Terminology and notation

All rings will be commutative with 1. A ring homomorphism $A \to B$ is called *essentially of finite type* (e.f.t.) if B is a localization of an A-algebra of finite type. We will also say that B is e.f.t. over A.

We will always denote by $\mathbf{k}$ a fixed algebraically closed field. All schemes will be assumed to be defined over $\mathbf{k}$ and locally noetherian and all sheaves will be quasi coherent unless otherwise specified. If X and Y are schemes we will write $X \times Y$ instead of $X \times_{\mathbf{k}} Y$. If S is a scheme and $s \in S$ we denote by $\mathbf{k}(s) = \mathcal{O}_{S,s}/m_{S,s}$ the residue field of S at s.

For all definitions not explicitly given we will refer to Hartshorne(1977).

Notation

As customary various categories will be denoted by indicating their objects within round parentheses when it will be clear what the morphisms in the category are. For instance (sets), (A-modules), etc. The class of objects of a category $\mathcal{C}$ will be denoted ob($\mathcal{C}$).

We will consider the following categories of $\mathbf{k}$-algebras:

$\mathcal{A}$	=	the category of local artinian $\mathbf{k}$-algebras with residue field $\mathbf{k}$
$\hat{\mathcal{A}}$	=	the category of complete local noetherian $\mathbf{k}$-algebras with residue field $\mathbf{k}$
$\mathcal{A}^*$	=	the category of local noetherian $\mathbf{k}$-algebras with residue field $\mathbf{k}$
($\mathbf{k}$-algebras)	=	the category of noetherian $\mathbf{k}$-algebras

Morphisms are unitary $\mathbf{k}$-homomorphisms, which are local in $\mathcal{A}, \hat{\mathcal{A}}$ and $\mathcal{A}^*$. For a given Λ in ob($\mathcal{A}^*$) we will consider the following:

$\mathcal{A}_\Lambda$	=	the category of local artinian Λ-algebras with residue field $\mathbf{k}$
$\mathcal{A}^*_\Lambda$	=	the category of local noetherian Λ-algebras with residue field $\mathbf{k}$

They are full subcategories of $\mathcal{A}$ and $\mathcal{A}^*$ respectively. If Λ is in $\hat{\mathcal{A}}$ then we will let

$\hat{\mathcal{A}}_\Lambda$ = the category of complete local noetherian Λ-algebras with residue field $\mathbf{k}$

which is a full subcategory of $\hat{\mathcal{A}}$. Moreover we will set:

$$\text{(schemes)} \quad = \quad \text{the category of schemes}$$

(i.e. of locally noetherian **k**-schemes) and

$$\text{(algschemes)} \quad = \quad \text{the category of algebraic schemes}$$

For a given scheme Z we set

$$\begin{array}{rcl} (\text{schemes}/Z) & = & \text{the category of } Z\text{-schemes} \\ (\text{algschemes}/Z) & = & \text{the category of algebraic } Z\text{-schemes} \end{array}$$

$h^i(X, \mathcal{F})$ denotes $\dim[H^i(X, \mathcal{F})]$ where $\mathcal{F}$ is a coherent sheaf on the complete scheme X. When no confusion is possible we will sometimes write $H^i(\mathcal{F})$ and $h^i(\mathcal{F})$ instead of $H^i(X, \mathcal{F})$ and $h^i(X, \mathcal{F})$ respectively.
$\coprod_i X_i$ denotes the disjoint union of the schemes X_i.

Introduction

> *La méthode générale consiste toujours à faire des constructions* formelles, *ce qui consiste essentiellement à faire de la* géométrie algébrique sur un anneau artinien, *et à en tirer des conclusions de nature "algébrique" en utilisant les trois théorèmes fondamentaux* (Grothendieck(1959), p. 11).

Deformation theory is a formalization of the Kodaira, Nirenberg, Spencer, Kuranishi (KNSK) approach to the study of small deformations of complex manifolds. Its main ideas are clearly outlined in the series of Bourbaki seminar exposes by Grothendieck which go under the name of "Fondements de la Géométrie Algébrique" (FGA); in particular they are explained in detail in Grothendieck(1960a) (see especially page 17), while the technical foundations are lied in Grothendieck(1959). The quotation at the top of this page gives a concise description of the method employed.

