) of X_0 parametrized by (or *over*) Δ is a cartesian diagram:

E:infdef1 (1.4)
$$\begin{tikzcd} X_0 \arrow[r, hook] \arrow[d] & X \arrow[d, "f"] \\ \star \arrow[r, "0"] & \Delta \end{tikzcd}$$

such that f is flat. The condition that (1.4) is cartesian means that it induces an isomorphism $X_0 \cong \star \times_\Delta X$. If $\Delta = \mathrm{Spec}(A)$, where A is in $\mathcal{A}$ then the above deformation will be called *infinitesimal*.
Assume that (1.4) is infinitesimal, i.e. that $\Delta = \mathrm{Spec}(A)$, where A is in $\mathcal{A}$. An *isomorphism* of (1.4) with another deformation of X_0 over $\mathrm{Spec}(A)$:

E:infdef2 (1.5)
$$\begin{tikzcd} X_0 \arrow[r, hook] \arrow[d] & Y \arrow[d, "g"] \\ \star \arrow[r] & \mathrm{Spec}(A) \end{tikzcd}$$

is an isomorphism $\phi : X \to Y$ which makes the following diagram commutative:

E:infdef3 (1.6)
$$\begin{tikzcd} X_0 \arrow[r, hook] \arrow[d, hook] & Y \arrow[d] \\ X \arrow[r] \arrow[ur, "\phi"] & \mathrm{Spec}(A) \end{tikzcd}$$

The notion of deformation is functorial. If $A \to B$ is a morphism in $\mathcal{A}$ then to a deformation (1.4) one associates its pullback:

$$\begin{tikzcd} X_0 \arrow[r] \arrow[d] & X \times_{\mathrm{Spec}(A)} \mathrm{Spec}(B) \arrow[d] \\ \star \arrow[r] & \mathrm{Spec}(B) \end{tikzcd}$$

which is a deformation of X_0 over $\mathrm{Spec}(B)$. Isomorphic deformations over $\mathrm{Spec}(A)$ are mapped to isomorphic deformations over $\mathrm{Spec}(B)$. Therefore we have a functor of Artin rings

$$\mathrm{Def}_{X_0} : \mathcal{A} \longrightarrow \mathrm{Sets}$$

defined by:

$$\mathrm{Def}_{X_0}(A) = \{\text{isom. classes of deformations of } X_0 \text{ over } \mathrm{Spec}(A)\}$$

The main goal of deformation theory is to study Def_{X_0}, and other similar functors of Artin rings. The hope is that it is of the form h_R for some complete local $\mathbf{k}$-algebra (i.e. that it is *prorepresentable*).

Note that $\mathrm{Def}_{X_0}(\mathrm{Spec}(A)) \neq \emptyset$ for all A because it contain at least the *trivial deformation*

$$\begin{tikzcd} X_0 \arrow[r] \arrow[d] & X_0 \times \mathrm{Spec}(A) \arrow[d] \\ \star \arrow[r] & \mathrm{Spec}(A) \end{tikzcd}$$

Definition 1.3.1. *X_0 is* rigid *if* $\mathrm{Def}_{X_0}(A)$ *contains only the trivial deformation for every A in $\mathcal{A}$.*

Definition 1.3.2. $\mathrm{Def}_{X_0}(T)$ *is the* tangent space *of* Def_{X_0}*, and is denoted by* $\mathbf{t}_{\mathrm{Def}_{X_0}}$*. Its elements are called* first order deformations *of X_0.*

L:isodef

Lemma 1.3.3. *Suppose given deformations* (1.4) *and* (1.5) *of X_0, and that there is a morphism $\phi : X \to Y$ making the diagram* (1.6) *commutative. Then ϕ is an isomorphism.*

Proof. The question is local, thus we may assume that $X_0 = \mathrm{Spec}(R_0)$; then $X = \mathrm{Spec}(R)$ and $Y = \mathrm{Spec}(S)$ (see footnote 1). We have the following situation:

E:isodef

$$\begin{tikzcd} S \arrow[rr, "\varphi"] \arrow[d] & & R \arrow[d] \\ S \otimes_A \mathbf{k} \arrow[r, equal] & R_0 \arrow[r, equal] & R \otimes_A \mathbf{k} \end{tikzcd} \tag{1.7}$$

We proceed by induction on $\dim_{\mathbf{k}}(A)$. If $\dim_{\mathbf{k}}(A) = 1$ there is nothing to prove. We consider $t \in \mathfrak{m}_A$ such that $\mathfrak{m}_A t = 0$. Then we have a small extension

$$0 \to \mathbf{k}t \longrightarrow A \longrightarrow A' \to 0$$

which tensored by φ gives:

$$\begin{tikzcd} 0 \arrow[r] & \mathbf{k}t \otimes_A R \arrow[r, "i"] & R \arrow[r] & A' \otimes_A R \arrow[r] & 0 \\ 0 \arrow[r] & \mathbf{k}t \otimes_A S \arrow[r, "j"] \arrow[u, "\varphi_t"] & S \arrow[r] \arrow[u, "\varphi"] & A' \otimes_A S \arrow[r] \arrow[u, "\varphi'"] & 0 \end{tikzcd}$$

By the inductive hypothesis φ' is an isomorphism. By the A-flatness of R and S the maps i and j are injective and φ_t is the isomorphism $S \otimes_A \mathbf{k} \cong R \otimes_A \mathbf{k}$ coming from diagram (1.7). Therefore φ is an isomorphism. $\square$

1.4. Nonsingular varieties

Let R_0 be a noetherian $\mathbf{k}$-algebra and $X_0 = \mathrm{Spec}(R_0)$ the corresponding affine scheme. Note that X_0 is not necessarily algebraic. An infinitesimal deformation of X_0 over $\mathrm{Spec}(A)$ is a cartesian diagram (1.4) such that $X = \mathrm{Spec}(R)$ is also affine.[1] Therefore an infinitesimal deformation of X_0 can be also described as a cartesian diagram of rings:

$$\begin{tikzcd} R \arrow[r] & R_0 \\ A \arrow[u] \arrow[r] & \mathbf{k} \arrow[u] \end{tikzcd}$$

i.e. a commutative diagram inducing an isomorphism $R \otimes_A \mathbf{k} \cong R_0$. Such a diagram will be also called an infinitesimal deformation of R_0. The functor Def_{X_0} can be identified with the functor Def_{R_0} of infinitesimal deformations of R_0.

P:rigidaffine

PROPOSITION 1.4.1. *If* $X_0 = \mathrm{Spec}(R_0)$ *is formally smooth then it is rigid.*

PROOF. Let

$$\begin{tikzcd} X_0 \arrow[r, hook] \arrow[d] & X \arrow[d, "f"] \\ \star \arrow[r] & \mathrm{Spec}(A) \end{tikzcd}$$

be an infinitesimal deformation of X_0. we have the following commutative diagram:

$$\begin{tikzcd} X_0 \arrow[r, hook] \arrow[d, hook] & X_0 \otimes \mathrm{Spec}(A) \arrow[d] \\ X \arrow[r] & \mathrm{Spec}(A) \end{tikzcd}$$

which is equivalent to the following one:

$$\begin{tikzcd} A \otimes_{\mathbf{k}} R_0 \arrow[r] & R_0 \\ A \arrow[u, "g"] \arrow[r] & R \arrow[u] \end{tikzcd}$$

g is formally smooth because $\mathbf{k} \to R_0$ is. Since the surjective homomorphism $R \to R_0$ has a nilpotent kernel there exists $h : A \otimes_{\mathbf{k}} R_0 \to R$ making the diagram

$$\begin{tikzcd} A \otimes_{\mathbf{k}} R_0 \arrow[r] \arrow[dr, "h"] & R_0 \\ A \arrow[u, "g"] \arrow[r] & R \arrow[u] \end{tikzcd}$$

---

[1] Actually this last condition is automatic at least in the algebraic case, because if $Z_0 \subset Z$ is an affine closed subscheme defined by a nilpotent ideal sheaf, then Z is affine as well. For the proof see [**Se**], Lemma 1.2.3

F:1

commutative (see Remark 1.1.5). This corresponds to a morphism $h^\sharp : X \to X_0 \otimes \mathrm{Spec}(A)$ making

$$\begin{tikzcd} X_0 \arrow[r, hook] \arrow[d, hook] & X_0 \otimes \mathrm{Spec}(A) \arrow[d] \\ X \arrow[ur, "h^\sharp"] \arrow[r] & \mathrm{Spec}(A) \end{tikzcd}$$

commutative. But $h^\sharp$ is an isomorphism, by Lemma 1.3.3. □

P:H1theta

PROPOSITION 1.4.2. *Let X_0 be a nonsingular algebraic variety. Then there is a natural identification*

$$\mathbf{t}_{\mathrm{Def}_{X_0}} = H^1(X_0, T_{X_0})$$

where T_{X_0} is the tangent sheaf of X_0.

