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Curves on surfaces of degree $2r$-δ in $\mathbf{P}^r$

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Introduction

In this paper we consider the problem of finding the values of d, g for which there exists a nonsingular irreducible and nondegenerate (i.e. not contained in a hyperplane) curve X of degree d and genus g in $\mathbf{P}^r$, the projective space over an algebraically closed $\mathbf{k}$ of *arbitrary characteristic.*

This problem has been completely solved in $\mathbf{P}^3$ by Gruson–Peskine [GP] in the case $\mathrm{char}(\mathbf{k}) = 0$, then extended to arbitrary characteristic by Hartshorne [Ha], and in $\mathbf{P}^4$ and $\mathbf{P}^5$ by Rathmann [Ra]. The approach of [GP], which has been generalized in [Ra], is divided into two parts. The first consists in constructing, on a quartic surface with a double line F, nonsingular curves of degree d and genus g for every (d, g) such that

$$0 \le g \le (d-1)^2/8.$$

A similar result has been proved by Mori [M] in complex projective 3-space for every d, g as above, and his construction has been extended in [Ra], proving the existence of smooth curves of degree d and genus g in $\mathbf{P}^r$ lying on a K-3 surface when

$$0 \le g \le d^2/2(2r-2) - (r-1)/4.$$

The second part of the approach of [GP] is a detailed study of curves on a nonsingular cubic surface, which implies the existence result in the range

$$(d-1)^2/8 < g \le d(d-3)/6.$$

We generalize the first construction of Gruson–Peskine and we prove the existence of nonsingular curves of degree d and genus g in $\mathbf{P}^r$ for all $r \ge 6$ in a

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wide range of (d, g) (see statement below). Our curves are constructed on certain rational surfaces which are all embeddings of one and the same surface S': this is the blow-up of $\mathbf{P}^2$ at nine points in general position. We exploit the rich geometry of S' very much in the same way as it is done in [GP] and [Ra], with the difference that, for technical reasons, we first work with the surface S obtained by blowing up nine points which are *not* in general position, but are base points of a generic pencil of cubics. Then we prove the main result using deformation theoretic arguments. The main consequence of our analysis of curves lying on the surface S is the following:

MAIN THEOREM. (i) *For every $r \geq 5$ there exists an embedding of S' as a nonsingular surface F^{2r-3} of degree $2r-3$ in $\mathbf{P}^r$, and for every (d, g) such that*

$$0 \leq g \leq (d-r)^2/2(2r-3)$$

there exists a nonsingular irreducible and nondegenerate curve X of degree d and genus g on F^{2r-3}

(ii) *For every $r \geq 7$ there exists an embedding of S' as a nonsingular surface F^{2r-4} of degree $2r-4$ in $\mathbf{P}^r$, and for every (d, g) such that*

$$0 \leq g \leq (d-r)^2/2(2r-4)$$

there exists a nonsingular irreducible and nondegenerate curve X of degree d and genus g on F^{2r-4}.

Clearly, the existence result for curves in $\mathbf{P}^5$, contained in part (i) of the above theorem, follows from [Ra]. In more detail, the content of the paper is the following.

In section 1 we prove preliminary general results on the surface S which are repeatedly used in the paper. Precisely we give a criterion (proposition 1) for a linear system on S to be base point free and such that the associated map to projective space realizes S as a nonsingular surface. From this result we directly deduce an ampleness criterion which can also be found in [H].

In section 2 we introduce the notion of δ-system on S, which is a δ-tuple, $\delta \geq 3$, of elements of Pic (S) satisfying certain conditions. This notion turns out to be a powerful tool in the study of curves lying on the surface S. The main result of this section (theorem 6) states that, if a δ-system exists on S, then S can be embedded in $\mathbf{P}^r$ with degree $2r-\delta$ for all $r \geq 2\delta - 1$, in such a way that it contains smooth nondegenerate curves of degree d and genus g for all (d, g) such that

$$(*)\quad 0 \le g \le (d-r)^2/2(2r-\delta).$$

Actually we prove a slightly better bound (see remark 1).

In section 3 we consider the problem of existence of δ-systems. It is easy to show that δ-systems do not exist for $\delta > 9$ (see remark 2). It is not difficult to find candidates for $3 \le \delta \le 9$, namely to find δ-tuples of classes of Pic (S) which satisfy all but the last of the defining conditions. To prove that the last condition is also satisfied boils down to finding certain lists of elements of Pic (S). These lists become increasingly long as δ grows, and this has forced us to consider the cases $\delta = 3,4$ only, in which we are able to exhibit them. Via theorem 6, this proves a result which differs from the main theorem only in the fact that the surface S appears instead of S' in its statement. In remark 2 we also deduce the existence of smooth rational surfaces of degree $2r-\delta$ in $\mathbf{P}^r$, $r \ge \delta - 1$, for $5 \le \delta \le 9$.

In section 4 we show how to extend to S' most of the previous results concerning the surface S. Of course, this and the results quoted above imply the main theorem. We also discuss linear normality and the Brill-Noether map for the curves we have constructed.

Relying on the results of this paper, the first author has proved in [C] an asymptotic existence result for smooth nondegenerate curves in $\mathbf{P}^r$ for *all* values of r, which essentially says that for $d \gg 0$ smooth curves of degree d and genus g exist when

$$0 \le g \le \varphi_r(d)$$

where $\varphi_r(d) \sim d^2/2(4r/3-1)$, improving a similar one of Rathmann [Ra].

After this work was completed we have become aware of a preprint of Pasarescu [P], where he claims the existence of smooth nondegenerate curves of degree d and genus g in $\mathbf{P}^r$ for all $r \ge 5$ and all d, g such that

$$0 \le g \le (d-1)^2/2(2r-2).$$

His proof appears incomplete to us as it stands (on page 9, line -4, the maximum is not necessarily attained at an *integer*, as needed). From the argument of Pasarescu it seems to us that only a weaker bound, which is worse than ours for every d, g, r, can be deduced.

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1. Preliminaries

As already stated in the introduction, we work over an algebraically closed field $\mathbf{k}$ of arbitrary characteristic. We denote by S the surface obtained by blowing up nine points $P_1, \ldots, P_9$ of $\mathbf{P}^2$ which are base points of a *generic* pencil of cubics; we let $\pi: S \to \mathbf{P}^2$ be the projection. Note that any cubic $C \subset \mathbf{P}^2$ containing $P_1, \ldots, P_9$ is reduced, irreducible and with at most one node and no other singularity.

Let's denote by $E_1, \ldots, E_9$ the exceptional curves (of the first kind) on S corresponding to $P_1, \ldots, P_9$, and by H the inverse image on S of a general line of $\mathbf{P}^2$.

We identify an invertible sheaf on S with its class in Pic (S). As a basis of Pic (S) we take the classes $\mathbf{o}(H)$, $\mathbf{o}(-E_1), \ldots, \mathbf{o}(-E_g)$; we will sometimes denote an element of Pic (S) by the 10-tuple of its coordinates with respect to this basis.

We have:

$$\omega_S = (-3, -1, \ldots, -1) = \mathbf{o}(-C)$$

where C is the proper transform of a cubic through $P_1, \ldots, P_g$.

We will use without further mention the obvious fact that if D is an irreducible curve on S such that $(D, \omega_S) = 0$, then $D \in |-\omega_S|$.

We will freely use the notion of 1-connectedness of an effective divisor on a surface. We will also use without further notice the following vanishing theorem, referring the reader to [R] for the proof.

VANISHING THEOREM: *If D is an effective 1-connected divisor on a projective nonsingular surface F such that $h^1(F, \mathbf{o}_F) = 0$, then*

$$H^1(F, \mathbf{o}_F(-D)) = (0).$$

PROPOSITION 1. *Let D be an effective divisor on S.*

a) *If D is 1-connected and $(D, \omega_S) \le -2$, then the linear system $|D|$ has no base points.*

b) *If D is 1-connected and $(D, \omega_S) \le -3$, then $|D|$ has no base points, the morphism $\varphi_D: S \to \mathbf{P}(H^0(S, \mathbf{o}(D))\check{})$ is an isomorphism of S onto its image, except possibly for the contraction of some exceptional curves, and the image $\varphi_D(S)$ is nonsingular. In particular, a general element of $|D|$ is irreducible and nonsingular.*

c) *If D is 1-connected and $(D, \omega_S) \le -3$, then D is very ample if and only if $(D, E) > 0$ for every exceptional curve E.*

d) *If* $|\omega_S(D)|$ *is not empty, contains an effective* 1*-connected divisor and* $(D, \omega_S) \leq -3$, *then* $|D|$ *is very ample.*

Proof. a) For every $C \in |-\omega_S|$ we have an exact sequence

$$0 \to \omega_S(D) \to \mathbf{o}(D) \to \mathbf{o}_C(D) \to 0.$$

Since, by the connectedness of D, $h^1(S, \omega_S(D)) = h^1(S, \mathbf{o}(-D)) = 0$, we see that the restriction map $H^0(S, \mathbf{o}(D)) \to H^0(C, \mathbf{o}_C(D))$ is surjective. Therefore $|D|$ cuts a complete series on C of degree $(D, C) \geq 2$, hence without base points (recall that every $C \in |-\omega_S|$ is reduced and irreducible). It follows that $|D|$ has no base points on C. Since $\dim(|-\omega_S|) > 0$, the conclusion follows.

Proof of b) and c). By part a), $|D|$ has no base points. Let's denote by $|D-p|$ the linear system consisting of the curves of $|D|$ passing through p, for a given point p.

CLAIM 1. *Let* p *be any point of* S; *if* $|D-p|$ *has a fixed part, then it consists of an exceptional curve* E *passing through* p. *Moreover* $(D, E) = 0$, *i.e.* E *is contracted to a point by the morphism* φ_D.

Proof of claim 1. The fixed divisor F of $|D-p|$ satisfies $(F, \omega_S) = -1$, because $|D|$ cuts a complete series on any $C \in |-\omega_S|$ and $|D-p|$ has codimension one in $|D|$. Therefore $F = E$ is reduced, irreducible and rational (because $|-\omega_S|$ cuts on E a series of dimension and degree one), hence it is an exceptional curve of the first kind. Moreover, since p is a fixed point of the linear series $|D-p|_C$ cut on C by $|D-p|$, and since $|D-p|_C$ has codimension at most one in $|D|_C$, necessarily E contains p. Since $|D|$ has no base points we have $(D, E) \geq 0$. If $(D, E) > 0$, then from the exact sequence

$$0 \to \mathbf{o}(D-E) \to \mathbf{o}(D) \to \mathbf{o}_E(D) \to 0$$

and from $h^0(S, \mathbf{o}(D-E)) = h^0(S, \mathbf{o}(D) \otimes \mathbf{I}_p) = h^0(S, \mathbf{o}(D)) - 1$ ($\mathbf{I}_p \subset \mathbf{o}_S$ the ideal sheaf of p) it follows that $|D|$ has base points on E; this is a contradiction. This proves the claim.

As a consequence we have:

CLAIM 2. *If* p *and* q *are distinct points on* S, $|D|$ *does not separate* p *and* q *if and only if* p *and* q *are both contained in an exceptional curve* E *such that* $(D, E) = 0$.