The first step of this formalization consists in studying infinitesimal deformations, and this is accomplished via the notion of "functor of Artin rings"; the study of such functors leads to the construction of "formal deformations". This method enhances the analogies between the analytic and the algebraic cases, and at the same time hides some delicate phenomena typical of the algebraic geometrical world. These phenomena become visible when one tries to pass from formal to algebraic deformations. The techniques of deformation theory have a variety of applications which make them an extremely useful tool, especially in understanding the local structure of schemes defined by geometrical conditions or by functorial constructions.

In this introduction we shall explain in outline the logical structure of deformation theory; for this purpose we will start by outlining the KNSK theory of small deformations of compact complex manifolds.

Given a compact complex manifold X, a *family of deformations* of X is a commutative diagram of holomorphic maps between complex manifolds

$$\begin{array}{cccc} & X & \subset & \mathcal{X} \\ \xi: & \downarrow & & \downarrow \pi \\ & \star & \xrightarrow{t_o} & B \end{array}$$

with π proper and smooth (i.e. with everywhere surjective differential), B connected and where $\star$ denotes the singleton space. We denote by $\mathcal{X}_t$ the fibre $\pi^{-1}(t)$, $t \in B$. It is a standard fact that, locally on B, $\mathcal{X}$ is differentiably a product so that π can be viewed as a family of complex structures on the differentiable manifold X_{diff}. The family ξ is *locally trivial* at t_o if there is a neighborhood $U \subset B$ of t_o such that we have $\pi^{-1}(U) \cong X \times U$ analytically.

Kodaira and Spencer started by defining, for every tangent vector $\frac{\partial}{\partial t} \in T_{t_o}B$ *the derivative of the family π along $\frac{\partial}{\partial t}$* as an element

$$\frac{\partial \mathcal{X}_t}{\partial t} \in H^1(X, T_X)$$

thus giving a linear map

$$\kappa : T_{t_o}B \to H^1(X, T_X)$$

called the *Kodaira-Spencer map* of the family π. They showed that if π is locally trivial at t_o then $\kappa(\frac{\partial}{\partial t}) = 0$ for all $\frac{\partial}{\partial t} \in T_{t_o}B$. Then they investigated the problem of classifying all small deformations of X, by constructing a "complete family" of deformations of X. A family ξ as above is called *complete* if for every other family of deformations of X:

$$\begin{array}{cccc} & X & \subset & \mathcal{Y} \\ \eta: & \downarrow & & \downarrow p \\ & \star & \xrightarrow{m_o} & M \end{array}$$

there is an open neighborhood $V \subset M$ and a commutative diagram

$$\begin{array}{ccccc} & & X & & \\ & \swarrow & & \searrow & \\ p^{-1}(V) & & \to & & \mathcal{X} \\ \downarrow & & & & \downarrow \\ V & & \to & & B \end{array}$$

inducing an isomorphism $p^{-1}(V) \cong V \times_B \mathcal{X}$. The family is called *universal* if it is complete and moreover the morphism $V \to B$ is unique locally around m_o for each family η as above. Kodaira and Spencer proved that if κ is surjective then the family ξ is complete. The following existence result was then proved:

THEOREM (Kodaira-Nirenberg-Spencer(1958)) *If $H^2(X, T_X) = 0$ then there exists a complete family of deformations of X whose Kodaira-Spencer map is an isomorphism. If moreover $H^0(X, T_X) = 0$ then such complete family is universal.*

Later Kuranishi generalized this result by showing that a complete family of deformations of X such that κ is an isomorphism exists without assumptions on $H^2(X, T_X)$ provided the base B is allowed to be an analytic space (Kuranishi(1964)).

We want to rephrase everything algebraically as far as possible. Let's fix an algebraically closed field **k** and consider an algebraic **k**-scheme X. A *local deformation*, or a *local family of deformations* of X is a cartesian diagram

$$\xi: \begin{array}{ccc} X & \to & \mathcal{X} \\ \downarrow & & \downarrow \pi \\ \mathrm{Spec}(\mathbf{k}) & \subset & S \end{array}$$

where π is a flat morphism, $S = \mathrm{Spec}(A)$ where A is a local **k**-algebra with residue field **k**, and X is identified with the fibre over the closed point. If X is nonsingular and/or projective we will require π to be smooth and/or projective. We say that ξ is *a deformation over* $\mathrm{Spec}(A)$ or over A. If in particular A is an artinian local **k**-algebra then we speak of an *infinitesimal deformation*.