PROOF. Let

$$\theta : \begin{tikzcd} X_0 \arrow[r] \arrow[d] & X \arrow[d] \\ \star \arrow[r] & T \end{tikzcd}$$

be a first order deformation of X_0. Then θ is locally trivial. In fact, if $\mathcal{U} = \{U_i = \mathrm{Spec}(R_i)\}$ is an affine cover of X_0, then, By Prop. 1.4.1, for all i there are T-isomorphisms

$$\theta_i : X_{|U_i} \cong \mathrm{Spec}(R_i) \times T = \mathrm{Spec}(R_i \otimes_{\mathbf{k}} \mathbf{k}[\epsilon])$$

restricting to the identity on U_i, and X is given by gluing data

$$\theta_{ij} := \theta_i \cdot \theta_j^{-1} : (U_i \cap U_j) \times T \longrightarrow (U_i \cap U_j) \times T$$

such that $\theta_{ij}\theta_{jk} = \theta_{ik}$ on $(U_i \cap U_j \cap U_k) \times T$ for all i, j, k (see [**H1**], ex. 1.22 p. 69). Since X_0 is separated we may assume that $U_i \cap U_j = \mathrm{Spec}(R_{ij})$ is affine for all i, j ([**H1**], Exercise II 4.3, p. 106). Then θ_{ij} corresponds to an automorphism t_{ij} of $R_{ij}[\epsilon] := R_{ij} \otimes_{\mathbf{k}} \mathbf{k}[\epsilon]$ inducing the identity on R_{ij}:

$$\begin{tikzcd} 0 \arrow[r] & \epsilon R_{ij} \arrow[r] & R_{ij}[\epsilon] \arrow[r] & R_{ij} \\ & & R_{ij}[\epsilon] \arrow[u, "t_{ij}"] \arrow[ur] & \end{tikzcd}$$

By Lemma 1.2.1 every t_{ij} is of the form $t_{ij} = \mathrm{id} + d_{ij}$, where

$$d_{ij} \in \mathrm{Der}_{\mathbf{k}}(R_{ij}[\epsilon], \epsilon R_{ij}) = \mathrm{Der}_{\mathbf{k}}(R_{ij}, R_{ij}) = \mathrm{Hom}_{R_{ij}}(\Omega_{R_{ij}}, R_{ij}) = \Gamma(U_i \cap U_j, T_{X_0})$$

The gluing conditions imply that $\{d_{ij}\} \in \mathcal{Z}^1(\mathcal{U}, T_{X_0})$ is a 1-cocycle and therefore defines an element $t(\theta) \in H^1(X_0, T_{X_0})$. Now patiently, but trivially, one checks that 1) $t(\theta)$ does not depend on the choice of $\mathcal{U}$, and 2) conversely, to every $t \in H^1(X_0, T_{X_0})$ there is associated a first order deformation θ of X_0 such that $t(\theta) = t$. □

REMARK 1.4.3. The $\mathbf{k}$-vector space structure on $\mathbf{t}_{\mathrm{Def}_{X_0}}$ can be reconstructed only using the properties of the functor, with an argument similar to the one used in Example 1.1.2. In the nonsingular case just observe that elements of $\mathrm{Def}_{X_0}(\mathbf{k}[\epsilon] \times_{\mathbf{k}} \mathbf{k}[\epsilon])$ can be identified with pairs $(\theta_1, \theta_2) \in H^1(X_0, T_{X_0}) \times H^1(X_0, T_{X_0})$.

From the proposition it follows that if $H^1(X_0, T_{X_0})$ is not finite dimensional then Def_{X_0} is not prorepresentable.
The previous proof shows that, for *any* X_0, not necessarily nonsingular, there is an identification between the set of first order *locally trivial* deformations of X_0 and $H^1(X_0, T_{X_0})$, where $T_{X_0} = \mathrm{Hom}(\Omega^1_{X_0}, \mathcal{O}_{X_0})$.

DEFINITION 1.4.4. *Consider a family of deformations of X_0 over a scheme Δ:*

$$\begin{tikzcd} X_0 \arrow[r, hook] \arrow[d] & X \arrow[d, "f"] \\ \star \arrow[r, "0"] & \Delta \end{tikzcd}$$

and let $\theta : T \to \Delta$ be a tangent vector to Δ at 0. Pulling back the above deformation we obtain a first order deformation of X_0:

$$\begin{tikzcd} X_0 \arrow[r, hook] \arrow[d] & X \times_\Delta T \arrow[d, "f_\theta"] \\ \star \arrow[r] & T \end{tikzcd}$$

Setting $\kappa_f(\theta) = f_\theta$ one obtains a linear map

$$\kappa_f : T_0\Delta \longrightarrow H^1(X_0, T_{X_0})$$

called Kodaira-Spencer map *of the given deformation.*

P:rigidity

PROPOSITION 1.4.5. *Let X_0 be a nonsingular algebraic variety. Then X_0 is rigid if and only if $H^1(X_0, T_{X_0}) = 0$.*

PROOF. The "only if" implication is obvious. Conversely, let's assume that $H^1(X_0, T_{X_0}) = 0$. We will prove that $\mathrm{Def}_{X_0}(A) = \{\text{trivial deformation}\}$ by induction on $\dim_{\mathbf{k}}(A)$. If $\dim_{\mathbf{k}}(A) = 1$ there is nothing to prove. Assume that $\dim_{\mathbf{k}}(A) = n \geq 2$. Let $t \in \mathfrak{m}_A$ such that $\mathfrak{m}_A t = 0$. Then we have a small extension:

$$0 \to \mathbf{k}t \longrightarrow A \longrightarrow A' \to 0$$

with $\dim_{\mathbf{k}}(A') = n - 1$. Let

$$\begin{tikzcd} X_0 \arrow[r] \arrow[d] & X \arrow[d] \\ \star \arrow[r] & \mathrm{Spec}(A) \end{tikzcd}$$

be a deformation of X_0 over $\mathrm{Spec}(A)$. By the induction hypothesis its pullback to $\mathrm{Spec}(A')$ is isomorphic to the trivial one. Therefore we have an isomorphism $X_0 \times \mathrm{Spec}(A') \cong X \times_{\mathrm{Spec}(A)} \mathrm{Spec}(A') := X'$. Composing with the inclusion $X' \subset X$ we find:

$$\begin{tikzcd} X_0 \times \mathrm{Spec}(A') \arrow[r, hook] \arrow[d, hook] & X_0 \times \mathrm{Spec}(A) \\ X \arrow[ur, dotted, "\phi"'] & \end{tikzcd}$$

and we need to find a dotted arrow ϕ. Since X_0 is nonsingular ϕ exists locally. In other words, we can choose an affine open cover $\mathcal{U} = \{U_i := \mathrm{Spec}(R_i)\}_{i \in I}$ of X_0

and isomorphisms $\phi_i : X_{|U_i} \to U_i \times \mathrm{Spec}(A)$, one for each $i \in I$. Let

$$\theta_{ij} := \phi_i \cdot \phi_j^{-1} : U_{ij} \times \mathrm{Spec}(A) \to U_{ij} \times \mathrm{Spec}(A)$$

Then, since each θ_{ij} restricts to the identity on $U_{ij} \times \mathrm{Spec}(A')$, it must be of the form:

$$\theta_{ij} = 1 + td_{ij}$$

where $d_{ij} \in \Gamma(U_{ij}, T_{X_0})$. Moreover, since $\theta_{ij} \cdot \theta_{jk} = \theta_{ik}$, we have $d_{ij} + d_{jk} = d_{ik}$; in other words

$$\{d_{ij}\} \in \mathcal{Z}^1(\mathcal{U}, \mathrm{Hom}(\Omega^1_X, t\mathcal{O}_{X_0}) = \mathcal{Z}^1(\mathcal{U}, T_{X_0})$$

If we can choose the isomorphisms ϕ_i so that θ_{ij} is the identity for each $i, j \in I$ then they will patch together to give an isomorphism ϕ, as required. A different choice of ϕ_i is of the form

$$\widetilde{\phi}_i = (1 + td_i) \cdot \phi$$

where $d_i \in \mathrm{Der}_{\mathbf{k}}(R_i, \mathbf{k}t) = \Gamma(U_i, T_{X_0})$; namely $\widetilde{\phi}_i$ is the composition of ϕ_i with an arbitrary automorphism $1 + td_i$ of $U_i \times \mathrm{Spec}(A)$ which restricts to the identity of $U_i \times \mathrm{Spec}(A')$. Correspondingly the θ_{ij} will be changed into

$$\begin{aligned}
\widetilde{\theta}_{ij} = \widetilde{\phi}_i \cdot \widetilde{\phi}_j^{-1} &= (1 + td_i) \cdot \phi_i \cdot \phi_j^{-1}(1 - td_j) \\
&= (1 + td_i) \cdot \theta_{ij}(1 - td_j) \\
&= (1 + td_i) \cdot (1 + td_{ij})(1 - td_j) \\
&= 1 + t(d_{ij} + d_i - d_j)
\end{aligned}$$

Since $H^1(X_0, T_{X_0}) = 0$ we can find $d_i \in \Gamma(U_i, T_{X_0})$ such that $d_{ij} = d_j - d_i$ for all i, j. Using them the $\widetilde{\phi}_i$'s will patch together to give ϕ as desired. $\square$

Examples 1.4.6. (1) From the Euler sequence

$$0 \to \mathcal{O}_{\mathbb{P}^n} \longrightarrow \mathcal{O}_{\mathbb{P}^n}(1)^{n+1} \longrightarrow T_{\mathbb{P}^n} \to 0$$

one computes that $H^1(\mathbb{P}^n, T_{\mathbb{P}^n}) = 0$. Proposition 1.4.5 implies that $\mathbb{P}^n$ is rigid.