Proof of claim 2. If p and q are not separated by $|D|$, then they cannot belong to the same $C \in |-\omega_S|$ because $|D|_C$ is very ample on C. If the general curve of $|D-p|$ is reducible then, by claim 1, it contains an exceptional curve E and, by the same claim, this curve contains both p and q. If the general curve M of $|D-p|$ is irreducible, then it passes simply through p, because on the curve $C \in |-\omega_S|$ containing p, $|D-p|_C$ has codimension one in $|D|_C$, hence it cannot have $2p$ as a fixed divisor. Similarly M passes simply through q. If M has genus g, then the degree of $|D|_M$ is at least $2g+1$; it follows that $|D|_M$ is very ample, therefore p and q are separated by $|D|_M$, and this is a contradiction.

Next we prove the following

CLAIM 3. *If p does not belong to an exceptional curve E such that $(D, E)=0$, then φ_D separates tangent vectors in p.*

Proof of claim 3. The general curve M of $|D-p|$ is irreducible and nonsingular in p. Since $2p$ is not a fixed divisor of $|D-p|_M$, because $|D-p|_M$ has codimension at most one in the complete and very ample $|D|_M$, the curves of $|D-p|$ are not all tangent to each other in p; this proves the claim.

Finally we prove

CLAIM 4. *If E is an exceptional curve such that $(D, E)=0$, then $\varphi_D(E)$ is a nonsingular point of the surface $\varphi_D(S)$. In particular $\varphi_D(S)$ is nonsingular.*

Proof of claim 4. On every curve $C \in |-\omega_S|$ the series $|D-E|_C$ coincides with the complete series $|D|_C-(E, C)$. Since we have $(D-E, \omega_S)=(D, \omega_S)+1 \leq -2$. It follows that $|D-E|$ has no base points on C, hence $|D-E|$ has no base points. The surface $\varphi_{D-E}(S)$ can be regarded as the projection of $\varphi_D(S)$ from the point $q=\varphi_D(E)$., Letting $\mu=\operatorname{mult}_q((\varphi_D(S)))$, we have

$$\begin{aligned}\deg(\varphi_D(S))-\mu &= \deg[\varphi_{D-E}(S)]\deg(\varphi_{D-E})\\ &= (D-E, D-E)=(D, D)-1=\deg(\varphi_D(S))-1,\end{aligned}$$

it follows that $\mu=1$, hence $\varphi_D(E)$ is nonsingular.

Assertions b) and c) of the proposition are clearly a consequence of claims 1), . . . , 4).

d) Since $|D|=|\omega_S(D)+C|$, $C \in |-\omega_S|$, it follows that $|D|$ contains a 1-connected divisor D'. Since, by a), $|\omega_S(D)|$ has no base points, for every exceptional curve E we have $(\omega_S(D), E) \geq 0$, and therefore

$$(D', E)=(D, E) \geq -(\omega_S, E)=1>0.$$

From c) it follows that $|D|=|D'|$ is very ample.

PROPOSITION 2. *Let D be an effective 1-connected divisor on S. Then every $D' \in |D|$ is 1-connected.*

Proof. Since D is effective we have $(D, \omega_S) \leq 0$. If $(D, \omega_S) = 0$ then $|D| = |-\omega_S|$ and the conclusion is clear. If $(D, \omega_S) = -1$ then $D = E + C_1 + \ldots + C_h$, with E exceptional curve and $C_1, \ldots, C_h \in |-\omega_S|$, and E is a fixed component of $|D|$. Then D' has a similar decomposition, hence it is 1-connected.

Suppose now that $(D, \omega_S) = -2$ and that $D' = A_1 + A_2$, with A_1, A_2 effective. Since $(A_i, \omega_S) \leq 0$ we have $-2 \leq (A_i, \omega_S) \leq 0$, $i = 1, 2$. If $(A_1, \omega_S) = 0$, $(A_2, \omega_S) = -2$ then $A_1 \in |-h\omega_S|$ for some $h \geq 1$, hence $(A_1, A_2) = 2h > 0$. Similar conclusion we have if $(A_1, \omega_S) = -2$, $(A_2, \omega_S) = 0$. If $(A_1, \omega_S) = -1 = (A_2, \omega_S)$ then $A_1 = E_1 + C_1 + \ldots + C_h$, $A_2 = E_2 + C_1' + \ldots + C_k'$, with E_1, E_2 exceptional curves and $C_1, \ldots, C_k' \in |-\omega_S|$. We now have $(A_1, A_2) > 0$ except in the case $A_1 = E + C$, $A_2 = E$. But then $(D, D) = (D', D') = 0$, hence $|D|$, being base point free by proposition 1, part a), is composed with a pencil; moreover, since D is 1-connected, $|D|$ is a pencil. But then, since $|D| = |2E + C|$ and C moves in a pencil, we see that $2E$ is a fixed divisor of $|D|$, and this contradicts proposition 1, part a).

Finally assume that $(D, \omega_S) \leq -3$. Since $(C, \omega_S(-D)) \leq -3 < 0$, we have

$$h^2(S, \mathbf{o}(D)) = h^0(S, \omega_S(-D)) = 0.$$

From the Riemann–Roch theorem we therefore obtain

$$\dim(|D|) \geq ((D, D) - (D, \omega_S))/2;$$

since, by proposition 1, part a), $|D|$ is base point free, we have $(D, D) \geq 0$ hence $\dim(|D|) \geq 2$. therefore $|D|$ is not a pencil; moreover $|D|$ cannot be composed with a pencil because D is 1-connected. It follows that $(D, D) > 0$. Now the conclusion follows from lemma 2 of [R].

We will say that an effective class $\mathbf{o}(D) \in \operatorname{Pic}(S)$, respectively a linear system $|D|$, is 1-connected if $|D|$ contains a 1-connected divisor, or equivalently, if every divisor of $|D|$ is 1-connected. The equivalence of the two conditions follows from proposition 2.

2. δ-systems on S

We will denote by **G** the group of Cremona isometries of S, namely the group of automorphisms of $\operatorname{Pic}(S)$ which

1) leave the semigroup of effective classes invariant,
2) preserve the intersection form (,),

3) leave the canonical class ω_S invariant.

We will need the following elementary result.

LEMMA 3. *If D is an effective 1-connected divisor and $\sigma \in \mathbf{G}$, then $\sigma(\mathbf{o}(D)) = \mathbf{o}(\bar{D})$, where $\bar{D}$ is an effective 1-connected divisor.*

Proof. Because of the defining property 1) of $\mathbf{G}$, an effective divisor $\bar{D}$ such that $\sigma(\mathbf{o}(D)) = \mathbf{o}(\bar{D})$ exists. Suppose that $\bar{D} = \bar{A}_1 + \bar{A}_2$, with $\bar{A}_1$ and $\bar{A}_2$ effective. Then, again by 1), $\sigma^{-1}(\bar{A}_1) = \mathbf{o}(A_i)$, with A_i effective, $i = 1,2$. Since $\mathbf{o}(D) = \mathbf{o}(A_1 + A_2)$, and D is 1-connected, from proposition 2 it follows that $(A_1, A_2) \geq 1$. Hence $(\bar{A}_1, \bar{A}_2) = (A_1, A_2) \geq 1$ and therefore $\bar{D}$ is 1-connected.

With an abuse of notation we will often write $\sigma(D)$ instead of $\sigma(\mathbf{o}(D))$, for a divisor D on S and $\sigma \in \mathbf{G}$. We will denote by $g(D)$ the arithmetic genus of D, namely

$$g(D) := (D, D + \omega_S)/2 + 1.$$

We give the following definition.

DEFINITION. Let $\delta \geq 2$ be an integer. An ordered δ-tuple $\{D_0, D_1, \ldots, D_{\delta-1}\}$ of effective classes in $\mathrm{Pic}(S)$ is called a *δ-system* if the following conditions are satisfied:

I) $(D_i, \omega_S) = -\delta$, $i = 0, \ldots, \delta - 1$.

II) $(D_i, D_i) = -\delta + 2i$, $i = 0, \ldots, \delta - 1$.

III) $D_0 - \omega_S, \ldots, D_{\delta-2} - \omega_S, D_{\delta-1}$ are effective and 1-connected.

IV) For every $0 \leq i,j \leq \delta - 1$, the number $(D_i, \sigma(D_j))$ assumes all integral values N such that

$$i + j + 2 - \delta \leq N,$$

when σ varies in $\mathbf{G}$.

Note that property IV) is clearly equivalent to the following:

V) for every $0 \leq i \leq j \leq \delta - 1$, all integral values $N \geq i + j + 2 - \delta$ are assumed by $(\sigma(D_1), \rho(D_j)))$ as σ, ρ vary in $\mathbf{G}$.

In this section we will investigate some of the consequences of the existence of a δ-system of divisors on S for some δ. In the next section we will construct such systems for $\delta = 3, 4$.

PROPOSITION 4. *Let $\{D_0, D_1, \ldots, D_{\delta-1}\}$ be a δ-system on S, with $\delta \geq 3$.*

Then

i) *for every* $\alpha \geq 1$ *the linear systems*

$$|D_0 - \alpha\omega_S|, \dots, |D_{\delta-2} - \alpha\omega_S|, \ |D_{\delta-1} - (\alpha-1)\omega_S|$$

are base point free and with general member irreducible and nonsingular; moreover for every $\alpha \geq 2$ *they are very ample.*

ii) *We have*:

$$(D_1 - \alpha\omega_S, D_1 - \alpha\omega_S) = (2\alpha - 1)\delta + 2i,$$

$$g(D_1 - \alpha\omega_S) = (\alpha - 1)\delta + i + 1,$$

$$\dim(|D_i - \alpha\omega_S|) = \alpha\delta + i \ (=g(D_i - \alpha\omega_S) + \delta - 1)$$

for $0 \leq i \leq \delta - 2$ *and for every* $\alpha \geq 1$, *and for* $i = \delta - 1$ *and for every* $\alpha \geq 0$.

Proof. From the defining property III) it follows that each of the linear systems $|D_0 - \alpha\omega_S|, \dots, |D_{\delta-2} - \alpha\omega_S|$, $|D_{\delta-1} - (\alpha-1)\omega_S|$ contains a 1-connected effective divisor for all $\alpha \geq 1$ and has intersection number equal to $-\delta \leq -3$ with ω_S. Applying parts b) and d) of proposition 1 we obtain i). The expressions for $(D_i - \alpha\omega_S, D_i - \alpha\omega_S)$ and $g(D_i - \alpha\omega_S)$ are obvious. The $\dim(|D_i - \alpha\omega_S|)$ is computed using the Riemann–Roch theorem on S, noting that

$$h^1(S, \mathbf{o}(D_1 - \alpha\omega_S)) = 0 = h^2(S, \mathbf{o}(D_i - \alpha\omega_S)).$$

The first equality follows because $|D_i - \alpha\omega_S - \omega_S|$ contains a 1-connected divisor when α and i assume the indicated values. The second equality is obvious because

$$h^2(S, \mathbf{o}(D_1 - \alpha\omega_S)) = h^0(S, \mathbf{o}(-(D_1 - (\alpha+1)\omega_S))$$

and $|D_1 - (\alpha+1)\omega_S| \neq \varnothing$. This proves ii).