The notion of local family has the fundamental property of being *funtorial*. Given two infinitesimal deformations of X:

$$\xi: \begin{array}{ccc} X & \to & \mathcal{X} \\ \downarrow & & \downarrow \pi \\ \mathrm{Spec}(k) & \subset & \mathrm{Spec}(A) \end{array} \quad \text{and} \quad \eta: \begin{array}{ccc} X & \to & \mathcal{Y} \\ \downarrow & & \downarrow \rho \\ \mathrm{Spec}(\mathbf{k}) & \subset & \mathrm{Spec}(A) \end{array}$$

parametrized by the same $\mathrm{Spec}(A)$, an isomorphism $\xi \cong \eta$ is defined to be a morphism $f : \mathcal{X} \to \mathcal{Y}$ of schemes over $\mathrm{Spec}(A)$ inducing the identity on the closed fibre, i.e. such that the following diagram

$$\begin{array}{ccccc} & & X & & \\ & \swarrow & & \searrow & \\ \mathcal{X} & & \xrightarrow{f} & & \mathcal{Y} \\ & \searrow & & \swarrow & \\ & & \mathrm{Spec}(A) & & \end{array}$$

is commutative. Consider the category

$$\mathcal{A}^* = (\text{noetherian local } \mathbf{k}\text{-algebras with residue field } \mathbf{k})$$

and its full subcategory

$$\mathcal{A} = (\text{artinian local } \mathbf{k}\text{-algebras with residue field } \mathbf{k})$$

One defines a covariant functor

$$\mathrm{Def}_X : \mathcal{A}^* \to (\text{sets})$$

by

$$\mathrm{Def}_X(A) = \{\text{local deformations of } X \text{ over Spec(A)}\}/(\text{isomorphism})$$

This is *the functor of local deformations* of X; its restriction to $\mathcal{A}$ is *the functor of infinitesimal deformations* of X. One may now ask whether Def_X is representable, namely if there is a noetherian local $\mathbf{k}$-algebra $\mathcal{O}$ and a local deformation

$$\upsilon : \begin{array}{ccc} X & \to & \mathcal{X}^\circ \\ \downarrow & & \downarrow p \\ \mathrm{Spec}(\mathbf{k}) & \subset & \mathrm{Spec}(\mathcal{O}) \end{array}$$

which is universal, i.e. such that any other local deformation ξ is obtained by pulling back υ under a unique $\mathrm{Spec}(A) \to \mathrm{Spec}(\mathcal{O})$.

The approach of Grothendieck to this problem was to formalize the method of Kodaira and Spencer, which consists in a formal construction followed by a proof of convergence. In the search for the universal deformation υ the formal construction corresponds to the construction of the sequence of its restrictions to the truncations $\mathrm{Spec}(\mathcal{O}/m_{\mathcal{O}}^{n+1})$:

$$u_n : \begin{array}{ccc} X & \to & \mathcal{X}_n^\circ \\ \downarrow & & \downarrow \\ \mathrm{Spec}(\mathbf{k}) & \to & \mathrm{Spec}(\mathcal{O}/m_{\mathcal{O}}^{n+1}) \end{array} \qquad n \geq 0$$

These are infinitesimal deformations of X because the rings $\mathcal{O}/m_{\mathcal{O}}^{n+1}$ are in $\mathcal{A}$. The sequence $\hat{u} = \{u_n\}$ can be considered as a formal approximation of υ. It is a special case of a formal deformation: more precisely, a *formal deformation* of X is given by a complete local $\mathbf{k}$-algebra R with residue field $\mathbf{k}$ and by a sequence of infinitesimal deformations

$$\xi_n : \begin{array}{ccc} X & \to & \mathcal{X}_n \\ \downarrow & & \downarrow \\ \mathrm{Spec}(\mathbf{k}) & \to & \mathrm{Spec}(R/m_R^{n+1}) \end{array} \qquad n \geq 0$$

such that $\xi_n \mapsto \xi_{n-1}$ under the truncation $R/m_R^{n+1} \to R/m_R^n$. In our case $R = \hat{\mathcal{O}}$. The goal of the formal step in deformation theory is the construction of $\hat{u}$ for a given X, i.e. of a formal deformation having a suitable universal property which is inherited from the corresponding property of υ, and which we do not need to specify now.