(2) If C is a projective nonsingular curve of genus g then $H^1(C, T_C) \cong H^0(C, \omega_C^2)^\vee$. Hence, by Riemann-Roch:

$$\dim(\mathbf{t}_{\mathrm{Def}_C}) = \begin{cases} 0 & g = 0 \\ 1 & g = 1 \\ 3g - 3 & g \geq 2 \end{cases}$$

1.5. The local Hilbert functor

Let $Y \subset X$ be a closed embedding of algebraic schemes. Given A in $\mathcal{A}$, an *infinitesimal deformation of* Y *in* X parametrized by (or over) $\mathrm{Spec}(A)$ is a closed embedding $\widetilde{Y} \subset X \times \mathrm{Spec}(A)$ such that

(i) $\widetilde{Y} \cap X = Y$, where here $X = X \times \star$,
(ii) the composition $\widetilde{Y} \subset X \times \mathrm{Spec}(A) \to \mathrm{Spec}(A)$ is flat.

Given a deformation of Y in X over $\mathrm{Spec}(A)$ and a morphism $\pi : A \to B$ in $\mathcal{A}$ one has an induced deformation $\widetilde{Y}_B = \widetilde{Y} \times_{\mathrm{Spec}(A)} \mathrm{Spec}(B) \subset X \times \mathrm{Spec}(B)$ of Y in X over $\mathrm{Spec}(B)$. This defines a functor of Artin rings:

$$H_{Y/X} : \mathcal{A} \longrightarrow \mathrm{Sets}$$

called the *local Hilbert functor of* X *at* Y or of Y in X. Clearly $H_{Y/X}(A) \neq \emptyset$ for all A because it contains at least the trivial deformation

$$Y \times \mathrm{Spec}(A) \subset X \times \mathrm{Spec}(A)$$

If $H_{Y/X}(A)$ contains only the trivial deformation for all A in $\mathcal{A}$ we say that Y is *rigid* in X.

P:HON

Proposition 1.5.1. *Let $Y \subset X$ be a closed embedding of algebraic schemes. There is a natural identification*

$$\mathbf{t}_{H_{Y/X}} = H^0(Y, N_{Y/X})$$

where $N_{Y/X} = \mathcal{H}om(\mathcal{I}_{Y/X}, \mathcal{O}_Y)$ is the normal sheaf *of Y in X.*

Proof. **Case 1: Y is a Cartier divisor in X.** Locally, we have an affine open subset $\mathrm{Spec}(R) \subset X$ and $f \in R$, not a 0-divisor, such that $Y = \mathrm{Spec}(S)$, where $S = R/(f)$. A first order deformation $\widetilde{Y}$ of Y in X is defined by an ideal $I \subset R[\epsilon] = R \otimes_{\mathbf{k}} \mathbf{k}[\epsilon]$ such that $\widetilde{S} := R[\epsilon]/I$ is $\mathbf{k}[\epsilon]$-flat and such that $I \otimes_{R[\epsilon]} R = (f)$.

Claim: $\widetilde{Y}$ is a Cartier divisor in $\mathrm{Spec}(R) \times T$.
Choose an element $g \in R$ such that $F = f + \epsilon g \in I$. Then F is not a 0-divisor in $R[\epsilon]$, because f is not a 0-divisor in R. It follows that $S' := R[\epsilon]/(F)$ is $\mathbf{k}[\epsilon]$-flat because tensoring the sequence

$$0 \longrightarrow R[\epsilon] \xrightarrow{F} R[\epsilon] \longrightarrow S' \longrightarrow 0$$

by $\mathbf{k}$ we obtain

$$0 \longrightarrow R \xrightarrow{f} R \longrightarrow S \longrightarrow 0$$

and therefore $\mathrm{Tor}_1^{\mathbf{k}[\epsilon]}(S', \mathbf{k}) = 0$. The inclusion $(F) \subset I$ induces a surjective homomorphism $S' \to \widetilde{S}$ and we have the commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & \epsilon S' \arrow[r] \arrow[d] & S' \arrow[r] \arrow[d] & S \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \epsilon \widetilde{S} \arrow[r] & \widetilde{S} \arrow[r] & S \arrow[r] & 0
\end{tikzcd}$$

Since $\epsilon S' = S = \epsilon \widetilde{S}$ the left vertical arrow is an isomorphism and therefore also $S' \to \widetilde{S}$ is an isomorphism. This proves that $I = (f + \epsilon g)$ and the Claim.
Any other $g' \in R$ such that $I = (f + \epsilon g')$ must be of the form

$$f + \epsilon g' = (f + \epsilon g)(1 + \epsilon h) = f + \epsilon(g + hf)$$

for some $h \in R$. Therefore g is determined by f only modulo (f). If we replace f by uf, $u \in R$, g is replaced by ug. In conclusion to the given first order deformation of Y in X there is associated a homomorphism $(f) \to S$. Conversely, to an element of $\mathrm{Hom}((f), S)$ one associates a first order deformation of Y in X by reversing the above argument.

This construction globalizes to the case of a Cartier divisor $Y \subset X$, not necessarily affine, as follows. By the previous argument a first order deformation of Y in X is a Cartier divisor $\widetilde{Y} \subset X \times T$. Then there is an affine open cover

$\mathcal{U} = \{U_i\}$ of X where Y is defined by a system $\{f_i \in \Gamma(U_i, \mathcal{O}_X)\}$, such that $f_{ij} := f_i f_j^{-1} \in \Gamma(U_{ij}, \mathcal{O}_X^*)$ for all i, j, and $\widetilde{Y}$ is defined by a system

$$\{F_i = f_i + \epsilon g_i \in \Gamma(U_i \times T, \mathcal{O}_{X \times T})\}$$

such that there are $g_{ij} \in \Gamma(U_{ij}, \mathcal{O}_X)$ satisfying the identities:

$$f_i + \epsilon g_i = (f_{ij} + \epsilon g_{ij})(f_j + \epsilon g_j)$$

Note that $f_{ij} + \epsilon g_{ij} \in \Gamma(U_{ij} \times T, \mathcal{O}_{X \times T}^*)$. The above identities can be written as:

$$g_i = f_{ij} g_j + g_{ij} f_j$$

and dividing by f_i:

$$\frac{g_i}{f_i} - \frac{g_j}{f_j} = g_{ij} f_{ij}^{-1}$$

Since $\frac{g_i}{f_i} \in \Gamma(U_i, \mathcal{O}_X(Y))$, the identities say that $\{\frac{g_i}{f_i}\}$ patch together to give a section of $H^0(Y, N_{Y/X})$, because their difference is an element of $\Gamma(U_{ij}, \mathcal{O}_X)$.
The system $\{F_i\}$ defines the same section of $N_{Y/X}$ as a system $\{F_i' = f_i + \epsilon g_i'\}$ if and only if $g_i' = g_i + h_i f_i$ for some $h_i \in \Gamma(U_i, \mathcal{O}_X)$. But then:

$$f_i + \epsilon g_i' = (f_i + \epsilon g_i)(1 + \epsilon h_i)$$

and, since $1 + \epsilon h_i$ is invertible, $\{F_i'\}$ defines the same divisor $\widetilde{Y}$ as $\{F_i\}$.
It remains to be checked that, conversely, every section of $N_{Y/X}$ defines a first order deformation of Y in X. This is an easy exercise.

Case 2: $Y \subset X$ is not necessarily a Cartier divisor. Assume first $X = \mathrm{Spec}(R)$ and $Y = \mathrm{Spec}(S)$, $S = R/I$. First order deformations of Y in X are in 1-1 correspondence with with ideals $\widetilde{I} \subset R[\epsilon]$ such that there is a commutative exact diagram:

$$\begin{tikzcd}
 & & \widetilde{I} \arrow[r] \arrow[d] & I \arrow[d] & \\
0 \arrow[r] & \epsilon R \arrow[r] \arrow[d] & R[\epsilon] \arrow[r] \arrow[d] & R \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \epsilon S \arrow[r] \arrow[d] & \widetilde{S} \arrow[r] \arrow[d] & S \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

In fact by the local criterion of flatness ([**Se**], Thm. A5) $\widetilde{S}$ is $\mathbf{k}[\epsilon]$-flat if and only if $\epsilon\widetilde{S} \cong S$. We can embed such diagram in the following one, whose rows are exact (but not the columns):

$$
\begin{tikzcd}
 & & & I \arrow[d] & \\
0 \arrow[r] & \epsilon R \arrow[r] \arrow[d] & R[\epsilon] \arrow[r] \arrow[d] & R \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \epsilon S \arrow[r] \arrow[d] & \mathcal{E} \arrow[r] \arrow[d] & R \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \epsilon S \arrow[r] & \widetilde{S} \arrow[r] & S \arrow[r] & 0
\end{tikzcd}
$$

Here the second row is the pushout of the first one induced by $\epsilon R \to \epsilon S$. From this we get the following commutative diagram with exact rows and columns:

E:hilb1 (1.8)

$$
\begin{tikzcd}
 & & I \arrow[r, equal] \arrow[d] & I \arrow[d] \arrow[dl, "s"] & \\
0 \arrow[r] & \epsilon S \arrow[r] \arrow[d, equal] & \mathcal{E} \arrow[r] \arrow[d, "\varphi"'] & R \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \epsilon S \arrow[r] & \widetilde{S} \arrow[r] \arrow[d] & S \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}
$$

(it is easy to check that $I \cong \ker(\varphi)$). Suppose that $s = s_0 : I \to \mathcal{E}$ is the map corresponding to the trivial deformation of Y in X. Then there is a 1-1 correspondence:

$$\{s : I \to \mathcal{E} \text{ making (1.8) commute}\} \longleftrightarrow \mathrm{Hom}(I, S)$$

defined by $s \mapsto s - s_0$. This proves the proposition in the affine case.