PROPOSITION 5. *Assume that a* δ*-system* $\{D_0, D_1, \dots, D_{\delta-1}\}$ *exists on* S *for some* $\delta \geq 3$. *For each* $r \geq 2\delta - 1$ *write* $r = n\delta + i$, $i \in \{0, \dots, \delta - 1\}$, *and let*

$$H_r := D_i - n\omega_S.$$

Then H_r *is very ample and for every* (d, g) *such that* $0 \leq g \leq d - r - 1$, *there exists* *an irreducible and nonsingular curve* $X \subset S$ *such that*

$$(H_r, X) = d, \; g(X) = g$$

and

$$h^0(S, \mathbf{o}(H_r - X)) = 0.$$

Proof. Note that, since $r \geq 2\delta - 1$, we have $n \geq 2$ except in the case $r = 2\delta - 1$ when $n = 1$, $i = \delta - 1$, hence H_r is very ample by proposition 4. We can write in a unique way

$$g = (\alpha - 1)\delta + j + 1$$

for some $j \in \{0, \ldots, \delta - 1\}$ and $\alpha \geq 0$ ($\alpha \geq 1$ if $0 \leq j \leq \delta - 2$ and $\alpha = 0$ if and only if $g = 0$). For every $\sigma \in \mathbf{G}$ we have

$$\begin{aligned}(H_r, \sigma(D_j - \alpha\omega_S)) &= (D_i - n\omega_S, \sigma(D_j) - \alpha\omega_S)\\ &= (D_i, \sigma(D_j)) + (\alpha + n)\delta\\ &= (D_i, \sigma(D_j)) + g + r + \delta - 1 - i - j.\end{aligned}$$

Since $d \geq g + r + 1$, we have

$$d - [g + r + \delta - 1 - i - j] \geq i + j + 2 - \delta,$$

hence, by the defining property IV), there exists $\sigma \in \mathbf{G}$ such that

$$(D_i, \sigma(D_j)) = d - [g + r + \delta - 1 - i - j],$$

equivalently such that

$$(H_r, \sigma(D_j - \alpha\omega_S)) = d.$$

Since by lemma 3 $|\sigma(D_j - \alpha\omega_S)|$ contains an effective 1-connected divisor, from proposition 1b) it follows that $|\sigma(D_j - \alpha\omega_S)|$ contains an irreducible and nonsingular curve X. From proposition 4 we deduce that this curve has genus

$$g(D_j - \alpha\omega_S) = (\alpha - 1)\delta + j + 1 = g.$$

By contradiction, let's assume that

$$(*) \quad h^0(S, \mathbf{o}(H_r - X)) \neq 0.$$

From (*) and $(H_r - X, \omega_S) = 0$ we deduce that $|H_r - X| = |-a\omega_S|$, $a \geq 0$. Therefore

$$|\sigma(D_j) - \alpha\omega_S| = |X| = |H_r + a\omega_S| = |D_i + (a-n)\omega_S|.$$

and it follows that $i = j$ and $\sigma(D_j) = D_i$. Then:

$$-\delta + 2i = (D_i, D_i) = (D_i, \sigma(D_j)) \geq 2i + 2 - \delta,$$

a contradiction. This concludes the proof.

Using proposition 5 we can prove the following consequence of the existence of a δ-system on S for some $\delta \geq 3$.

THEOREM 6. *Assume that a δ-system $\{D_0, D_1, \ldots, D_{\delta-1}\}$ exists on S for some $\delta \geq 3$ and let $r \geq 2\delta - 1$ be an integer. Then there exists an embedding of S as a nonsingular surface F of degree $2r - \delta$ in $\mathbf{P}^r$, and for every (d, g) such that*

$$0 \leq g \leq (d-r)^2/2(2r-\delta) \tag{1}$$

there exists a nonsingular irreducible and nondegenerate curve X of degree d and genus g on F.

Proof. Writing $r = n\delta + i$, the embedding of S is given by the very ample class $H_r = D_i - n\omega_S$ considered in proposition 5. If $0 \leq g \leq d - r - 1$ the assertion has already been proved (proposition 5). Note that if a nonsingular irreducible and nondegenerate curve Z lies on F then a general element $Z' \in |Z + H_F|$ (H_F a hyperplane section of F) is a nonsingular irreducible curve, by proposition 1b), and is obviously nondegenerate. If Z has (degree, genus) $= (d, g)$, then Z' has (degree, genus) $= (d', g')$ given by

$$(d', g') = (d + 2r - \delta, g + d + r - \delta).$$

The inverse of the transformation

$$\begin{aligned} d' &= d + 2r + \delta \\ g' &= g + d + r - \delta \end{aligned} \tag{2}$$

is

$$\begin{aligned} d &= d' - 2r + \delta \\ g &= g' - d' + r. \end{aligned} \tag{3}$$

Applying (2) s times we obtain the transformation

$$
\begin{aligned}
d^{(s)} &= d + s(2r-\delta) \\
g^{(s)} &= g + sd + s(s-1)(2r-\delta)/2 + s(r-\delta)
\end{aligned}
\tag{2^s}
$$

whose inverse is:

$$
\begin{aligned}
d &= d^{(s)} - s(2r-\delta) \\
g &= g^{(s)} - sd^{(s)} + s(s+1)(2r-\delta)/2 - s(r-\delta).
\end{aligned}
\tag{3^s}
$$

In the plane with coordinates d, g represent with integral points the couples (d, g) for which we want to prove the theorem. They fill the region R under the parabola K with equation

$$g = (d-r)^2/2(2r-\delta).$$

We know that the theorem is true for all the points strictly below the line L_0 with equation $g = d - r$ and above the d-axis, by proposition 5. Let's denote by V_0 the set of these points. The transformation (2) maps the d-axis into L_0 and maps L_0 into the line $L_1 : g = 2d - 4r + \delta$. Therefore (2) maps V_0 in the set of points (d', g') such that

$$d' - r \leq g' < 2d' - 4r + \delta.$$

Since the theorem is also true in V_0, we see that the transformation (2) and the remark made at the beginning allow us to extend the validity of the theorem to $V_0 \cup V_1$, where V_1 is defined by:

$$0 \leq g \leq 2d - 4r + \delta.$$

Note that the two lines L_0 and L_1 have in common the point $P_1 = (3r-\delta, 2r-\delta)$. For every $s \geq 1$ let's denote by L_s the line whose equation is

$$g = (s+1)d - s(s+3)(2r-\delta)/2 + s(r-\delta) - r,$$

and which is the image of L_0 under (2^s). The line $g = 0$ will be denoted L_{-1}; it is transformed by (2^s) into L_{s-1}.

By induction we deduce that for every $k \geq 0$ the theorem is true in

$V_0 \cup \ldots \cup V_k$ where V_s is the region defined by the inequalities:

$$0 \leq g < (s+1)d - s(s+3)(2r-\delta)/2 + s(r-\delta) - r.$$

Note that $L_s \cap L_{s-1}$ is the point

$$P_s = ((2s+1)r - s\delta,\ s(s+1)(2r-\delta)/2).$$

In particular $P_0 = (r, 0)$. Note that the expression $g - (d-r)^2/2(2r-\delta)$ is zero in P_0. If we prove that it is positive in all the points $P_s \geq 1$, then these points are above the parabola K. Since K is concave upwards, it follows that the segments $P_{s-1}P_s$ lie above K and the theorem is true. In P_s we have:

$$\begin{aligned}(d-r)^2/2(2r-\delta) &= (2sr - s\delta)^2/2(2r-\delta)\\ &= s^2(2r-\delta)^2/2(2r-\delta) = s^2(2r-\delta)/2.\end{aligned}$$

Therefore in P_5:

$$g - (d-r)^2/2(2r-\delta) = s(s+1)(2r-\delta)/2 - s^2(2r-\delta)/2 = s(2r-\delta)/2,$$

which is positive for all $s \geq 1$. This concludes the proof.

Remark 1. Note that the proof of theorem 6 actually shows the existence of curves X of degree d and genus g on S for all (d, g) located below the polygon whose vertices are the points P_s considered in the proof. This region is slightly larger than that defined by (1).

3. Existence of δ-systems for $\delta = 3,4$.

In this section we discuss the existence of δ-systems on S. We will show that δ-systems exist for $\delta = 3,4$.

The basis $\mathbf{o}(H)$, $\mathbf{o}(-E_1), \ldots, \mathbf{o}(-E_9)$ identifies $\mathrm{Pic}(S)$ with $\mathbf{Z}^{10}$ and the intersection form(,) with the inner product $x_0^2 - \sum_1^9 x_i^2$. Consider the elements of $\mathrm{Pic}(S)$:

$$r_1 = (1, 1, 1, 1, 0, \ldots, 0),\quad r_2 = (0, -1, 1, 0, \ldots, 0),$$
$$r_3 = (0, 0, -1, 1, 0, \ldots, 0), \ldots$$
$$r_8 = (0, \ldots, 0, -1, 1, 0),\quad r_9 = (0, \ldots, 0, -1, 1).$$

Letting:

$$f_i(x) = x + (x, r_i)r_i, \quad i = 1, \dots, 9,$$

we obtain elements $f_1, \dots, f_g \in \mathbf{G}$. Recall that $f_1, f_2, \dots, f_9$ act on an element $(x_0, x_1, \dots, x_9) \in \operatorname{Pic}(S)$ in the following way:

$$f_1(x_0, x_1, \dots, x_9) = (x_0 + h, x_1 + h, x_2 + h, x_3 + h, x_4, \dots, x_9),$$
$$h = x_0 - x_1 - x_2 - x_3.$$

$$f_2(x_0, x_1, \dots, x_9) = (x_0, x_2, x_1, x_3, \dots, x_9),$$

$$f_3(x_0, x_1, \dots, x_9) = (x_0, x_1, x_3, x_2, x_4, \dots, x_9),$$
$$\dots$$
$$f_g(x_0, x_1, \dots, x_9) = (x_0, x_1, \dots, x_7, x_9, x_8).$$

In particular note that combining $f_2, \dots, f_9$ we can obtain any permutation of $x_1, \dots, x_9$. We will also consider, for every $z \in \omega_S^\perp$, the element $\tau_z \in \mathbf{G}$ defined as follows:

$$\tau_z(x) = x - (x, z)\omega_S + (x, \omega_S)z - (z, z)(x, \omega_S)\omega_S/2.$$

The following lemma generalizes lemma 1.4.1 of [Ra].

LEMMA 7. *Let $x, y \in \operatorname{Pic}(S)$ be such that $(x, \omega_S) = (y, \omega_S) = -\delta$, for some $\delta \geq 1$, and such that $x - y = (u_0, u_1, \dots, u_g)$ satisfies either one of the following two conditions*:

i) $u_1 = u_2, \; u_3 = u_4, \; u_5 = u_6, \; u_7 = u_8$.

ii) $u_1 = u_2, \; u_3 = u_4, \; u_5 = u_6, \; u_7 - u_8 = \delta$.

Then $(\rho(y), \sigma(x))$ assumes all integer values $(y, x) + n\delta^2$, $n \geq 0$, as ρ, σ vary in $\mathbf{G}$.

