DEFORMATIONS OF SCHEMES

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Preface

In some sense deformation theory is as old as algebraic geometry itself: this is because all algebro-geometric objects can be "deformed" by suitably varying the coefficients of their defining equations, and this has of course always been known by the classical geometers. Nevertheless a correct understanding of what "deforming" means leads into the technically most difficult parts of our discipline. It is fair to say that such technical obstacles have had a vast impact on the crisis of the classical language and on the development of the modern one, based on the theory of schemes and on cohomological methods.

The modern point of view originates from the seminal work of Kodaira and Spencer on local deformations of complex analytic manifolds and from its formalization and translation into the language of schemes given by Grothendieck. Here I will not recount the history of the subject since good surveys already exist (e.g. Seshadri(1975), Catanese(1988), Palamodov(1990)). Today, while this area is rapidly developing, a comprehensible text containing the basic results of what we can call "classical deformation theory" seems to be missing. Moreover a number of technicalities and of "well known" facts are scattered in a vast literature as folklore, sometimes with proofs available only in the complex analytic category. This book is an attempt to fill such a gap, at least partially. More precisely it aims at giving an account with complete proofs of the results and techniques which are needed to understand the local deformation theory of algebraic schemes over an algebraically closed field, thus providing the tools needed for example in the local study of Hilbert schemes and moduli. The existing monographs, like Artin(1976), Illusie(1971), Kollar(1996), Sernesi(1986), all aim at goals different from the above.

For these reasons my approach has been to work exclusively in the category of locally noetherian schemes over a fixed algebraically closed field **k**, rather than switching back and forth between the algebraic and the analytic category. I tried to make the text self contained as much as possible, but without forgetting that all the technical ideas and prerequisites come from [SGA1] and [FGA]: therefore the reader is advised to read this text keeping a copy of them on his table. In any case a good familiarity with Hartshorne(1977) and with a standard text in commutative algebra like Eisenbud(1995) or Matsumura(1986) will be generally sufficient; the classical Serre(2000) and Zariski-Samuel(1960) will be also useful. A good acquaintance with homological algebra is assumed throughout.

One of the difficulties of the subject is that it needs a great deal of technical results, which make it hard to maintain a proper balance between generality and understandability. In order to overcome this problem I tried to keep the technicalities to a minimum, and I introduced the main deformation problems in an elementary fashion in Chapter II; they are then reconsidered as functors of Artin rings in Chapter III, where the main results of formal deformation theory are proved. In particular I did not treat cotangent complexes and functors, nor the approach to deformations via differential graded Lie algebras. Although the central issue of the book is Grothendieck's formal deformation theory, I considered necessary to include a Chapter on Hilbert schemes and Quot schemes, since it would be impossible to give examples and meaningful applications without them, and because of the lack of an appropriate reference. Deformation theory is closely tied with classical algebraic geometry because some of the issues which had remained controversial and unclear in the old language have found a natural explanation using the methods discussed here. I have included a Section on the theory of Severi varieties of plane singular curves which gives a good illustration of this point.

Unfortunately many other important topics and results have been omitted because of lack of space, energy and competence. In particular I did not include the construction of any global moduli spaces/stacks, which would have taken me too far from the main theme.

The book is organized in the following way. Chapter I gives a concise treatment of the algebraic tools which are fundamental in deformation theory: algebra extensions and their scheme-theoretic version, formal smoothness and obstruction theory. Chapter II is an introduction to the most important deformation problems. The treatment is concrete and elementary, and many examples are given. Chapter III deals with the theory of "functors of Artin rings", the abstract tool for the study of formal deformation theory. The main result of this theory is Schlessinger's Theorem. We then reconsider the deformation problems of Chapter II as defining functors of Artin rings and we apply Schlessinger's Theorem to them, thus deriving the existence of formal (semi)universal deformations. We also discuss the delicate relation between prorepresentability and automorphisms, and between formal and algebraic deformations. Chapter IV is devoted to the construction and general properties of Hilbert schemes and Quot schemes. It ends with a Section on Severi varieties, whose existence is proved in the case of nodal curves. In an Appendix we have collected several topics which are well known and standard but we felt it would be convenient for the reader to have them available here.

Most Sections end with Notes which give bibliographical information, propose exercises and sometimes report on further developments. The bibliography at the end is certainly not complete, but it also includes references not quoted in the text and related with deformation theory.

I am aware of several weaknesses of the present version: I am working at the next one. In the meantime I will be very grateful to all those who, with their comments and suggestions, will help me to improve it. I will appreciate of being informed of any errors (`sernesi@matrm3.mat.uniroma3.it`).