The above discussion globalizes in a straightforward way and gives the proposition in the general case as well. $\square$

Given $Y \subset X$ closed embedding, let $\mathcal{Y} \subset X_0 \times \mathrm{Spec}(A)$ be an infinitesimal deformation of Y in X parametrized by A in $\mathcal{A}$. Then we have an associated deformation of Y over $\mathrm{Spec}(A)$:

$$
\begin{tikzcd}
Y \arrow[r, hook] \arrow[d] & \mathcal{Y} \arrow[d] \\
\star \arrow[r] & \mathrm{Spec}(A)
\end{tikzcd}
$$

Therefore we have a map:

$$\Phi(A) : H_{Y/X}(A) \longrightarrow \mathrm{Def}_Y(A)$$

Since this correspondence is clearly functorial, it defines a natural transformation of functors:

$$\Phi : H_{Y/X} \longrightarrow \mathrm{Def}_Y$$

called *forgetful*, because it is defined by forgetting the embedding $Y \subset X$.

PROPOSITION 1.5.2. *Let $Y \subset X$ be a closed embedding of nonsingular varieties, and consider the normal sequence:*

$$0 \to T_Y \longrightarrow T_{X|Y} \longrightarrow N_{Y/X} \to 0$$

Then the coboundary map

$$H^0(Y, N_{Y/X}) \longrightarrow H^1(Y, T_Y)$$

coincides with

$$\Phi(\mathbf{k}[\epsilon]) : \mathbf{t}_{H_{Y/X}} \longrightarrow \mathbf{t}_{\mathrm{Def}_Y}$$

PROOF. [**Se**], Prop. 3.2.9 p. 132. □

1.6. The local Picard functor

Let X be a scheme, and let $\mathrm{Pic}(X) := H^1(X, \mathcal{O}_X^*)$ be the *Picard group* of X, consisting of the isomorphism classes of invertible sheaves on X. For each A in $\mathcal{A}$ consider the trivial deformation of X over A:

$$\begin{tikzcd} X \arrow[r, "\iota"] \arrow[d] & X_A = X \times \mathrm{Spec}(A) \arrow[d] \\ * \arrow[r] & \mathrm{Spec}(A) \end{tikzcd}$$

ι induces an isomorphism $X \cong X_A \otimes_{\mathrm{Spec}(A)} \mathrm{Spec}(\mathbf{k}) = X_A \otimes_A \mathbf{k}$. Fix an invertible sheaf L on X and denote by $[L] \in \mathrm{Pic}(X)$ its isomorphism class. A *family of deformations* (or simply a *deformation*) of L over $\mathrm{Spec}(A)$ (over A for brevity) is an invertible sheaf L_A on X_A such that $\iota^* L_A = L$. Given another deformation L'_A of L over A, an *isomorphism of deformations* is an isomorphism of invertible sheaves $L_A \cong L'_A$ inducing the identity $1_L : \iota^* L_A \to \iota^* L'_A$.
Given $A \longrightarrow B$ in $\mathcal{A}$ and a deformation L_A of L over A, we have a cartesian diagram:

$$\begin{tikzcd} X_B \arrow[r, "f"] \arrow[d] & X_A \arrow[d] \\ \mathrm{Spec}(B) \arrow[r] & \mathrm{Spec}(A) \end{tikzcd}$$

and $L_B := f^* L_A$ is a deformation of L over B. Clearly $L_A \cong L'_A$ implies $L_B \cong L'_B$. Therefore we have a well defined functor of Artin rings

$$P_{[L]} : \mathcal{A} \longrightarrow \text{sets}, \quad P_{[L]}(A) = \{L_A : \text{ deform.s of } L \text{ over } A\}/\text{isomorphism}$$

called *local Picard functor* defined by L.

PROPOSITION 1.6.1.

$$\mathbf{t}_{P_{[L]}} = H^1(X, \mathcal{O}_X)$$

PROOF. Suppose that L is given by a set of transitions functions $\{f_{ij} \in \Gamma(U_{ij}, \mathcal{O}_X^*)\}$ with respect to an open covering $\mathcal{U} = \{U_i\}_{i \in I}$, such that $f_{ij} f_{jk} = f_{ik}$ on U_{ijk}, and let L_ϵ be a deformation of L over T. Then L_ϵ is defined by transition functions $\{\widetilde{f}_{ij} \in \Gamma(U_{ij}, \mathcal{O}_{X_\epsilon}^*)\}$ such that

E:cocycle

$$\widetilde{f}_{ij} \widetilde{f}_{jk} = \widetilde{f}_{ik} \tag{1.9}$$

on U_{ijk}. Since

$$\mathcal{O}^*_{X_\epsilon} = \mathcal{O}^*_X + \epsilon\mathcal{O}_X$$

we can write:

$$\widetilde{f}_{ij} = f_{ij}(1 + \Phi_{ij})$$

for suitable $\Phi_{ij} \in \Gamma(U_{ij}, \mathcal{O}_X)$. The cocycle condition (2.2) gives:

$$\Phi_{ij} + \Phi_{ij} = \Phi_{ik}$$

Therefore the collection $\{\Phi_{ij}\}$ defines an element $z(L_\epsilon) \in H^1(X, \mathcal{O}_X)$. It is easy to check that $z(L_\epsilon)$ depends only on the isomorphism class of L_ϵ. Conversely, to any $z \in H^1(X, \mathcal{O}_X)$ one can associate an isomorphism class of deformations of L over T by just reversing the previous argument. □

Observe that $\mathbf{t}_{P_{[L]}}$ does not depend on L, but only on X. The reason is because the group structure on $\mathrm{Pic}(X)$ makes $P_{[L]}$ isomorphic to any other $P_{[L']}$.

PROPOSITION 1.6.2. *Let D be an effective Cartier divisor on X. Then we have a natural transformation:*

$$\psi : H_{D/X} \longrightarrow P_{\mathcal{O}_X(D)}$$

associating to a deformation $D_A \subset X_A$ over A of D the class $[\mathcal{O}_{X_A}(D_A)]$. Then the differential of ψ:

$$d\psi : H^0(D, \mathcal{O}_D(D)) \longrightarrow H^1(X, \mathcal{O}_X)$$

is the coboundary map coming from the exact sequence:

$$0 \to \mathcal{O}_X \longrightarrow \mathcal{O}_X(D) \longrightarrow \mathcal{O}_D(D) \to 0$$

PROOF. Exercise. □

CHAPTER 2

Obstructions

2.1. What is an obstruction

Recall from §1.1 that a **k**-algebra R is formally smooth if and only if for every small extension $\pi : A \to \overline{A}$ the natural map $h_R(A) \to h_R(\overline{A})$ is surjective. More generally, a functor of Artin rings F is smooth if and only if

E:surjsm

$$F(A) \to F(\overline{A}) \tag{2.1}$$

is surjective for every π. A similar definition can be give for a homomorphism of k-algebras $f : S \to R$ and for a natural transformation $f : F \to G$ of functors of Artin rings; namely f is smooth if and only if the natural map

$$F(A) \longrightarrow F(\overline{A}) \times_{G(\overline{A})} G(A)$$

is surjective for all π as above.

If the functor $F : \mathcal{A} \to$ **sets** is given, we need a systematic procedure to investigate the failure of the surjectivity of (2.1) as π varies. The procedure consists in associating to F a vector space $v(F)$ in such a way that to every small extension $\pi : A \to \overline{A}$ and to every $\xi \in F(\overline{A})$ there corresponds an element of $\xi_v(\pi) \in v(F)$ which vanishes if and only ξ is in the image of (2.1). The element $\xi_v(\pi)$ will be called *the obstruction to lift ξ over A*. We start by illustrating this idea in two specific examples.

2.2. Obstructions to deformations of nonsingular varieties

Suppose given a nonsingular algebraic variety X_0 and a surjection $\pi : A \to \overline{A}$ in $\mathcal{A}$. We want to find conditions for a deformation of X_0 over $\mathrm{Spec}(\overline{A})$:

$$\bar{\xi} : \begin{array}{ccc} X_0 & \longrightarrow & \overline{X} \\ \downarrow & & \downarrow \\ \star & \longrightarrow & \mathrm{Spec}(\overline{A}) \end{array}$$

to be in the image of the map

$$\mathrm{Def}_{X_0}(\pi) : \mathrm{Def}_{X_0}(A) \longrightarrow \mathrm{Def}_{X_0}(\overline{A})$$

i.e. to be the restriction of a deformation over $\mathrm{Spec}(A)$. In other words, we want to know if there exists a deformation

$$\xi : \begin{array}{ccc} X_0 & \longrightarrow & X \\ \downarrow & & \downarrow \\ \star & \longrightarrow & \mathrm{Spec}(A) \end{array}$$

such that

$$\begin{tikzcd}
X_0 \arrow[r] \arrow[d] & \overline{X} \arrow[r, "\cong"] \arrow[d] & X \times_{\mathrm{Spec}(A)} \mathrm{Spec}(\overline{A}) \arrow[dl] \\
\star \arrow[r] & \mathrm{Spec}(\overline{A}) &
\end{tikzcd}$$