Proof. It suffices to show that the conclusion holds taking ρ = identity and $\sigma = \tau_z$, $z \in \omega_S^\perp$. For every $z \in \omega_S^\perp$ we have:

$$\begin{aligned}(y, \tau_z(x)) &= (y, x - (x, z)\omega_S + (x, \omega_S)z - (z, z)(x, \omega_S)\omega_S/2) \\ &= (y, x) - (x, z)(y, \omega_S) + (x, \omega_S)(y, z) - (z, z)(x, \omega_S)(y, \omega_S)/2 \\ &= (y, x) + \delta(x - y, z) - (z, z)\delta^2/2.\end{aligned}$$

Suppose that we are in case i). Then, taking

$$z = (0, a, -a, b, -b, c, -c, d, -d, 0)$$

we have $(x-y, z)=0$ and $-(z, z)/2=a^2+b^2+c^2+d^2$ and the conclusion follows from the fact that every positive integer is the sum of four squares. Suppose now that we are in case ii). Taking

$$z=(0, a, -a, b, -b, c, -c, 0, 0, 0)$$

we obtain $(x-y, z)=0$ and $-(z, z)/2=a^2+b^2+c^2$. This takes care of the cases in which $n \equiv 1,2,3,5,6 \pmod 8$, because every such integer n is the sum of three squares (see [5]). If $n \equiv 0,4,7 \pmod 8$ then $n-2>0$; we can write

$$n-2=a^2+b^2+c^2$$

and we take

$$z=(0, a, -a, b, -b, c, -c, 1, -1, 0).$$

We obtain $(x-y, z)=\delta$, and therefore:

$$(y, \tau_z(x))=(y, x)+\delta^2-(z, z)\delta^2/2=(y, x)+\delta^2+\delta^2(n-1).$$

This concludes the proof.

The existence of a 3-system is our next result. One half of the computations of this theorem are already in [Ra], where the divisors D_0, D_1, D_2 of theorem 8 are also considered.

THEOREM 8. *The classes*

$$D_0=(0, 0, 0, 0, 0, 0, 0, -1, -1, -1)$$

$$D_1=(1, 1, -1, 0, 0, 0, 0, 0, 0, 0)$$

$$D_2=(1, 0, 0, 0, 0, 0, 0, 0, 0, 0)$$

are a 3-*system on* S.

Proof. The defining properties I), II) and III) are obviously satisfied. Let's prove property V). For every $0 \leq i \leq j \leq 2$ and for every

$$i+j-1 \leq N \leq i+j+7$$

we will find elements of the form $\sigma(D_i)$, $\rho(D_j)$, σ, $\rho \in \mathbf{G}$, such that:

1) $(\sigma(D_i), \rho(D_j)) = N$,
2) $\sigma(D_i) - \rho(D_j)$ satisfies either one of conditions i), ii) of lemma 7.

Then property V), and the conclusion, will follow from lemma 7 applied to $x = \sigma(D_i)$ and $y = \rho(D_j)$. We give a list of such elements below. Each $\sigma(D_i)$ and $\rho(D_j)$ is obtained from D_i and D_j respectively by acting with a combination of the elements $f_1, \dots, f_g$ of $\mathbf{G}$. The tables are the following:

$\sigma(D_0)$	$\rho(D_0)$	$(\sigma(D_0), \rho(D_0))$
(0, −1, −1, 0, 0, 0, 0, 0, 0, −1)	(0, 0, 0, −1, −1, 0, 0, 0, 0, −1)	−1
(0, −1, −1, 0, 0, 0, 0, 0, 0, −1)	(1, 0, 0, 1, 1, −1, −1, 0, 0, 0)	0
(0, 0, 0, 0, 0, 0, 0, −1, −1, −1)	(2, 1, 1, 0, 0, 0, 0, −1, 2, 0)	1
(0, −1, −1, 0, 0, 0, 0, 0, 0, −1)	(1, 1, 1, 0, 0, −1, −1, 0, 0, 0)	2
(0, −1, −1, 0, 0, 0, 0, 0, 0, −1)	(2, 1, 1, −1, −1, 1, 1, 0, 0, 1)	3
(0, 0, 0, 0, 0, 0, 0, −1, −1, −1)	(3, 1, 1, −1, −1, 1, 1, 1, 1, 2)	4
(2, 1, 1, −1, −1, 1, 1, 0, 0, 1)	(2, 1, 1, 1, 1, −1, −1, 0, 0, 1)	5
(2, 1, 1, 1, 1, −1, −1, 0, 0, 1)	(3, 1, 1, −1, −1, 1, 1, 1, 1, 2)	6
(1, −1, −1, 1, 1, 0, 0, 0, 0, 0)	(3, 1, 1, −1, −1, 1, 1, 1, 1, 2)	7

$\sigma(D_0)$	$\rho(D_1)$	$(\sigma(D_0), \rho(D_1))$
(0, 0, −1, −1, −1, 0, 0, 0, 0, 0)	(1, 1, 0, 0, 0, 0, 0, 0, 0, −1)	0
(0, −1, −1,0, 0, 0, 0, 0, 0, −1)	(2, 1, 1, 1, 1, 0, 0, 0, 0, −1)	1
(2, 1, 1, 1, 0, 1, 1, −1, −1, 0)	(1, 0, 0, 0, −1, 0, 0, 0, 0, 1)	2
(2, 1, 1, 0, 0, 1, 1, −1, −1, 1)	(2, 1, 1, 1, 1, 0, 0, 0, 0, −1)	3
(1, −1, −1, 0, 0, 0, 0, 1, 1, 0)	(3, 1, 1, 1, 1, 0, 0, −1, 2, 1)	4
(1, −1, −1, 0, 0, 1, 1, 0, 0, 0)	(3, 1, 1, 1, 1, 0, 0, −1, 2, 1)	5
(3, 1, 1, −1, −1, 1, 1, 1, 1, 2)	(3, 1, 1, 1, 1, 0, 0, −1, 2, 1)	6
(2, 1, 1, 0, 0, 1, 1, −1, −1, 1)	(2, 0, 0, 1, 1, 0, 0, 1, 1, −1)	7
(3, 1, 1, −1, −1, 1, 1, 1, 1, 2)	(2, 1, 1, 1, 1, 0, 0, 0, 0, −1)	8

$\sigma(D_0)$	$\rho(D_2)$	$(\sigma(D_0), \rho(D_2))$
(0, −1, −1, −1, 0, 0, 0, 0, 0, 0)	(4, 0, 0, 1, 2, 2, 2, 1, 1, 0)	1
(0, −1, −1, −1, 0, 0, 0, 0, 0, 0)	(2, 1, 1, 0, 1, 0, 0, 0, 0, 0)	2
(0, −1, −1, −1, 0, 0, 0, 0, 0, 0)	(4, 1, 1, 1, 2, 2, 2, 0, 0, 0)	3
(0, −1, −1, −1, 0, 0, 0, 0, 0, 0)	(6, 1, 1, 2, 3, 3, 3, 1, 1, 0)	4
(0, −1, −1, −1, 0, 0, 0, 0, 0, 0)	(4, 2, 2, 1, 2, 1, 1, 0, 0, 0)	5
(0, −1, −1, −1, 0, 0, 0, 0, 0, 0)	(6, 3, 3, 0, 1, 2, 2, 2, 2, 0)	6

$(0, -1, -1, -1, 0, 0, 0, 0, 0, 0)$	$(8, 3, 3, 1, 2, 4, 4, 2, 2, 0)$	7
$(0, -1, -1, -1, 0, 0, 0, 0, 0, 0)$	$(6, 3, 3, 2, 3, 1, 1, 1, 1, 0)$	8
$(0, -1, -1, -1, 0, 0, 0, 0, 0, 0)$	$(8, 4, 4, 1, 2, 3, 3, 2, 2, 0)$	9

$\sigma(D_1)$	$\rho(D_1)$	$(\sigma(D_1), \rho(D_1))$
$(1, 1, 0, -1, 0, 0, 0, 0, 0, 0)$	$(1, 0, -1, 0, 1, 0, 0, 0, 0, 0)$	1
$(1, 1, 0, 0, 0, 0, 0, 0, 0, -1)$	$(1, 0, -1, 0, 0, 0, 0, 0, 0, 1)$	2
$(2, 1, 1, 1, 1, 0, 0, 0, 0, -1)$	$(2, 0, 0, 0, 0, 1, 1, 1, 1, -1)$	3
$(2, 1, 1, 0, 0, 0, 0, 1, 1, -1)$	$(3, 1, 1, 0, 0, 1, 1, -1, 2, 1)$	4
$(2, 1, 1, 1, 1, 0, 0, 0, 0, -1)$	$(3, 1, 1, 0, 0, 1, 1, -1, 2, 1)$	5
$(2, 1, 1, 1, 1, 0, 0, 0, 0, -1)$	$(3, 0, 0, 0, 0, 1, 1, 2, 2, 0)$	6
$(3, 0, 0, 2, 2, 1, 1, 0, 0, 0)$	$(3, 0, 0, 0, 0, 1, 1, 2, 2, 0)$	7
$(3, 0, 0, 2, 2, 0, 0, 1, 1, 0)$	$(3, 1, 1, 0, 0, 1, 1, -1, 2, 1)$	8
$(3, 1, 1, 2, 2, 0, 0, 0, 0, 0)$	$(3, 0, 0, 0, 0, 1, 1, 2, 2, 0)$	9

$\sigma(D_1)$	$\rho(D_2)$	$(\sigma(D_1), \rho(D_2)$
$(1, 0, 0, -1, 0, 0, 0, 0, 0, 1)$	$(2, 0, 0, 1, 0, 1, 1, 0, 0, 0)$	2
$(1, 1, 0, -1, 0, 0, 0, 0, 0, 0)$	$(3, 1, 0, 1, 2, 1, 1, 0, 0, 0)$	3
$(3, 1, 1, 2, 2, 0, 0, 0, 0, 0)$	$(2, 1, 1, 0, 0, 0, 0, 0, 0, 1)$	4
$(2, 0, 0, 1, 1, 1, 1, 0, 0, -1)$	$(2, 1, 1, 0, 0, 0, 0, 0, 0, 1)$	5
$(3, 1, 1, 2, 2, 0, 0, 0, 0, 0)$	$(2, 0, 0, 0, 0, 1, 1, 0, 0, 1)$	6
$(3, 1, 1, 2, 2, 0, 0, 0, 0, 0)$	$(3, 1, 1, 0, 0, 1, 1, 0, 0, 2)$	7
$(2, 0, 0, 1, 1, 1, 1, 0, 0, -1)$	$(3, 1, 1, 0, 0, 0, 0, 1, 1, 2)$	8
$(3, 1, 1, 2, 2, 0, 0, 0, 0, 0)$	$(3, 0, 0, 0, 0, 1, 1, 1, 1, 2)$	9
$(3, 1, 1, 2, 2, 0, 0, 0, 0, 0)$	$(4, 1, 0, 0, 1, 1, 1, 1, 3)$	10

$\sigma(D_2)$	$\rho(D_2)$	$(\sigma(D_2), \rho(D_2))$
$(1, 0, 0, 0, 0, 0, 0, 0, 0, 0)$	$(3, 1, 1, 1, 1, 0, 0, 0, 0, 2)$	3
$(1, 0, 0, 0, 0, 0, 0, 0, 0, 0)$	$(4, 1, 1, 1, 1, 1, 1, 0, 0, 3)$	4
$(1, 0, 0, 0, 0, 0, 0, 0, 0, 0)$	$(5, 1, 1, 1, 1, 1, 1, 1, 1, 4)$	5
$(2, 1, 1, 0, 0, 0, 0, 0, 0, 1)$	$(4, 1, 1, 1, 1, 1, 1, 0, 3, 0)$	6
$(2, 1, 1, 0, 0, 0, 0, 0, 0, 1)$	$(5, 1, 1, 1, 1, 1, 1, 1, 4, 1)$	7
$(3, 1, 1, 1, 1, 0, 0, 0, 0, 2)$	$(4, 1, 1, 1, 1, 1, 1, 0, 3, 0)$	8
$(3, 1, 1, 1, 1, 0, 0, 0, 0, 2)$	$(5, 1, 1, 1, 1, 1, 1, 1, 4, 1)$	9
$(4, 1, 1, 1, 1, 1, 1, 0, 0, 3)$	$(4, 1, 1, 1, 1, 1, 1, 0, 3, 0)$	10
$(4, 1, 1, 1, 1, 1, 1, 0, 0, 3)$	$(5, 1, 1, 1, 1, 1, 1, 1, 4, 1)$	11

This concludes the proof of theorem 7.