Observe that in trying to perform the formal step we will at best succeed in describing $\hat{\mathcal{O}}$ and not $\mathcal{O}$. Since a formal deformation consists of infinitesimal deformations, for the construction of $\hat{u}$ we will only need to work with the covariant functor

$$\mathrm{Def}_X : \mathcal{A} \to (\text{sets})$$

A covariant functor $F : \mathcal{A} \to (\text{sets})$ is called a *functor of Artin rings*. To every complete local $\mathbf{k}$-algebra R we can associate a functor of Artin rings h_R by

$$h_R(A) = \mathrm{Hom}_{\mathcal{A}}(R, A)$$

A functor of this form is called *prorepresentable*. By categorical general nonsense one shows that a formal deformation $\hat{\xi}$ defines a morphism of functors (a natural transformation) $h_R \to \mathrm{Def}_X$ and that this morphism is an isomorphism precisely when $\hat{\xi}$ is universal. Therefore we see that the search for $\hat{u}$ is a problem of prorepresentability of Def_X. More generally, to every local deformation problem there corresponds a functor of Artin rings F analogous to Def_X; the task of constructing a formal universal deformation for the given problem consists in showing that F is prorepresentable, producing the ring R prorepresenting F and the formal universal deformation defining the isomorphism $h_R \to F$. This is the scheme of approach to the formal part of every local deformation problem as it was outlined by Grothendieck. What one needs is to find criteria for the prorepresentability of a functor of Artin rings; we will also need to consider properties weaker than prorepresentability satisfied by more general classes of functors coming from interesting deformation theoretic problems. Necessary and sufficient conditions of prorepresentability are given by Schlessinger's Theorem.

After having solved the problem of existence of a formal universal deformation (by means of necessary and sufficient conditions for its existence) one still has to decide whether $\mathcal{O}$ and v exist and to find them. To pass from $\hat{\mathcal{O}}$ to $\mathcal{O}$ is the analogous of the convergence step in the Kodaira-Spencer theory, and it is a very difficult problem which has no solution in general. The search for $\mathcal{O}$ is the *algebraization problem*. Under reasonably general assumptions one shows that there exists a deformation v over an *algebraic local ring* (i.e. the henselization of a local **k**-algebra essentially of finite type) which does not quite represent the functor Def_X but at least has a universal associated formal deformation. The further property of representing Def_X is not in general satisfied by $(\mathcal{O}, v)$, being related with the existence of nontrivial automorphisms of X. This part of the theory is largely due to the work of M. Artin, and based on the notions of *effectivity* of a formal deformation and of *local finite presentation* of a functor, already introduced by Grothendieck. The main technical tool is Artin' approximation theorem.

Chapter I. Technical tools

In this preliminary Chapter we introduce some of the tools, mostly algebraic, which are necessary for the study of deformation theory. They consist essentially in the infinitesimal techniques needed for analizing smooth and etale morphisms. We will refer to Hartshorne(1977), Eisenbud(1995) and to the Appendix for other basic notions used in the book.

I.1. EXTENSIONS

Let $A \to R$ be a ring homomorphism. An A-*extension* of R (or of R by I) is an exact sequence

$$(R', \varphi) \qquad 0 \to I \to R' \xrightarrow{\varphi} R \to 0$$

where R' is an A-algebra and φ is a homomorphism of A-algebras whose kernel I is an ideal of R' satisfying $I^2 = (0)$. This condition implies that I has a structure of R-module. (R', φ) is also called *an extension of A-algebras.*

If (R', φ) and (R'', ψ) are A-extensions of R by I, an A-homomorphism $\xi : R' \to R''$ is called an *isomorphism of extensions* if the following diagram commutes:

$$\begin{array}{ccccccc} 0 \to & I & \to & R' & \to & R & \to 0 \\ & \| & & \downarrow \xi & & \| & \\ 0 \to & I & \to & R'' & \to & R & \to 0 \end{array}$$

Such a ξ is necessarily an isomorphism of A-algebras. More generally, given A-extensions (R', φ) and (R'', ψ) of R, not necessarily having the same kernel, a homomorphism of A-algebras $r : R' \to R''$ such that $\psi r = \varphi$ is called a *homomorphism of extensions.*

The following Lemma is immediate.