By Lemma 1.1.1 we can reduce to consider the case where $\ker(\pi) = (t)$, with $\dim(t) = 1$.
Let $\mathcal{U} = \{U_i\}$ be an affine open cover of X_0. Then

$$\begin{tikzcd}
U_i \arrow[r, hook] \arrow[d] & \overline{X}_{|U_i} \arrow[d] \\
\star \arrow[r] & \mathrm{Spec}(\overline{A})
\end{tikzcd}$$

is a deformation of the nonsingular affine U_i, therefore it is trivial, for each i. Choose isomorphisms

$$\phi_i : \overline{X}_{|U_i} \longrightarrow U_i \times \mathrm{Spec}(\overline{A})$$

for each i. Then for each i, j they induce an automorphism on the double intersection $U_{ij} := U_i \cap U_j$:

$$\begin{tikzcd}
U_{ij} \times \mathrm{Spec}(\overline{A}) \arrow[rr, "\phi_j \cdot \phi_i^{-1}"] \arrow[dr] & & U_{ij} \times \mathrm{Spec}(\overline{A}) \arrow[dl] \\
& \mathrm{Spec}(\overline{A}) &
\end{tikzcd}$$

which reduces to the identity on $U_{ij} \times \star$. The automorphisms $\overline{\theta}_{ji} := \phi_j \cdot \phi_i^{-1}$ satisfy the conditions

E:cocycle

$$\overline{\theta}_{kj} \cdot \overline{\theta}_{ji} = \overline{\theta}_{ki} \tag{2.2}$$

when restricted to $U_{ijk} \times \mathrm{Spec}(\overline{A})$, where we denote $U_{ijk} = U_i \cap U_j \cap U_k$. Conversely, $\overline{X}$ is determined by the data $\{\overline{\theta}_{ji}\}$.
Similarly the family $X \longrightarrow \mathrm{Spec}(A)$ exists if and only if there are $\mathrm{Spec}(A)$-automorphisms

$$\begin{tikzcd}
U_{ij} \times \mathrm{Spec}(A) \arrow[rr, "\theta_{ji}"] \arrow[dr] & & U_{ij} \times \mathrm{Spec}(A) \arrow[dl] \\
& \mathrm{Spec}(A) &
\end{tikzcd}$$

such that

(a) $\theta_{kj} \cdot \theta_{ji} = \theta_{ki}$ on $U_{ijk} \times \mathrm{Spec}(A)$,
(b) they coincide with the $\overline{\theta}_{ji}$'s when restricted to $U_{ij} \times \mathrm{Spec}(\overline{A})$.

Automorphisms θ_{ji} satisfying condition (b) exist. Consider in fact the diagram:

$$\begin{tikzcd}
U_{ij} \times \mathrm{Spec}(A) \arrow[rr] \arrow[drr, dotted] & & \mathrm{Spec}(A) \\
U_{ij} \times \mathrm{Spec}(\overline{A}) \arrow[u, hook] \arrow[r, "\overline{\theta}_{ji}"] & U_{ij} \times \mathrm{Spec}(\overline{A}) \arrow[r, hook] & U_{ij} \times \mathrm{Spec}(A) \arrow[u]
\end{tikzcd}$$

The dotted arrow exists by the smoothness of U_{ij} and we can choose it as a θ_{ji}. Condition (a) is also satisfied by the θ_{ji}'s if and only if the automorphisms

$$\delta_{ijk} := \theta_{ki}^{-1} \cdot \theta_{kj} \cdot \theta_{ji}$$

are the identity of $U_{ijk} \times \mathrm{Spec}(A)$ for all i, j, k. Their restrictions to $U_{ijk} \times \mathrm{Spec}(\overline{A})$ are the identity, because of (2.2). Then, by Lemma 1.2.1, we have $\delta_{ijk} = \mathrm{id} + d_{ijk}$, with $d_{ijk} \in \Gamma(U_{ijk}, \Theta_{X_0})$. By construction the system $\{d_{ijk}\}$ is a 2-cocycle with coefficients in T_{X_0}. A different choice of the θ_{ji}'s is of the form $\widetilde{\theta}_{ji} = \theta_{ji} \cdot \delta_{ji}$, where δ_{ji} corresponds to a $d_{ji} \in \Gamma(U_{ij}, T_{X_0})$. Then

$$\widetilde{\theta}_{ki}^{-1} \cdot \widetilde{\theta}_{kj} \cdot \widetilde{\theta}_{ji} =: \widetilde{\delta}_{ijk} = \mathrm{id} + d_{ijk} + d_{ji} + d_{kj} - d_{ki}$$

In other words $\{d_{ijk}\}$ and $\{\widetilde{d}_{ijk}\}$ define the same element $o(\overline{\xi}, \pi) \in H^2(X_0, T_{X_0})$.

Claim: $o(\overline{\xi}, \pi) = 0$ if and only if X exists.

In fact X exists if and only if we can choose the θ_{ji}'s so that (a) is satisfied. Clearly $o(\overline{\xi}, \pi) = 0$ if (a) is satisfied. Conversely, if $o(\overline{\xi}, \pi) = 0$ then we can choose $\{d_{ji}\} \in \mathcal{C}^1(\mathcal{U}, T_{X_0})$ such that

$$d_{ijk} = d_{ji} + d_{kj} - d_{ki}$$

and we can take $\widetilde{\theta}_{ji} = \theta_{ji} \cdot \delta_{ji}^{-1}$. Then

$$\widetilde{\theta}_{ki}^{-1} \cdot \widetilde{\theta}_{kj} \cdot \widetilde{\theta}_{ji} = \mathrm{id} + d_{ijk} - (d_{ji} + d_{kj} - d_{ki}) = \mathrm{id}$$

This proves the Claim. We have proved the following

P:H2T

PROPOSITION 2.2.1. *Given a nonsingular variety* X_0, *a small extension*

$$0 \longrightarrow \mathbf{k}t \longrightarrow A \xrightarrow{\pi} \overline{A} \longrightarrow 0$$

and a deformation $\overline{\xi}$ *of* X_0 *over* $\mathrm{Spec}(\overline{A})$, *there is an element* $o(\overline{\xi}, \pi) \in H^2(X_0, T_{X_0})$ *associated to these data which vanishes if and only if* $\overline{\xi}$ *is the restriction of a deformation* ξ *of* X_0 *over* $\mathrm{Spec}(A)$.

2.3. Obstructions to deformations of a closed subscheme

In this section we consider a closed subscheme Y of a scheme X and we consider a problem analogous to the one discussed in the previous section for nonsingular varieties. Namely we consider a small extension

$$0 \longrightarrow \mathbf{k}t \longrightarrow \widetilde{A} \xrightarrow{\pi} \overline{A} \longrightarrow 0$$

and a deformation $\overline{Y} \subset X \times \mathrm{Spec}(\overline{A})$ of Y in X over $\mathrm{Spec}(\overline{A})$. We want to find the obstruction to the existence of a closed subscheme $\widetilde{Y} \subset X \times \mathrm{Spec}(\widetilde{A})$ such that $\overline{Y} = \widetilde{Y} \cap (X \times \mathrm{Spec}(\overline{A}))$.

We assume that $X = \mathrm{Spec}(R)$ and $Y = \mathrm{Spec}(S) = \mathrm{Spec}(R/I)$ are affine. The given deformation consists of an exact sequence:

$$0 \to \overline{I} \longrightarrow R \otimes \overline{A} \longrightarrow \overline{S} \to 0$$

We want to find the obstruction to the existence of $\widetilde{I} \subset R\otimes\widetilde{A}$ such that the following diagram is commutative:

$$\begin{tikzcd}
 & & \widetilde{I} \arrow[r] \arrow[d] & \overline{I} \arrow[d, "\iota"] \\
0 \arrow[r] & tR \arrow[r] \arrow[d, "\sigma"'] & R\otimes\widetilde{A} \arrow[r] \arrow[d] & R\otimes\overline{A} \arrow[d] \\
0 \arrow[r] & tS \arrow[r] & \widetilde{S} \arrow[r] & \overline{S}
\end{tikzcd}$$

Let

$$\begin{tikzcd}
0 \arrow[r] & tR \arrow[r] \arrow[d, "\sigma"'] & R\otimes\widetilde{A} \arrow[r] \arrow[d] & R\otimes\overline{A} \arrow[d, equal] & \\
0 \arrow[r] & tS \arrow[r] & \mathcal{E} \arrow[r] & R\otimes\overline{A} \arrow[r] & 0
\end{tikzcd}$$

be the pushout diagram determined by σ. Then $\widetilde{S}$ exists if and only if there is lifting $\widetilde{\iota} : \overline{I} \to \mathcal{E}$ of ι:

$$\begin{tikzcd}
\xi : & 0 \arrow[r] & tS \arrow[r] \arrow[d, equal] & F \arrow[r] \arrow[d] & \overline{I} \arrow[r] \arrow[d, "\iota"] \arrow[dl, "\widetilde{\iota}"'] & 0 \\
 & 0 \arrow[r] & tS \arrow[r] \arrow[d, equal] & \mathcal{E} \arrow[r] \arrow[d] & R\otimes\overline{A} \arrow[r] \arrow[d] & 0 \\
 & 0 \arrow[r] & tS \arrow[r] & \widetilde{S} \arrow[r] & \overline{S} \arrow[r] & 0
\end{tikzcd}$$