Now we will prove the existence of a 4-system on S.

THEOREM 9. *The classes*

$$D_0 = (0, -1, -1, -1, -1, 0, 0, 0, 0, 0)$$

$$D_1 = (1, -1, -1, 0, 0, 0, 0, 0, 0, 1)$$

$$D_2 = (1, 0, 0, 0, 0, 0, 0, 0, 0, -1)$$

$$D_3 = (2, 1, 1, 0, 0, 0, 0, 0, 0, 0)$$

are a 4-*system on* S.

Proof. Also in this case it is obvious that properties I), II) and III) are satisfied. We will proceed as in the proof of theorem 8: by applying lemma 7, for every $0 \leq i \leq j \leq 3$ and for every

$$i + j - 2 \leq N \leq i + j + 13$$

it will suffice to find elements of the form $\sigma(D_i)$, $\rho(D_j)$, σ, $\rho \in \mathbf{G}$, such that:

1) $(\sigma(D_i), \rho(D_j)) = N$,

2) $\sigma(D_i) - \rho(D_j)$ satisfies either one of conditions i), ii) of lemma 7. A list of such elements is given below:

$\sigma(D_0)$	$\rho(D_0)$	$(\sigma(D_0), \rho(D_0))$
(0, −1, −1, −1, −1, 0, 0, 0, 0, 0)	(0, 0, 0, 1, −1, −1, −1, 0, 0, 0)	−2
(0, 0, 0, −1, −1, 0, −1, 0, −1, 0)	(2, 0, 0, −1, −1, 1, 0, 2, 1, 0)	−1
(0, −1, 1, −1, −1, 0, 0, 0, 0, 0)	(0, 0, 0, 0, 0, −1, −1, −1, −1, 0)	0
(0, 0, 0, −1, −1, 0, 0, 0, −1, −1)	(3, 0, 0, −1, −1, 1, 1, 2, 1, 2)	1
(1, −1, −1, 0, 0, 1, 1, 0, 0, −1)	(1, 1, 1, −1, −1, 0, 0, 0, 0, −1)	2
(0, −1, −1, 0, 0, 0, 0, 0, −1, −1)	(3, 0, 0, −1, −1, 1, 1, 2, 1, 2)	3
(1, −1, −1, 1, 1, 0,1 0, 0, 0, −1)	(1, 1, 1, −1, −1, 0, 0, 0, 0, −1)	4
(0, 0, 0, 0, 0, −1, −1, 0, −1, −1)	(3, 0, 0, −1, −1, 1, 1, 2, 1, 2)	5
(1, 1, 1, −1, −1, 0, 0, 0,0, −1)	(2, −1, −1, 0, 0, 1, 1, 0, 0, 2)	6
(2, 1, 0. 0, 0, −1, −1, 2, 1, 0)	(2, 0, −1, 0, 0, 1, 1, 0, −1, 2)	7
(1, 1, 1, −1, −1, 0, 0, 0, 0, −1)	(2, −1, −1, 1, 1, 0, 0, 0, 0, 2)	8
(1, 1, 1, −1, −1, 0, 0, 0, −1, 0)	(4, −1, −1, 1, 1, 1, 1, 2, 1, 3)	9
(1, 1, 1, −1, −1, 0, 0, 0, 0, −1)	(3, −1, −1, 2, 2, 1, 1, 0, 0, 1)	10
(1, 1, 1, −1, −1, 0, 0, 0, −1, 0)	(5, −1, −1, 1, 1, 2, 2, 3, 2, 2)	11

(1, 1, 1, −1, −1, 0, 0, 0, 0, −1)	(5, −1, −1, 1, 1, 2, 2, 2, 2, 3)	12
(1, 1, 1, 0, 0, −1, −1, 0, −1, 0)	(5, −1, −1, 1, 1, 2, 2, 3, 2, 2)	13

$\sigma(D_0)$	$\rho(D_1)$	$(\sigma(D_0), \rho(D_1))$
(1, 1, 1, −1, −1, 0, 0, 0, −1, 0)	(1, 0, 0, −1, −1, 0, 0, 1, 0, 0)	−1
(1, 1, 1, −1, −1, 0, 0, 0, 0, −1)	(1, 0, 0, −1, −1, 0, 0, 0, 0, 1)	0
(1, 1, 1, −1, −1, 0, 0, 0, −1, 0)	(1, 0, 0, 0, 0, −1, −1, 1, 0, 0)	1
(1, 1, 1, −1, −1, 0, 0, 0, 0, −1)	(1, 0, 0, 0, 0, −1, −1, 0, 0, 1)	2
(1, 1, 1, −1, −1, 0, 0, 0, −1, 0)	(1, −1, −1, 0, 0, 0, 0, 1, 0, 0)	3
(1, 1, 1, −1, −1, 0, 0, 0, 0, −1)	(1, −1, −1, 0, 0, 0, 0, 0, 0, 1)	4
(0, −1, −1, 0, 0, 0, 0, 0, −1, −1)	(3, 1, 1, 0, 0, 0, 0, 0, 3, 0)	5
(1, 0, 0, 1, 1, −1, −1, 0, −1, 0)	(3, 1, 1, 0, 0, 0, 0, 0, 3, 0)	6
(0, −1, −1, 0, 0, 0, 0, −1, 0, −1)	(4, 2, 2, 0, 0, 0, 0, 0, 1, 3)	7
(0, −1, −1, 0, 0, 0, 0, 0, −1, −1)	(4, 2, 2, 0, 0, 0, 0, 0, 3, 1)	8
(0, −1, −1, 0, 0, 0, 0, 0, −1, −1)	(5, 3, 3, 0, 0, 0, 0, 2, 1, 2)	9
(2, −1, −1, 1, 1, 0, 0, 0, 0, 2)	(5, 3, 3, 2, 2, 0, 0, 0, 0, 1)	10
(1, −1, −1, 0, 0, 1, 1, −1, 0, 0)	(5, 3, 3, 2, 2, 0, 0, 0, 1, 0)	11
(1, −1, −1, 0, 0, 1, 1, 0, 0, −1)	(5, 3, 3, 2, 2, 0, 0, 0, 0, 1)	12
(0, 0, 0, −1, −1, 0, 0, 0, −1, −1)	(10, 0, 0, 4, 4, 5, 5, 3, 2, 3)	13
(2, −1, −1, 0, 0, 1, 1, 0, 0, 2)	(5, 3, 3, 2, 2, 0, 0, 0, 0, 1)	14

$\sigma(D_0)$	$\rho(D_2)$	$(\sigma(D_0), \rho(D_2))$
(0, −1, −1, −1, −1, 0, 0, 0, 0, 0)	(1, 0, 0, 0, 0, 0, 0, 0, 0, −1)	0
(0, −1, −1, 0, 0, 0, 0, −1, 0, −1)	(2, 1, 1, 0, 0, 0, 0, 0, 1, −1)	1
(1, 0, 0, 1, 1, −1, −1, −1, 0, 0)	(2, 1, 1, 0, 0, 0, 0, 0, 1, −1)	2
(2, 0, 0, −1, −1, 0, 0, 1, 2, 1)	(2, 1, 1, 0, 0, 0, 0, 0, 1, −1)	3
(1, −1, −1, 1, 1, 0, 0, −1, 0, 0)	(2, 1, 1, 0, 0, 0, 0, 0, 1, −1)	4
(2, 0, 0, −1, −1, 0, 0, 2, 1, 1)	(2, 0, 0, 1, 1, 0, 0, 1, 0, −1)	5
(2, 0, 0, −1, −1, 1, 1, 2, 0, 0)	(2, 1, 1, 0, 0, 0, 0, −1, 1, 0)	6
(2, 0, 0, −1, −1, 0, 0, 1, 2, 1)	(3, 1, 1, 1, 1, 0, 0, −1, 0, 2)	7
(2, 0, 0, −1, −1, 0, 0, 2, 1, 1)	(3, 1, 1, 1, 1, 0, 0, −1, 2, 0)	8
(3, 1, 1, −1, −1, 0, 0, 2, 1, 2)	(3, 1, 1, 1, 1, 0, 0, −1, 2, 0)	9
(1, 1, 0, −1, −1, 0, 0, 1, 0, −1)	(5, 0, −1, 2, 2, 1, 1, 2, 1, 3)	10
(3, 0, 0, −1, −1, 1, 1, 2, 1, 2)	(3, 1, 1, 1, 1, 0, 0, −1, 2, 0)	11
(3, 0, 0, −1, −1, 1, 1, 2, 1, 2)	(4, 2, 2, 1, 1, 0, 0, −1, 2, 1)	12
(4, −1, −1, 1, 1, 1, 1, 2, 3, 1)	(4, 1, 1, 1, 1, 1, 1, −1, 0, 3)	13
(3, −1, −1, 0, 0, 1, 1, 2, 1, 2)	(4, 2, 2, 1, 1, 0, 0, −1, 2, 1)	14
(3, −1, −1, 0, 0, 1, 1, 2, 1, 2)	(6, 2, 2, 2, 2, 3, 3, 0, −1, 1)	15