(I.1.1) LEMMA *Let (R', φ) be an extension as above. Given an A-algebra B and two A-homomorphisms $f_1, f_2 : B \to R'$ such that $\varphi f_1 = \varphi f_2$ the induced map $f_2 - f_1 : B \to I$ is an A-derivation. In particular, given two homomorphisms of extensions*

$$r_1, r_2 : (R', \varphi) \to (R'', \psi)$$

the induced map $r_2 - r_1 : R' \to \ker(\psi)$ is an A-derivation.

The A-extension (R', φ) is called *trivial* if it has a section, that is if there exists a homomorphism of A-algebras $\sigma : R \to R'$ such that $\varphi\sigma = 1_R$. We also say that (R', φ) *splits*, and we call σ a *splitting.*

Given an R-module I, a trivial A-extension of R by I can be constructed considering the A-algebra $R\tilde{\oplus}I$ whose underlying A-module is $R \oplus I$ and with multiplication defined by:

$$(r, i)(s, j) = (rs, rj + si)$$

The first projection

$$p : R\tilde{\oplus}I \to R$$

defines an A-extension of R by I which is trivial: a section q is given by $q(r) = (r, 0)$.

The sections of p can be identified with the A-derivations $d : R \to I$. Indeed, if we have a section $\sigma : R \to R\tilde{\oplus}I$ with $\sigma(r) = (r, d(r))$ then for all $r, r' \in R$:

$$\sigma(rr') = (rr', d(rr')) = \sigma(r)\sigma(r') = (r, d(r))(r', d(r')) = (rr', rd(r') + r'd(r))$$

and if $a \in A$ then:

$$\sigma(ar) = (ar, d(ar)) = a\sigma(r) = a(r, d(r)) = (ar, ad(r))$$

hence $d : R \to I$ is an A-derivation. Conversely every A-derivation $d : R \to I$ defines a section $\sigma_d : R \to R\tilde{\oplus}I$ by $\sigma_d(r) = (r, d(r))$.

Every trivial A-extension (R', φ) of R by I is isomorphic to $(R\tilde{\oplus}I, p)$. If $\sigma : R \to R'$ is a section an isomorphism $\xi : R\tilde{\oplus}I \to R'$ is given by:

$$\xi((r, i)) = \sigma(r) + i$$

and its inverse is

$$\xi^{-1}(r') = (\varphi(r'), r' - \sigma\varphi(r'))$$

An A-extension (P, f) of R will be called *versal* if for every other A-extension (R', φ) of R there is a homomorphism of extensions $r : (P, f) \to (R', \varphi)$. If $R = P/I$ where P is a polynomial algebra over A then

$$0 \to I/I^2 \to P/I^2 \to R \to 0$$

is a versal A-extension of R.

(I.1.2) EXAMPLES

(i) Every A-extension of A is trivial because by definition it has a section. Therefore it is of the form $A\tilde{\oplus}V$ for an A-module V. In particular, if t is an indeterminate the A-extension $A[t]/(t^2)$ of A is trivial, and is denoted $A[\epsilon]$ (where $\epsilon = t \bmod (t^2)$ satisfies $\epsilon^2 = 0$). The corresponding exact sequence is:

$$0 \to (\epsilon) \to A[\epsilon] \to A \to 0$$

$A[\epsilon]$ is called the *algebra of dual numbers* over A.

(ii) Assume that K is a field. If R is a local K-algebra with residue field K a K-extension of R by K is called a *small extension* of R. Let

$$(R', f) \qquad 0 \to (t) \to R' \xrightarrow{f} R \to 0$$

be a small K-extension; in other words $t \in m_{R'}$ is annihilated by $m_{R'}$ so that (t) is a K-vector space of dimension one.

(R', f) is trivial if and only if the surjective linear map induced by f:

$$f_1 : \frac{m_{R'}}{m_{R'}^2} \to \frac{m_R}{m_R^2}$$

is not bijective.
Indeed for the trivial K-extension

$$0 \to (\epsilon) \to R\tilde{\oplus}(\epsilon) \to R \to 0$$

we have $\epsilon \in m_{R\tilde{\oplus}(\epsilon)} \setminus m^2_{R\tilde{\oplus}(\epsilon)}$, hence the map f_1 is not injective because $f_1(\epsilon) = 0$.
Conversely, if f_1 is not injective choose a vector subspace $V \subset R'$ such that $R' = V \oplus (t)$. Since $t \in m_{R'} \backslash m^2_{R'}$ (because f_1 is not injective), V is a subring mapped isomorphically onto R by f. The inverse of $f_{|V}$ is a section of f, therefore (R', f) is trivial.