In fact the cokernel of $\widetilde{\iota}$ will be an $\widetilde{S}$ satisfying the requirements. In the above diagram the first row is the pullback of the second with respect to ι. It is clear that the existence of $\widetilde{\iota}$ is equivalent to the splitting of ξ, which is an element $o(\overline{Y}, \pi)$ of

$$\mathrm{Ext}^1_{\overline{S}}(\overline{I}, S) \cong \mathrm{Ext}^1_S(I, S) \cong \mathrm{Ext}^1_S(I/I^2, S)$$

The argument just given extends in a straightforward way to a general, not necessarily affine, closed embedding of schemes. We have thus proved the following

PROPOSITION 2.3.1. *Let Y be a closed subscheme of X with ideal sheaf $\mathcal{I} \subset \mathcal{O}_X$, and let $\pi : \widetilde{A} \to \overline{A}$ be a small extension. To every deformation $\overline{Y} \subset X \times \mathrm{Spec}(\overline{A})$ of Y in X over $\mathrm{Spec}(\overline{A})$, there is associated an element $o(\overline{Y}, \pi) \in \mathrm{Ext}^1_{\mathcal{O}_Y}(\mathcal{I}/\mathcal{I}^2, \mathcal{O}_Y)$ which vanishes if and only if there is a deformation $\widetilde{Y} \subset X \times \mathrm{Spec}(\widetilde{A})$ such that $\overline{Y} = \widetilde{Y} \cap (X \times \mathrm{Spec}(\overline{A}))$.*

Note that if Y is a local complete intersection in X then

$$\mathrm{Ext}^1_{\mathcal{O}_Y}(\mathcal{I}/\mathcal{I}^2, \mathcal{O}_Y) = H^1(Y, N_{Y/X})$$

2.4. Obstruction theory of a local ring

We now proceed further, trying to get a better control of obstructions. We start from the case of prorepresentable functors. Assume that we have a noetherian complete local $\mathbf{k}$-algebra $(R, \mathfrak{m}_R)$ with residue field $\mathbf{k}$. By Cohen's Theorem 1.1.13

we can write it as a quotient $R = P/J$, where P is a formally smooth $\mathbf{k}$-algebra, and $J \subset \mathfrak{m}_P^2$. This means in particular that $\mathbf{t}_P \cong \mathbf{t}_R$. The $\mathbf{k}$-vector space

$$o(R) := (J/\mathfrak{m}_P J)^\vee$$

is called *the obstruction space of* R. Since $\dim(o(R))$ equals the minimal number of generators of J, the following inequalities hold:

E:ineqdim
$$\dim(\mathbf{t}_R) \geq \dim(R) \geq \dim(\mathbf{t}_R) - \dim(o(R)) \tag{2.3}$$

Consider a diagram

E:basicobstr1
$$\begin{tikzcd} & & & R \arrow[d, "\overline{\varphi}"] & \\ 0 \arrow[r] & \mathbf{k}t \arrow[r] & A \arrow[r, "\pi"] & \overline{A} \arrow[r] & 0 \end{tikzcd} \tag{2.4}$$

consisting of a small extension π and a $(\overline{\varphi} : R \to \overline{A}) \in h_R(\overline{A})$. It can be included in the following one:

$$\begin{tikzcd} 0 \arrow[r] & J \arrow[r] \arrow[d] & P \arrow[r, "q"] \arrow[d, "\phi"] & R \arrow[r] \arrow[d, "\overline{\varphi}"] & 0 \\ 0 \arrow[r] & \mathbf{k}t \arrow[r] & A \arrow[r, "\pi"] & \overline{A} \arrow[r] & 0 \end{tikzcd}$$

where ϕ is a lifting of $\overline{\varphi} \cdot q$. Since π is small, the left vertical map defines an element $o(\overline{\varphi}, \pi) \in \mathrm{Hom}(J/\mathfrak{m}_P J, \mathbf{k}t) \cong o(R)$. Observe that $o(\overline{\varphi}, \pi)$ is independent of the choice of ϕ. In fact any other choice is of the form $\widetilde{\phi} = \phi + d$, where $d : P \to \mathbf{k}t$ is a derivation; since $J \subset \mathfrak{m}_P^2$, it follows that $d(J) \subset \mathfrak{m}_A(t) = 0$ and therefore $\phi_{|J} = \widetilde{\phi}_{|J}$. We call $o(\overline{\varphi}, \pi)$ the *obstruction to lift* $\overline{\varphi}$ *to* A. The following are true:

(1) $o(\overline{\varphi}, \pi) = 0$ if and only if there is a lifting $\varphi : R \to A$ of $\overline{\varphi}$ such that $\phi = \varphi q$. In particular, $(J/\mathfrak{m}_P J)^\vee = 0$ if and only if R is formally smooth.

(2) For every $(\overline{\varphi} : R \to \overline{A}) \in h_R(\overline{A})$ the map

$$\mathrm{Ex}_{\mathbf{k}}(\overline{A}, \mathbf{k}) \longrightarrow o(R), \quad \pi \mapsto o(\overline{\varphi}, \pi)$$

is $\mathbf{k}$-linear. This is easy to check.

(3) To every homomorphism $f : S \to R$ in $\widehat{\mathcal{A}}$ there is associated a linear map:

$$o(f) : o(R) \longrightarrow o(S)$$

called *the obstruction map* of f. For every diagram (2.4) we have:

E:basicobstr2
$$o(f)(o(\overline{\varphi}, \pi)) = o(\overline{\varphi} \cdot f, \pi) \tag{2.5}$$

If $S = Q/J$, with Q formally smooth and $J \subset \mathfrak{m}_Q^2$, The map $o(f)$ is the dual of the map $J/\mathfrak{m}_Q J \to I/\mathfrak{m}_P I$ induced by f. The identity (2.5) is obvious by construction.

L:obstr1

LEMMA 2.4.1. *Let R be in $\widehat{\mathcal{A}}$. For every $n \geq 0$ consider the canonical surjection $p_n : R \to R_n := R/\mathfrak{m}_R^{n+1}$, and the induced map:*

$$\mathrm{Ex}_{\mathbf{k}}(R_n, \mathbf{k}) \longrightarrow o(R), \quad \pi \mapsto o(p_n, \pi)$$

Then for all $n >> 0$ this map is surjective.

Proof. We may assume that R is not formally smooth, otherwise the lemma is trivially true. Suppose that $R = P/J$, where P is a formally smooth $\mathbf{k}$-algebra and $J \subset \mathfrak{m}_P^2$. Let $\{g_1, \dots, g_k\}$ be a minimal set of generators of J, and let $n \gg 0$ so that no g_j belongs to $\mathfrak{m}_P^{n+1}$. Then R_n is in $\mathcal{A}$ and we will prove that every element of $o(R)$ is of the form $o(p_n, \pi)$ where $\pi : A \to R_n$ is a small extension. Consider the following diagram with exact rows:

$$\begin{tikzcd}
 & 0 \arrow[r] & J \arrow[r] \arrow[d, hook] & P \arrow[r] \arrow[d, equal] & R \arrow[r] \arrow[d, "p_n"] & 0 \\
\zeta_n : & 0 \arrow[r] & (J, \mathfrak{m}_P^{n+1}) \arrow[r] & P \arrow[r] & R_n \arrow[r] & 0
\end{tikzcd}$$

The hypothesis on n implies that the left vertical map induces an injection

$$\frac{J}{\mathfrak{m}_P J} \subset \frac{(J, \mathfrak{m}_P^{n+1})}{(\mathfrak{m}_P J, \mathfrak{m}_P^{n+2})}$$

and therefore a surjection $o(R_n) \to o(R)$. Choose any $\lambda \in o(R)$ and let $\Lambda \in o(R_n)$ be such that

$$\lambda : \qquad \frac{J}{\mathfrak{m}_P J} \longrightarrow \frac{(J, \mathfrak{m}_P^{n+1})}{(\mathfrak{m}_P J, \mathfrak{m}_P^{n+2})} \xrightarrow{\Lambda} \mathbf{k}$$

From the diagram:

$$\begin{tikzcd}
 & 0 \arrow[r] & J \arrow[r] \arrow[d, hook] & P \arrow[r] \arrow[d, equal] & R \arrow[r] \arrow[d, "p_n"] & 0 \\
\zeta_n : & 0 \arrow[r] & (J, \mathfrak{m}_P^{n+1}) \arrow[r] \arrow[d, "\Lambda"'] & P \arrow[r] \arrow[d] & R_n \arrow[r] \arrow[d, equal] & 0 \\
\Lambda_*\zeta_n : & 0 \arrow[r] & \mathbf{k} \arrow[r] & L \arrow[r, "\pi"] & R_n \arrow[r] & 0
\end{tikzcd}$$

where $\Lambda_*\zeta_n$ is the pushout of ζ_n via Λ, we see that $\lambda = o(p_n, \pi)$. □

The following elementary proposition is extremely useful in DT:

L:smoothbyobstr

Proposition 2.4.2. *The following conditions are equivalent for a homomorphism $f : S \to R$ in $\widehat{\mathcal{A}}$:*

(i) *f is formally smooth.*

(ii) *$df : \mathbf{t}_R \to \mathbf{t}_S$ is surjective and $o(f) : o(R) \to o(S)$ is injective.*

Proof. (i) $\Rightarrow$ (ii). The surjectivity of df is in Corollary 1.1.7. Let $\lambda \in \ker(o(f))$. by Lemma 2.4.1 there exist small extension $\pi : A \to \overline{A}$ and $\overline{\varphi} : R \to \overline{A}$ such that $\lambda = o(\overline{\varphi}, \pi)$. Since $\lambda \in \ker(o(f))$ there is a commutative diagram:

E:formsmdiagram

$$\begin{tikzcd}
R \arrow[r, "\overline{\varphi}"] & \overline{A} \\
S \arrow[u, "f"] \arrow[r] & A \arrow[u, "\pi"']
\end{tikzcd} \tag{2.6}$$

From the formal smoothness of f it follows that $\overline{\varphi}$, as a homomorphism of S-algebras, has a lifting $\varphi : R \to A$. In particular φ is a lifting of $\overline{\varphi}$ as a homomorphism of $\mathbf{k}$-algebra, thus $\lambda = o(\overline{\varphi}, \pi) = 0$.