$\sigma(D_0)$	$\rho(D_3)$	$(\sigma(D_0), \rho(D_3))$
$(1, 1, 1, -1, 0, 0, 0, -1, 0)$	$(3, 1, 1, 0, 0, 0, 0, 1, 0, 2)$	1
$(0, -1, -1, -1, -1, 0, 0, 0, 0)$	$(2, 1, 1, 0, 0, 0, 0, 0, 0, 0)$	2
$(1, 0, 0, -1, -1, 1, 1, 0, -1, 0)$	$(3, 1, 1, 0, 0, 0, 0, 1, 0, 2)$	3
$(0, -1, -1, 0, 0, 0, 0, 0, -1, -1)$	$(3, 1, 1, 0, 0, 0, 0, 1, 0, 2)$	4
$(1, -1, -1, 0, 0, 1, 1, 0, -1, 0)$	$(3, 1, 1, 0, 0, 0, 0, 1, 0, 2)$	5
$(1, -1, -1, 0, 0, 1, 1, 0, -1, 0)$	$(3, 1, 1, 0, 0, 0, 0, 2, 1, 0)$	6
$(1, -1, -1, 0, 0, 0, 1, -1, 0)$	$(3, 1, 1, 0, 1, 0, 0, 0, 2, 0)$	7
$(3, -1, -1, 0, 0, 1, 1, 2, 2, 1)$	$(2, 1, 1, 0, 0, 0, 0, 0, 0, 0)$	8
$(1, -1, -1, 1, 1, 0, 0, -1, 0)$	$(4, 2, 2, 0, 0, 0, 0, 2, 1, 1)$	9
$(0, -1, -1, -1, -1, 0, 0, 0, 0, 0)$	$(6, 3, 3, 2, 2, 2, 2, 0, 0, 0)$	10
$(1, -1, -1, 0, 0, 1, 1, 0, -1, 0)$	$(5, 2, 2, 1, 1, 0, 0, 3, 2, 0)$	11
$(1, -1, -1, 0, 0, 1, 1, 0, -1, 0)$	$(5, 2, 2, 1, 1, 0, 0, 0, 3, 2)$	12
$(0, 0, 0, -1, -1, 0, 0, -1, 0, -1)$	$(9, 3, 3, 4, 4, 0, 0, 3, 4, 2)$	13
$(2, -1, -1, 0, 0, 1, 1, 2, 0, 0)$	$(5, 2, 2, 1, 1, 0, 0, 0, 2, 3)$	14
$(1, 0, 0, -1, -1, 1, 1, 0, -1, 0)$	$(7, 3, 3, 2, 2, 0, 0, 1, 4, 2)$	15
$(4, -1, -1, 1, 1, 1, 1, 3, 1, 2)$	$(4, 2, 2, 0, 0, 1, 1, 0, 2, 0)$	16

$\sigma(D_1)$	$\rho(D_1)$	$(\sigma(D_1), \rho(D_1))$
$(1, -1, -1, 0, 0, 0, 0, 0, 0, 1)$	$(1, 0, 0, -1, -1, 0, 0, 0, 0, 1)$	0
$(1, -1, 0, 0, 0, 0, 0, -1, 0, 1)$	$(1, 0, 1, 0, 0, 0, 0, -1, 0, -1)$	1
$(1, -1, -1, 0, 0, 0, 0, 0, 0, 1)$	$(3, 1, 1, 0, 0, 0, 0, 0, 0, 3)$	2
$(2, 1, 0, 0, 0, 0, 0, 2, 0, -1)$	$(3, 1, 0, 0, 0, 0, 0, 1, 3, 0)$	3
$(2, 0, -1, 0, 0, 0, 0, 2, 0, 1)$	$(3, 1, 0, 0, 0, 0, 0, 1, 3, 0)$	4
$(2, 1, 0, 0, 0, 0, 0, 2, -1, 0)$	$(2, 0, -1, 0, 0, 0, 0, 0, 1, 2)$	5
$(4, 2, 2, 0, 0, 0, 0, 0, 1, 3)$	$(4, 0, 0, 2, 2, 0, 0, 0, 1, 3)$	6
$(1, -1, -1, 0, 0, 0, 0, 1, 0, 0)$	$(4, 2, 2, 0, 0, 0, 0, 1, 0, 3)$	7
$(1, -1, -1, 0, 0, 0, 0, 1, 0, 0)$	$(4, 2, 2, 0, 0, 0, 0, 0, 3, 1)$	8
$(3, 1, 1, 0, 0, 0, 0, 3, 0, 0)$	$(5, 3, 3, 2, 2, 0, 0, 0, 1, 0)$	9
$(2, 0, 0, 0, 0, 0, 0, 1, -1, 2)$	$(4, 2, 2, 0, 0, 0, 0, 1, 3, 0)$	10
$(4, 0, 0, 2, 2, 0, 0, 0, 1, 3)$	$(5, 3, 3, 2, 2, 0, 0, 0, 1, 0)$	11
$(3, 0, 0, 0, 0, 1, 1, 0, 0, 3)$	$(5, 3, 3, 2, 2, 0, 0, 0, 0, 1)$	12
$(4, 2, 2, 0, 0, 0, 0, 3, 0, 1)$	$(4, 0, 0, 2, 2, 0, 0, 0, 1, 3)$	13
$(2, 0, 0, 0, 0, -1, 0, 2, 0, 1)$	$(6, 2, 2, 0, 0, 3, 4, 0, 2, 1)$	14
$(3, 0, 0, 0, 0, 1, 1, 3, 0, 0)$	$(5, 3, 3, 2, 2, 0, 0, 0, 1, 0)$	15

$\sigma(D_1)$	$\rho(D_2)$	$(\sigma(D_1), \rho(D_2))$
(1, −1, −1, 0, 0, 0, 0, 1, 0, 0)	(1, 0, 0, 0, 0, 0, 0, 0, −1, 0)	1
(1, −1, −1, 0, 0, 0, 0, 0, 0, 1)	(1, 0, 0, 0, 0, 0, 0, 0, −1)	2
(2, 0, 0, 0, 0, 0, 0, 2, 1, −1)	(1, 0, 0, 0, 0, 0, 0, 0, −1, 0)	3
(2, 0, 0, 0, 0, 0, 0, 2, −1, 1)	(1, 0, 0, 0, 0, 0, 0, −1, 0, 0)	4
(2, 1, 0, 0, 0, 0, 0, 2, −1, 0)	(2, 1, 0, 1, 1, 0, 0, −1, 0, 0)	5
(3, 1, 1, 0, 0, 0, 0, 0, 0, 3)	(1, 0, 0, 0, 0, 0, 0, 0, 0, −1)	6
(3, 1, 1, 0, 0, 0, 0, 3, 0, 0)	(2, 1, 1, 0, 0, 0, 0, −1, 0, 1)	7
(3, 1, 0, 0, 0, 0, 0, 3, 0, 1)	(2, 1, 0, 1, 1, 0, 0, −1, 0, 0)	8
(3, 1, 1, 0, 0, 0, 0, 3, 0, 0)	(2, 0, 0, 1, 1, 0, 0, −1, 0, 1)	9
(3, 1, 1, 0, 0, 0, 0, 3, 0, 0)	(3, 1, 1, 1, 1, 0, 0, −1, 0, 2)	10
(5, 0, 0, 2, 2, 3, 3, 0, 1, 0)	(3, 1, 1, 1, 1, 0, 0, −1, 0, 2)	11
(3, 0, 0, 0, 0, 1, 1, 3, 0, 0)	(3, 1, 1, 1, 1, 0, 0, −1, 0, 2)	12
(3, 0, 0, 0, 0, 1, 0, 3, 0, 1)	(4, 2, 2, 1, 1, 2, 1, −1, 0, 0)	13
(5, 0, 0, 2, 2, 3, 3, 1, 0, 0)	(4, 2, 2, 1, 1, 0, 0, 2, 1, −1)	14
(5, 0, 0, 2, 2, 3, 3, 1, 0, 0)	(4, 2, 2, 0, 0, 1, 1, −1, 2, 1)	15
(9, 4, 4, 4, 4, 0, 0, 3, 1, 3)	(2, 0, 0, 0, 0, 1, 1, 1, −1, 0)	16

$\sigma(D_1)$	$\rho(D_3)$	$(\sigma(D_1), \rho(D_3))$
(1, 0, 0, −1, −1, 0, 0, 0, 0, 1)	(2, 1, 1, 0, 0, 0, 0, 0, 0, 0)	2
(1, −1, −1, 1, 0, 0, 0, 0, 0, 0)	(3, 1, 1, 2, 1, 0, 0, 0, 0, 0)	3
(1, −1, −1, 0, 0, 0, 0, 0, 0, 1)	(2, 1, 1, 0, 0, 0, 0, 0, 0, 0)	4
(1, 0, 0, −1, −1, 1, 0, 0, 0, 0)	(4, 1, 1, 1, 1, 1, 0, 0, 0, 3)	5
(3, 0, 0, 1, 1, 0, 0, 0, 0, 3)	(2, 1, 1, 0, 0, 0, 0, 0, 0, 0)	6
(3, 1, 1, 0, 0, 0, 0, 3, 0, 0)	(3, 1, 1, 0, 0, 0, 0, 0, 1, 2)	7
(5, 3, 3, 2, 2, 0, 0, 0, 1, 0)	(3, 1, 1, 0, 0, 0, 0, 0, 1, 2)	8
(3, 0, 0, 1, 1, 0, 0, 3, 0, 0)	(3, 1, 1, 0, 0, 0, 0, 0, 1, 2)	9
(3, 1, 1, 0, 0, 0, 0, 3, 0, 0)	(4, 1, 1, 1, 1, 0, 0, 0, 1, 3)	10
(1, −1, −1, 0, 0, 1, 0, 0, 0, 0)	(7, 3, 3, 2, 2, 2, 1, 0, 0, 4)	11
(3, 1, 1, 0, 0, 0, 0, 3, 0, 0)	(4, 0, 0, 1, 1, 1, 1, 0, 1, 3)	12
(4, 2, 2, 0, 0, 0, 0, 3, 0, 1)	(4, 0, 0, 1, 1, 1, 1, 0, 1, 3)	13
(5, 3, 3, 2, 2, 0, 0, 1, 0, 0)	(4, 1, 1, 0, 0, 1, 1, 0, 3, 1)	14
(5, 3, 3, 2, 2, 0, 0, 0, 1, 0)	(4, 0, 0, 1, 1, 1, 1, 0, 1, 3)	15
(5, 3, 3, 2, 2, 0, 0, 1, 0, 0)	(4, 0, 0, 1, 1, 1, 1, 0, 3, 1)	16
(9, 4, 4, 4, 4, 3, 3, 1, 0, 0)	(2, 0, 0, 0, 0, 0, 0, 1, 0, 1)	17

$\sigma(D_2)$	$\rho(D_2)$	$(\sigma(D_2), \rho(D_2))$
(1, 0, 0, 0, −1, 0, 0, 0, 0, 0)	(2, 1, 1, 1, 0, 0, 0, 0, 0, −1)	2
(2, 1, 1, 0, −1, 0, 0, 0, 0, 1)	(2, 1, 1, 1, 0, 0, 0, 0, 0, −1)	3
(1, 0, 0, 0, 0, 0, 0, 0, 0, −1)	(3, 2, 2, 0, 0, 0, 0, 0, 0, 1)	4
(2, 1, 1, 0, −1, 0, 0, 0, 0, 1)	(2, 1, 1, 1, 0, 0, 0, 0, 0, −1)	5
(2, 1, 1, 1, 0, 0, 0, 0, 0, −1)	(3, 0, 0, 2, 1, 0, 0, 0, 0, 2)	6
(3, 2, 2, 0, 0, 0, 0, 1, 0, 0)	(3, 0, 0, 0, 0, 0, 0, 2, 1, 2)	7
(3, 0, 0, 1, 0, 2, 2, 0, 0, 0)	(3, 0, 0, 1, 0, 0, 0, 2, 2, 0)	8
(6, 3, 3, 2, 2, 0, 0, 3, 0, 1)	(1, 0, 0, 0, 0, 0, 0, −1, 0, 0)	9
(6, 3, 3, 2, 2, 0, 0, 3, 0, 1)	(2, 0, 0, 1, 1, 0, 0, −1, 0, 1)	10
(5, 3, 3, 1, 1, 0, 0, 0, 1, 2)	(2, 0, 0, 0, 0, 1, 1, 0, 1, −1)	11
(6, 3, 3, 2, 2, 0, 0, 3, 1, 0)	(4, 1, 1, 2, 2, 0, 0, −1, 1, 2)	12
(3, 0, 0, 0, 0, 2, 2, 1, 0, 0)	(4, 2, 2, 1, 1, 0, 0, −1, 2, 1)	13
(6, 3, 3, 2, 2, 0, 0, 3, 0, 1)	(2, 0, 0, 0, 0, 1, 1, −1, 0, 1)	14
(4, 0, 0, 2, 2, 1, 1, 2, −1, 1)	(4, 2, 2, 0, 0, 1, 1, 1, 2, −1)	15
(6, 3, 3, 2, 2, 0, 0, 3, 0, 1)	(4, 1, 1, 0, 0, 2, 2, 1, 2, −1)	16
(6, 3, 3, 2, 2, 0, 0, 1, 0, 3)	(3, 0, 0, 0, 0, 2, 2, 1, 0, 0)	17

$\sigma(D_2)$	$\rho(D_3)$	$(\sigma(D_2), \rho(D_3))$
(2, 1, 1, 1, 0, 0, 0, 0, −1, 0)	(2, 0, 0, 1, 0, 0, 0, 1, 0, 0)	3
(2, 1, 1, 1, 0, 0, 0, 0, 0, −1)	(2, 0, 0, 1, 0, 0, 0, 0, 0, 1)	4
(2, 1, 1, 1, 0, 0, 0, 0, 0, −1)	(3, 1, 1, 1, 0, 0, 0, 0, 0, 2)	5
(3, 2, 2, 0, 0, 0, 0, 0, 0, 1)	(2, 0, 0, 1, 1, 0, 0, 0, 0, 0)	6
(2, 0, 0, 1, 0, 1, 1, 0, 0, −1)	(3, 1, 1, 1, 0, 0, 0, 0, 0, 2)	7
(3, 0, 0, 1, 0, 0, 0, 2, 2, 0)	(3, 1, 1, 1, 0, 0, 0, 0, 0, 2)	8
(1, 0, 0, 0, 0, 0, 0, 0, −1, 0)	(6, 2, 2, 2, 2, 0, 0, 0, 3, 3)	9
(4, 1, 1, 2, 2, 0, 0, 1, −1, 2)	(3, 1, 1, 0, 0, 0, 0, 0, 2, 1)	10
(3, 0, 0, 0, 0, 2, 2, 1, 0, 0)	(4, 1, 1, 1, 1, 0, 0, 1, 0, 3)	11
(4, 0, 0, 2, 2, 1, 1, 1, −1, 2)	(3, 1, 1, 0, 0, 0, 0, 0, 2, 1)	12
(5, 3, 3, 1, 1, 0, 0, 1, 0, 2)	(3, 0, 0, 0, 0, 1, 1, 2, 1, 0)	13
(4, 0, 0, 2, 2, 1, 1, 1, −1, 2)	(4, 2, 2, 1, 1, 0, 0, 0, 2, 0)	14
(6, 2, 2, 2, 2, 3, 3, 0, −1, 1)	(3, 1, 1, 0, 0, 0, 0, 2, 1, 0)	15
(4, 0, 0, 2, 2, 1, 1, 1, −1, 2)	(4, 2, 2, 0, 0, 1, 1, 0, 2, 0)	16
(13, 2, 2, 6, 6, 4, 4, 7, 2, 2)	(2, 0, 0, 0, 0, 0, 0, 1, 0, 1)	17
(5, 3, 3, 1, 1, 0, 0, 2, 0, 1)	(4, 0, 0, 1, 1, 2, 2, 0, 2, 0)	18

$\sigma(D_3)$	$\rho(D_3)$	$(\sigma(D_3), \rho(D_3))$
(2, 1, 1, 0, 0, 0, 0, 0, 0, 0)	(2, 0, 0, 1, 1, 0, 0, 0, 0, 0)	4
(3, 1, 1, 0, 0, 0, 0, 1, 0, 2)	(3, 1, 1, 0, 0, 0, 0, 2, 1, 0)	5
(3, 1, 1, 0, 0, 1, 1, 0, 0, 2, 0, 1)	(3, 1, 1, 0, 0, 0, 0, 0, 2, 1)	6
(3, 0, 0, 1, 1, 0, 0, 1, 0, 2)	(3, 1, 1, 0, 0, 0, 0, 2, 1, 0)	7
(3, 0, 0, 1, 1, 0, 0, 2, 0, 1)	(3, 1, 1, 0, 0, 0, 0, 0, 2, 1)	8
(5, 0, 0, 2, 2, 1, 1, 3, 0, 2)	(3, 0, 0, 0, 0, 1, 1, 0, 1, 2)	9
(3, 1, 1, 0, 0, 0, 0, 2, 0, 1)	(4, 1, 1, 0, 0, 2, 2, 0, 2, 0)	10
(5, 0, 0, 2, 2, 1, 1, 3, 0, 2)	(3, 1, 1, 0, 0, 0, 0, 0, 1, 2)	11
(3, 1, 1, 0, 0, 0, 0, 2, 0, 1)	(4, 0, 0, 1, 1, 2, 2, 0, 2, 0)	12
(5, 1, 1, 2, 2, 2, 3, 0, 0, 0)	(4, 0, 0, 1, 1, 0, 1, 1, 1, 3)	13
(5, 0, 0, 2, 2, 1, 1, 2, 0, 3)	(4, 2, 2, 0, 0, 1, 1, 2, 0, 0)	14
(5, 2, 2, 1, 1, 2, 3, 0, 0, 0)	(4, 0, 0, 1, 1, 0, 1, 1, 1, 3)	15
(5, 0, 0, 2, 2, 1, 1, 2, 0, 3)	(4, 1, 1, 0, 0, 2, 2, 0, 2, 0)	16
(11, 3, 3, 5, 5, 3, 4, 1, 0, 5)	(2, 0, 0, 0, 0, 0, 1, 1, 0, 0)	17
(5, 0, 0, 2, 2, 1, 1, 2, 0, 3)	(4, 2, 2, 0, 0, 1, 1, 0, 2, 0)	18
(13, 4, 4, 6, 6, 5, 5, 3, 2, 0)	(3, 0, 0, 1, 1, 0, 0, 2, 1, 0)	19

This concludes the proof of theorem 9.

We can now state the following theorem, which is a straightforward consequence of theorems 6, 8 and 9.

THEOREM 10. (*i*) *For every $r \geq 5$ there exists an embedding of S as a nonsingular surface F^{2r-3} of degree $2r-3$ in $\mathbf{P}^r$, and for every (d, g) such that*

$$0 \leq g \leq (d-r)^2/2(2r-3)$$

there exists a nonsingular irreducible and nondegenerate curve X of degree d and genus g on F^{2r-3}.

(ii) *For every $r \geq 7$ there exists an embedding of S as a nonsingular surface F^{2r-4} of degree $2r-4$ in $\mathbf{P}^r$, and for every (d, g) such that*

$$0 \leq g \leq (d-r)^2/2(2r-4)$$

there exists a nonsingular irreducible and nondegenerate curve X of degree d and genus g on F^{2r-4}.

Note that theorem 10 differs from our main theorem, as stated in the introduction, only in that the surface S appears instead of S'. In the next section we will show how to deduce the main theorem from theorem 10.

Remark 2. Arguing as in proposition 5 it is easy to see that if there exists a δ-tuple $D_0, \ldots, D_{\delta-1}$ of classes of Pic (S) satisfying conditions I), II), III) of the definition of δ-system, then S can be embedded in $\mathbf{P}^r$ as a smooth linearly normal surface of degree $2r-\delta$ for all $r \geq \delta - 1$. The following is a list of such δ-tuples for $5 \leq \delta \leq 9$:

$\delta = 5$:
$D_0 = (0, -1, -1, -1, 0, 0, 0, 0, 0, 0)$ $D_1 = (1, 1, -1, -1, 0, 0, 0, 0, 0, 0)$
$D_2 = (1, -1, -1, 0, 0, 0, 0, 0, 0, 0)$ $D_3 = (2, 1, 1, -1, 0, 0, 0, 0, 0, 0)$
$D_4 = (2, 1, 0, 0, 0, 0, 0, 0, 0, 0)$

$\delta = 6$:
$D_0 = (0, -1, -1, -1, -1, -1, -1, 0, 0, 0)$ $D_1 = (1, 1, -1, -1, -1, -1, 0, 0, 0, 0)$
$D_2 = (1, -1, -1, -1, -1, 0, 0, 0, 0, 0)$ $D_3 = (2, 1, 1, -1, -1, 0, 0, 0, 0, 0)$
$D_4 = (2, 1, -1, 0, 0, 0, 0, 0, 0, 0)$ $D_5 = (3, 2, 1, 0, 0, 0, 0, 0, 0, 0)$

$\delta = 7$:
$D_0 = (0, -1, -1, -1, -1, -1, -1, -1, 0, 0)$ $D_1 = (1, 1, -1, -1, -1, -1, -1, 0, 0, 0)$
$D_2 = (1, -1, -1, -1, -1, 0, 0, 0, 0, 0)$ $D_3 = (2, 1, 1, -1, -1, -1, 0, 0, 0, 0)$
$D_4 = (2, 1, -1, -1, 0, 0, 0, 0, 0, 0)$ $D_5 = (3, 2, 1, -1, 0, 0, 0, 0, 0, 0)$
$D_6 = (3, 2, 0, 0, 0, 0, 0, 0, 0, 0)$

$\delta = 8$:
$D_0 = (0, -1, -1, -1, -1, -1, -1, -1, -1, 0)$ $D_1 = (1, 1, -1, -1, -1, -1, -1, -1, 0, 0)$
$D_2 = (1, -1, -1, -1, -1, -1, 0, 0, 0, 0)$ $D_3 = (2, 1, 1, -1, -1, -1, -1, 0, 0, 0)$
$D_4 = (2, 1, -1, -1, -1, 0, 0, 0, 0, 0)$ $D_5 = (3, 2, 1, -1, -1, 0, 0, 0, 0, 0)$
$D_6 = (3, 2, -1, 0, 0, 0, 0, 0, 0, 0)$ $D_7 = (4, 3, 1, 0, 0, 0, 0, 0, 0, 0)$

$\delta = 9$:
$D_0 = (0, -1, -1, -1, -1, -1, -1, -1, -1, -1)$; $D_1 = (1, 1, -1, -1, -1, -1, -1, -1, -1, 0)$
$D_2 = (1, -1, -1, -1, -1, -1, -1, 0, 0, 0)$ $D_3 = (2, 1, 1, -1, -1, -1, -1, -1, -1, 0, 0)$
$D_4 = (2, 1, -1, -1, -1, -1, 0, 0, 0, 0)$ $D_5 = (3, 2, 1, -1, -1, -1, 0, 0, 0, 0)$
$D_6 = (3, 2, -1, -1, 0, 0, 0, 0, 0, 0)$ $D_7 = (4, 3, 1, -1, 0, 0, 0, 0, 0, 0)$
$D_8 = (4, 3, 0, 0, 0, 0, 0, 0, 0, 0)$

On the other hand it is clear that there are no such δ-tuples for $\delta \geq 10$: indeed, letting $D = |D_0 - \omega_S|$, $\varphi_D(S)$ is a smooth surface of degree δ in $\mathbf{p}^\delta$ with elliptic hyperplane sections.