(ii) $\Rightarrow$ (i). Suppose given a commutative diagram (2.6), with π a small extension. Then $o(\overline{\varphi}, \pi) \in \ker(o(f)) = (0)$. Therefore there exists a lifting $\varphi : R \to A$ of $\overline{\varphi}$ as a homomorphism of **k**-algebras. In other words in the diagram

$$\begin{tikzcd} R \arrow[r, "\overline{\varphi}"] \arrow[dr, "\varphi"] & \overline{A} \\ S \arrow[u, "f"] \arrow[r, "\psi"'] & A \arrow[u, "\pi"'] \end{tikzcd}$$

we have $\overline{\varphi} = \pi \cdot \varphi$, but not necessarily $\psi = \varphi \cdot f$. But $\delta := \psi - \varphi \cdot f : S \to \mathbf{k}$ is a **k**-derivation. By the surjectivity of df there is a **k**-derivation $\tilde{\delta} : R \to \mathbf{k}$ such that $\delta = \tilde{\delta} \cdot f$. Therefore

$$\psi = \varphi \cdot f + \tilde{\delta} \cdot f = (\varphi + \tilde{\delta}) \cdot f$$

Replacing φ by $\tilde{\varphi} := \varphi + \tilde{\delta}$ we obtain a commutative diagram:

$$\begin{tikzcd} R \arrow[r, "\overline{\varphi}"] \arrow[dr, "\tilde{\varphi}"] & \overline{A} \\ S \arrow[u, "f"] \arrow[r, "\psi"'] & A \arrow[u, "\pi"'] \end{tikzcd}$$

proving that f is formally smooth. $\square$

In practise it is difficult to compute $o(R)$ or its dimension. What one can do is to introduce a weaker notion of obstruction space, which turns out to be often computable.

D:obsthring

Definition 2.4.3. *Let R be in $\widehat{\mathcal{A}}$. An* obstruction theory *for R is a* **k***-vector space $v(R)$ satisfying the following conditions. For every $(\overline{\varphi} : R \to \overline{A}) \in h_R(\overline{A})$ there is a* **k***-linear map:*

$$\mathrm{Ex}_{\mathbf{k}}(\overline{A}, \mathbf{k}) \longrightarrow v(R)$$

whose kernel consists of the extensions $\pi : A \to \overline{A}$ such that $\overline{\varphi}$ has a lifting $\varphi : R \to A$. The space $v(R)$ is called an obstruction space *of R.*

The usefulness of this notion is due to the following

P:inclobstr

Proposition 2.4.4. *Let R in $\widehat{\mathcal{A}}$ and let $v(R)$ be an obstruction space for R. Then there is a natural linear inclusion $v : o(R) \subset v(R)$.*

Proof. Let $n \geq 0$ be the smallest integer such that the map

$$\gamma_n : \mathrm{Ex}_{\mathbf{k}}(R_n, \mathbf{k}) \longrightarrow o(R), \quad \pi \mapsto o(p_n, \pi)$$

is surjective (see Lemma 2.4.1). Then we have linear maps:

$$\begin{tikzcd} \mathrm{Ex}_{\mathbf{k}}(R_n, \mathbf{k}) \arrow[r] \arrow[dr, "\gamma_n"'] & v(R) \\ & o(R) \end{tikzcd}$$

Since both maps have the same kernel and γ_n is surjective, there is induced a linear inclusion $o(R) \to v(R)$. $\square$

Corollary 2.4.5. *Let R be in $\widehat{\mathcal{A}}$ and let $v(R)$ be an obstruction space for R. Then:*

E:ineqdim2

$$\dim(\mathbf{t}_R) \geq \dim(R) \geq \dim(\mathbf{t}_R) - \dim(v(R)) \tag{2.7}$$

Proof. Since $o(R) \subset v(R)$, (2.7) follows directly from (2.3). $\square$

2.5. Obstruction theory for functors of Artin rings

In this section all functors of Artin rings satisfy the conditions H_0 and H_ϵ (see §3.2). Therefore they have a tangent space.

Definition 2.5.1. *Let F be a functor of Artin rings. An* obstruction theory *for F is a $\mathbf{k}$-vector space $v(F)$ such that, for every $\overline{A}$ in $\mathcal{A}$ and $\overline{\xi} \in F(\overline{A})$ there is a $\mathbf{k}$-linear map*

$$\overline{\xi}_v : \mathrm{Ex}_{\mathbf{k}}(\overline{A}, \mathbf{k}) \longrightarrow v(F)$$

whose kernel consists of the extensions $\pi : A \to \overline{A}$ such that

$$\overline{\xi} \in \mathrm{Im}[F(A) \longrightarrow F(\overline{A})]$$

The map $\overline{\xi}_v$ is called obstruction map *of $\overline{\xi}$, and $v(F)$ an* obstruction space *for F.*

Clearly this definition generalizes Definition 2.4.3.

P:obstrfunctor

Proposition 2.5.2. (i) *F has (0) as an obstruction space if and only if it is smooth.*

(ii) *Let $f : F \longrightarrow G$ be a smooth morphism of functors of Artin rings. If $v(G)$ is an obstruction space for G then it is also an obstruction space for F.*

Proof. (i) is obvious.

(ii) Consider $\overline{A}$ in $\mathcal{A}$ and $\overline{\xi} \in F(\overline{A})$, and let

$$F(f)(\overline{\xi})_v : \mathrm{Ex}_{\mathbf{k}}(\overline{A}, \mathbf{k}) \longrightarrow v(G)$$

be the obstruction map of $F(f)(\overline{\xi}) \in G(\overline{A})$. If $(\pi : A \to \overline{A}) \in \ker(F(f)(\overline{\xi})_v)$ then there is $\eta \in G(A)$ such that $\eta \mapsto F(f)(\overline{\xi})$. From the smoothness of f it follows that

$$F(A) \longrightarrow F(\overline{A}) \times_{G(\overline{A})} G(A)$$

is surjective. Therefore there exists $\xi \in F(A)$ which maps to $(\overline{\xi}, \eta)$. Therefore $\xi \mapsto \overline{\xi}$ under the map $F(A) \to F(\overline{A})$.
Conversely, suppose that, for a given $\overline{\xi} \in F(\overline{A})$, the extension $\pi : A \to \overline{A}$ is such that $\xi \in \mathrm{Im}[F(A) \to F(\overline{A})]$. Then, since by functoriality we have the commutative diagram

$$\begin{tikzcd}
F(A) \arrow[r] \arrow[d] & F(\overline{A}) \arrow[d] \\
G(A) \arrow[r] & G(\overline{A})
\end{tikzcd}$$

it follows that

$$F(f)(\xi) \mapsto F(f)(\overline{\xi})$$

therefore $F(f)(\overline{\xi}) \in \ker(F(f)(\overline{\xi})_v)$. Therefore $(F(f)(\overline{\xi})_v)$ is an obstruction map for $\overline{\xi}$, and $v(G)$ is an obstruction space for F. $\square$

CHAPTER 3

Formal deformation theory

3.1. Formal elements of a functor of Artin rings

We are now ready to proceed with the program outlined in the Introduction, namely the local study of a given class $\mathcal{M}$ of geometric objects. This will be done by studying functors of Artin rings more closely, with the purpose of understanding whether such a functor F is prorepresentable, i.e. if it is isomorphic to one of the form h_R, where R is a complete local $\mathbf{k}$-algebra in $\widehat{\mathcal{A}}$, or has some weaker property. In order to understand what this means precisely we start by considering a morphism of functors (a natural transformation) $\Phi : h_R \to F$, with F such that $F(\mathbf{k})$ consists of only one element.
For each $n \geq 0$ the canonical homomorphism $p_n : R \to R/\mathfrak{m}_R^{n+1} =: R_n$ is an element of $h_R(R_n)$, thus it defines an element $\xi_n := \Phi(p_n) \in F(R_n)$. The sequence $\{\xi_n\}$ is compatible, in the sense that $\xi_n \mapsto \xi_{n-1}$ under the map $F(R_n) \to F(R_{n-1})$ induced by $R_n \to R_{n-1}$. Therefore $\{\xi_n\}$ can be identified with an element $\widehat{\xi}$ of $\widehat{F}(R) := \varprojlim F(R_n)$. Conversely, given $\widehat{u} = \{u_n\} \in \widehat{F}(R)$ we can define a morphism of functors $\Phi : h_R \to F$ as follows. Let A in $\mathcal{A}$ and $(\alpha : R \to A) \in h_R(A)$. Then, if $n \gg 0$ we have a factorization $\alpha : R \to R_n \xrightarrow{\alpha_n} A$, and we define

$$\Phi(\alpha) = F(\alpha_n)(u_n)$$

Clearly this definition, which is a variant of Yoneda Lemma, does not depend on n. In fact, letting $p_{n+1,n} : R_{n+1} \to R_n$:

$$F(\alpha_{n+1})(u_{n+1}) = F(\alpha_n \cdot p_{n+1,n})(u_{n+1}) = F(\alpha_n)\left[(F(p_{n+1,n})(u_{n+1})\right] = F(\alpha_n)(u_n)$$

for all $n \gg 0$. Therefore we have an identification between $\widehat{F}(R)$ and the set of morphisms from h_R to F. A pair $(R, \widehat{u})$ consisting of a complete local $\mathbf{k}$-algebra R in $\widehat{\mathcal{A}}$ and an element $\widehat{u} \in \widehat{F}(R)$ is called a *formal element* of F over R. Therefore the above argument shows that there is a 1-1 correspondence between formal elements of F over R and morphisms of functors $h_R \to F$.