4. Remarks

1) One of our main technical tools has been proposition 1, whose proof uses very strongly the geometrical properties of the surface S, particularly the fact that

the points $P_1, \ldots, P_9$ are not in general position, but are base points of a generic pencil of cubics. It is pretty clear that proposition 1 cannot be generalized to a surface S' obtained by blowing up 9 points of $\mathbf{P}^2$ in general position. Nevertheless it is not difficult to see that our other main results generalize to S'. This can be done in the following way.

Let $P_1, \ldots, P_9 \in \mathbf{P}^2$ be the points that define S, and let M be a general line through P_9. In $\mathbf{P}^2 \times M$ denote by Γ the diagonal curve, whose support is $\{(p, p): p \in M\}$. Let $\mathbf{S}$ be the blow-up of $\mathbf{P}^2 \times M$ along $P_1 \times M \cup \ldots \cup P_8 \times M \cup \Gamma$, and let $q: \mathbf{S} \to \mathbf{P}^2 \times M$ be the projection, and $\pi: \mathbf{S} \to M$ be the composition of q with the second projection $\mathbf{P}^2 \times M \to M$. Clearly π is a smooth family of projective surfaces, whose fibre over a point $p \in M$ is the surface $\mathbf{S}(p)$ obtained from $\mathbf{P}^2$ after blowing up $P_1, \ldots, P_8$ and p. In particular $\mathbf{S}(P_9) = S$. Note that for all p in some open neighborhood of P_9 the points $P_1, \ldots, P_8, p$ are contained in a unique cubic curve C_p which is nonsingular, hence they are in general position.

In Pic ($\mathbf{S}$) consider the classes

$$\mathbf{o}(\mathbf{H}), \mathbf{o}(-\mathbf{E}_1), \ldots, \mathbf{o}(-\mathbf{E}_8), \mathbf{o}(-\mathbf{E}_p),$$

where $\mathbf{H} = q^*(\mathbf{o}(1))$, and $\mathbf{E}_1, \ldots, \mathbf{E}_8, \mathbf{E}_p$ are the exceptional surfaces coming from the curves $P_1 \times M, \ldots, P_8 \times M, \Gamma$ respectively. It is clear that every element $\mathbf{o}(D)$ of Pic (S), being a linear combination of $\mathbf{o}(H)$, $\mathbf{o}(-E_1), \ldots, \mathbf{o}(-E_9)$, extends to an element $\mathbf{o}(\mathbf{D}) \in$ Pic ($\mathbf{S}$) which is the corresponding combination of the above classes; hence, by restriction, it defines a divisor class $\mathbf{o}(D_p) \in$ Pic ($\mathbf{S}(p)$) for all $p \in M$.

Suppose that $\mathbf{o}(D) \in$ Pic (S) is such that

$$(*) \quad h^1(S, \mathbf{o}(D)) = 0 = h^2(S, \mathbf{o}(D)),$$

From the upper-semicontinuity principle it follows that there is an open neighborhood U_D of P_9 in M such that for all $p \in U_D$

$$h^1(S, \mathbf{o}(D_p)) = 0 = h^2(S, \mathbf{o}(D_p)).$$

If moreover the linear system $|D|$ has no base points and contains an irreducible and nonsingular curve, then, after possibly shrinking U_D, the same is true of $|D_p|$ for all $p \in U_D$. Indeed the base point freeness of $|D|$ implies that the natural map

$$\pi^*[\pi_* \mathbf{o}(\mathbf{D})] \to \mathbf{o}(\mathbf{D})$$

is surjective in a neighborhood of $\mathbf{S}(P_9) = S$. From the base change properties it then follows that $|D_p|$, is base point free for all p in that neighborhood. Condition

(*) implies that $\pi_*\mathbf{o}(\mathbf{D})$ is locally free of rank $h^0(S, \mathbf{o}(D))$ on some open set U containing P_9. As a consequence we have that, if $X \in |D|$ is a general element, it can be extended to a relative effective Cartier divisor $\mathbf{X}$ on $\pi^{-1}(U)$. And if X is a nonsingular curve, then it follows from the flatness of $\mathbf{X}$ over U that the restriction X_p of $\mathbf{X}$ to $\mathbf{S}(p)$ is also a nonsingular curve for all p in some open set $V \subset U$ containing P_9.

Suppose in addition that $\mathbf{o}(D)$ is very ample; then it is easy to show that, after possibly shrinking U_D, $\mathbf{o}(D_p)$ is very ample for all $p \in U_D$. Indeed, on $\pi^{-1}(U_D)$ the natural map $\pi^*\pi_*\mathbf{o}(\mathbf{D}) \to \mathbf{o}(\mathbf{D})$ is surjective, hence it defines a U_D-morphism

$$\varphi : \pi^{-1}(U_D) \to \mathbf{P}(\pi_*\mathbf{o}(\mathbf{D})) =: \mathbf{P}$$

which restricts on every fibre $\mathbf{S}(p)$, $p \in U_D$, to the morphism

$$\varphi_p : \mathbf{S}(p) \to \mathbf{P}(H^0(\mathbf{S}(p), \mathbf{o}(D_p))\check{\ })$$

defined by the linear system $|D_p|$. For $p = P_9$ this is a closed embedding, because $\mathbf{o}(D)$ is very ample; hence there is an open $V \subset U_D$ such that the restriction of φ to $\pi^{-1}(V)$ is finite and such that $\mathbf{o_P} \to \varphi_*\mathbf{o_S}$ is an isomorphism; equivalently φ is a closed embedding of $\pi^{-1}(V)$ in $\mathbf{P}$ and this means that $\mathbf{o}(D_p)$ is very ample for all $p \in V$.

These remarks can be applied to $D = D_i - \alpha\omega_S$ to conclude that propositions 4 and 5 generalize to S' with no changes. As a consequence of this we have that theorem 6 is still true if we replace S by S'. Clearly lemma 7 extends to S', and the proofs of theorems 8 and 9 extend word by word to S'. Consequently theorem 10 also extends. In particular *the main theorem, as stated in the introduction, is true.*

2) Suppose that D is a 1-connected effective divisor on a projective nonsingular surface F such that $h^1(F, \mathbf{o}_F) = 0$, and let H be a divisor on F such that $|D + H|$ contains an irreducible nonsingular curve C. From the exact sequence

$$0 \to \mathbf{o}_F(-D) \to \mathbf{o}_F(H) \to \mathbf{o}_C(H) \to 0$$

and from the Ramanujam' vanishing theorem (see section 1) it follows that

$$H^0(F, \mathbf{o}_F(H)) \cong H^0(C, \mathbf{o}_C(H)).$$

We apply this remark to the surface S, equipped with a δ-system (e.g. a 3-system or a 4-system), and we take H to be one of the very ample divisors H_r and C' any of the curves X of degree d and genus g such that $0 \le g \le d - r - 1$, as described in proposition 5. It follows that the curves X of degree d and genus g constructed in theorem 6 satisfy $h^0(X, \mathbf{o}_X(H)) = r + 1$ (i.e. are "linearly normal") if

$$d - r \le g \le (d - r)^2/2(2r - \delta).$$

In particular this applies to the curves of theorem 10 and, by upper-semicontinuity, to those of the main theorem which satisfy the corresponding inequalities for $\delta = 3,4$.

3) Let $C \subset \mathbf{P}^r$ be a nonsingular irreducible and nondegenerate curve of degree n, $\mathbf{o}(H_C)$ the hyperplane section line bundle and ω_C the canonical bundle. Assume that $h^0(C, \mathbf{o}(H_C)) = r + 1$. The natural map

$$\mu_0(C): H^0(C, \mathbf{o}(H_C)) \otimes H^0(C, \omega_C(-H_C)) \to H^0(C, \omega_C)$$

is called the *Brill-Noether map* of $C \subset \mathbf{P}^r$.
The map $\mu_0(C)$ is relevant to the study of the scheme $W^r_n(C)$ of linear series of degree n and dimension at least r on C. In particular, the surjectivity of $\mu_0(C)$ is equivalent to the fact that $\mathbf{o}(H_C)$ is an isolated point of $W^r_n(C)$ with reduced structure. We will check this last property on some of the curves we have constructed.

Again suppose that the surface S is equipped with a δ-system, and let $F \subset \mathbf{P}^r$ be the nonsingular embedding of degree $2r - \delta$ of S given by theorem 6.

Let $X \subset F$ be a nonsingular nondegenerate curve of degree d and genus g as constructed in the proof of theorem 6. Assume that $g \ge 2d - 4r + \delta$ (with the notation of theorem 6, this means that (d, g) lies above the line L_1). Then it follows from the proof of theorem 6 that $X \in |C' + mH_r|$, for some $m \ge 2$ and for some irreducible, nonsingular and nondegenerate C' of degree d' and genus g' such that $0 \le g' \le d' - r - 1$. We will write $H_r = H$.

Since, by remark 2) above, $h^0(X, \mathbf{o}_C(H)) = r + 1$, we can consider the Brill-Noether map of $X \subset \mathbf{P}^r$. *We claim that $\mu_0(X)$ is surjective.* Indeed consider the following commutative diagram:

$$\begin{array}{ccc}
\mu: H^0(F, \mathbf{o}_F(H)) \otimes H^0(F, \omega_F(X - H)) & \to & H^0(F, \omega_F(X)) \\
\downarrow & & \downarrow q \\
\mu_0(X): H^0(X, \mathbf{o}_X(H_X)) \otimes H^0(X, \omega_X(-H_X)) & \to & H^0(X, \omega_X).
\end{array}$$

Since q is surjective, it suffices to show that μ is surjective, and for this purpose it is enough to show that the sheaf $\omega_F(X - H)$ is 0-regular with respect to

$\mathbf{o}(H)$ (see [M]). This amounts to check that:

$$H^1(F, \omega_F(X-2H)) = (0)$$

and

$$H^2(F, \omega_F(X-3H)) = (0).$$

The first condition follows from the vanishing theorem because $|X-2H| = |C' + (m-2)H|$ contains a 1-connected divisor. The second condition is equivalent to

$$H^0(F, \mathbf{o}_F(3H-X)) = (0),$$

which is true because

$$|3H-X| = |(3-m)H - C'|,$$

and this is clearly empty if $m \geq 3$, and likewise empty for $m = 2$ because C' is nondegenerate.

Of course this remark applies to the curves of theorem 10, taking $\delta = 3$ or 4, and, by upper-semicontinuity, it extends to the curves of the main theorem.

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