Definition 3.1.1. *In the above situation, if $(R, \widehat{u})$ corresponds to an isomorphism $\Phi_{\widehat{u}} : h_R \cong F$, i.e. F is prorepresented by R, we call $\widehat{u}$ a* universal formal element *of F. We call $\widehat{u}$* versal *if $\Phi_{\widehat{u}}$ is smooth. If $\widehat{u}$ is versal and moreover it induces a bijection $\mathbf{t}_R \to \mathbf{t}_F$ then we call $\widehat{u}$ a* semiuniversal element *of F; in this case we say that F has a* proprepresentable hull.

The following implications hold:

$$\widehat{u}\text{ universal} \Rightarrow \widehat{u}\text{ semiuniversal} \Rightarrow \widehat{u}\text{ versal}$$

and none of the inverse implications holds. In fact there is a subtle difference between a prorepresentable functor of Artin rings F and one having a prorepresentable hull. This point will be addressed in the next section.

P:functversusring

PROPOSITION 3.1.2. *Suppose that the functor of Artin rings F has a versal formal element $(R, \widehat{u})$. Then F is smooth if and only if R is formally smooth.*

PROOF. It follows from Lemma 1.1.10. □

S:schlessinger

3.2. Schlessinger's conditions

A prorepresentable functor $F \cong h_R$ has the following properties:

H_0) $F(\mathbf{k})$ consists of one element (the canonical quotient $R \to R/\mathfrak{m}_R = \mathbf{k}$).

Let

E:leftexactdiag

$$\begin{tikzcd} A' \arrow[dr] & & A'' \arrow[dl] \\ & A & \end{tikzcd} \tag{3.1}$$

be a diagram in $\mathcal{A}$ and consider the natural map:

$$\alpha : F(A' \times_A A'') \longrightarrow F(A') \times_{F(A)} F(A'')$$

Then:

H_ℓ (left exactness) For every diagram (3.1) α is bijective (straightforward to check).

H_f) $F(\mathbf{k}[\epsilon])$ has a structure of finite dimensional $\mathbf{k}$-vector space.

It can be proved that, conversely, if a functor of Artin rings has properties $H_0), H_\ell), H_f)$ then F is prorepresentable. Unfortunately condition H_ℓ is in general difficult to check. One needs sufficient conditions for prorepresentability, or for having a prorepresentable hull, which are easy to check. This issue is addressed by Schlessinger's theorem.

THEOREM 3.2.1 (Schlessinger). *Let F be a functor of Artin rings satisfying H_0). Then*

(i) *F has a semiuniversal element if and only if it satisfies the following conditions:*
$\overline{H}$) If $A'' \to A$ is a small extension then α is surjective.
H_ϵ) If $A = \mathbf{k}$ and $A'' = \mathbf{k}[\epsilon]$ then α is bijective.
H_f) $\dim(\mathbf{t}_F) < \infty$.

(ii) *F is prorepresentable if and only if it also satisfies the following additional condition:*
H) The natural map

$$F(A' \times_A A') \longrightarrow F(A') \times_{F(A)} F(A')$$

is bijective for every small extension $A' \to A$.

We will not give the full proof (we refer to [**H2**] or [**Se**] for it). We will rather discuss the meaning of condition H).

Assume that the conditions $\overline{H}, H_\epsilon, H_f$ are satisfied by the functor F. Then, by part (i) of the previous theorem, F has a semiuniversal element. What is needed for F to be prorepresentable is the property: for every small extension $\pi : A \to \overline{A}$ the non-empty fibres of $F(\pi)$ are torsors under the action of $\mathbf{t}_F$. We have discussed this point in §1.2, and we saw that such action can be defined for every functor F satisfying H_0 and H_ϵ as the composition:

$$\tau : \mathbf{t}_F \times F(A) \xrightarrow{\alpha^{-1}} F(\mathbf{k}[\epsilon] \times_{\mathbf{k}} A) \xrightarrow{F(b)} F(A)$$

where

$$b : \mathbf{k}[\epsilon] \times_{\mathbf{k}} A \longrightarrow A, \quad (x + y\epsilon, a) \mapsto a + yt$$

The map

$$\gamma : \mathbf{k}[\epsilon] \times_{\mathbf{k}} A \to A \times_{\overline{A}} A, \quad (x + y\epsilon, a) \mapsto (a + yt, a)$$

is an isomorphism. Consider the composition:

$$\beta : \mathbf{t}_F \times F(A) \xrightarrow{\alpha^{-1}} F(\mathbf{k}[\epsilon] \times_{\mathbf{k}} A) \xrightarrow{F(\gamma)} F(A \times_{\overline{A}} A) \xrightarrow{\overline{\alpha}} F(A) \times_{F(\overline{A})} F(A)$$

It acts as $(\theta, f) \mapsto (\tau(\theta, f), f)$, and this shows that τ preserves the fibres of $F(\pi)$. By $\overline{H}$, the map $\overline{\alpha}$ is surjective: this guarantees that the action τ is transitive on the fibres of $F(\pi)$. If we want it to be also faithful we need the bijectivity of $\overline{\alpha}$, namely we need condition H. This explains the meaning of H.

P:rigidfunctor

PROPOSITION 3.2.2. *Assume that F has a semiuniversal deformation, and that $\mathbf{t}_F = (0)$. Then F is the constant functor $F(A) = \{$one element$\}$. In particular F is prorepresentable.*

PROOF. From the previous discussion we know that $\mathbf{t}_F$ acts transitively on the fibres of $F(\pi)$ for every small extension π. Since $\mathbf{t}_F = (0)$ all such fibres consist of one element. Therefore $F = h_{\mathbf{k}}$ is prorepresentable. □

Proposition 1.4.5 is a special case of 3.2.2.

3.3. Automorphisms and prorepresentability

Let X be an algebraic scheme. Then it is easy to check that Def_X satisfies $H_0, \overline{H}, H_\epsilon$. If X is projective or affine with isolated singularities then it also satisfies H_f, hence Def_X has a semiuniversal deformation. For prorepresentability we need the further condition H. In the affine case this condition is not always satisfied (see [**Se**], Example 2.6.8(i), p. 95). In the projective case we have the following:

P:h0tX

PROPOSITION 3.3.1. *Let X be a projective nonsingular scheme such that $H^0(X, T_X) = 0$. Then Def_X is prorepresentable.*

COROLLARY 3.3.2. *Let C be a projective nonsingular curve. Then Def_C is prorepresentable.*

PROOF. If C has genus $g \geq 2$ then $H^0(C, T_C) = 0$ and we apply Prop. 3.3.1. If $g = 0$ then $H^1(C, T_C) = 0$ and C is rigid, by Prop. 3.2.2, so Def_C is prorepresentable. For $g = 1$ see [**Se**], Prop. 2.6.5, p. 93). □

3.4. Consequences of obstruction theory

PROPOSITION 3.4.1. *Suppose that F has a formal versal element $(R, \widehat{u})$ and has a finite dimensional obstruction space $v(F)$. Then:*

$$\dim(\mathbf{t}_R) \geq \dim(R) \geq \dim(\mathbf{t}_R) - \dim(v(F))$$

PROOF. It follows from (2.7) and Proposition 2.5.2: □

Definition 3.4.2. *Let* $f : F \to G$ *be a morphism of functors of Artin rings having semiuniversal formal elements and obstruction spaces* $v(F)$ *and* $v(G)$*, respectively. An* obstruction map *for* f *is a linear map* $o(f) : v(F) \to v(G)$ *such that, for every* $\overline{A}$ *in* $\mathcal{A}$ *and each* $\xi \in F(\overline{A})$ *the following diagram commutes:*

$$\begin{tikzcd} & \mathrm{Ex}_{\mathbf{k}}(\overline{A}, \mathbf{k}) \arrow[dl] \arrow[dr] & \\ v(F) \arrow[rr, "o(f)"] & & v(G) \end{tikzcd}$$

Proposition 3.4.3. *Let* $f : F \to G$ *be a morphism of functors of Artin rings having semiuniversal formal elements and finite dimensional obstruction spaces* $v(F)$ *and* $v(G)$ *respectively. Suppose that there is an obstruction map* $o(f) : v(F) \to v(G)$*. Consider the following conditions:*

(i) *df is surjective.*
(ii) *$o(f)$ is injective.*

If (i) and (ii) hold then f is smooth. If (ii) holds, but not necessarily (i), and G is smooth then F is smooth.

Proof. From Prop. 3.1.2, the assertions hold if and only if they hold for the corresponding prorepresentable functors. Then one can apply Prop. 2.4.2 and 2.4.4. (See [**Se**], Prop. 2.3.6, p. 59). □

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