For example, it follows from this criterion that the extension of K-algebras

$$0 \to \frac{(t^n)}{(t^{n+1})} \to \frac{K[t]}{(t^{n+1})} \to \frac{K[t]}{(t^n)} \to 0 \qquad n \geq 2$$

is non trivial.

(iii) Let K be a field. The K-algebra

$$K[\epsilon, \epsilon'] := K[t, t']/(t, t')^2$$

is a K-extension of $K[\epsilon]$ by K in two different ways. The first

$$0 \to (\epsilon') \to K[\epsilon, \epsilon'] \xrightarrow{p_\epsilon} K[\epsilon] \to 0$$

is a trivial extension, isomorphic to $p^*((K[\epsilon'], p'))$:

$$\begin{array}{ccccccc} 0 \to & (\epsilon') & \to & K[\epsilon] \times_K K[\epsilon'] & \to & K[\epsilon] & \to 0 \\ & \| & & \downarrow & & \downarrow p & \\ 0 \to & (\epsilon') & \to & K[\epsilon'] & \xrightarrow{p'} & K & \to 0 \end{array}$$

The isomorphism is given by

$$\begin{array}{ccc} K[\epsilon, \epsilon'] & \longrightarrow & K[\epsilon] \times_K K[\epsilon'] \\ a + b\epsilon + b'\epsilon' & \longmapsto & (a + b\epsilon, a + b'\epsilon') \end{array}$$

The second way is by "sum":

$$\begin{array}{ccccc} 0 \to & (\epsilon - \epsilon') & \to K[\epsilon, \epsilon'] & \xrightarrow{+} & K[\epsilon] & \to 0 \\ & & a + b\epsilon + b'\epsilon' & \mapsto & a + (b + b')\epsilon & \end{array}$$

We leave it as an exercise to show that $(K[\epsilon,\epsilon'],+)$ is isomorphic to $(K[\epsilon,\epsilon'],p_\epsilon)$.

* * * * * *

The module $\mathrm{Ex}_A(R,I)$

Let $A \to R$ be a ring homomorphism. In this subsection we will show how to give an R-module structure to the set of isomorphism classes of extensions of an A-algebra R by a module I, closely following the analogous theory of extensions in an abelian category as explained for example in Chapter III of MacLane(1967).

Let (R',φ) be an A-extension of R by I and $f: S \to R$ a homomorphism of A-algebras. We can define an A-extension $f^*(R',\varphi)$ of S by I, called the *pullback* of (R',φ) by f, in the following way:

$$\begin{array}{lccccccccc}
f^*(R',\varphi): & 0 \to & I & \to & R' \times_R S & \to & S & \to 0 \\
 & & \| & & \downarrow & & \downarrow f & \\
(R',\varphi): & 0 \to & I & \to & R' & \to & R & \to 0
\end{array}$$

where $R' \times_R S$ denotes the fibered product defined in the usual way.

Let $\lambda: I \to J$ be a homomorphism of R-modules. The *pushout* of (R',φ) by λ is the A-extension $\lambda_*(R',\varphi)$ of R by J defined by the following commutative diagram:

$$\begin{array}{ccccccc}
0 \to & I & \xrightarrow{\alpha} & R' & \xrightarrow{\varphi} & R & \to 0 \\
 & \downarrow \lambda & & \downarrow & & \| & \\
0 \to & J & \to & R' \coprod_I J & \to & R & \to 0
\end{array}$$

where

$$R' \coprod_I J = \frac{R' \tilde{\oplus} J}{\{(-\alpha(i), \lambda(i)), i \in I\}}$$

For every A-algebra R and for every R-module I denote by $\mathrm{Ex}_A(R,I)$ the set of isomorphism classes of A-extensions of R by I. If (R',φ) is such an extension we will denote by $[R',\varphi] \in \mathrm{Ex}_A(R,I)$ its class.

Using the operations of pullback and pushout it is possible to define a structure of R-module on $\mathrm{Ex}_A(R,I)$.

If $r \in R$ and $[R',\varphi] \in \mathrm{Ex}_A(R,I)$ we define

$$r[R',\varphi] = [r_*(R',\varphi)]$$

where $r: I \to I$ is the multiplication by r.

Given $[R',\varphi], [R'',\psi] \in \mathrm{Ex}_A(R,I)$, to define their sum we use the following diagram:

$$\begin{array}{ccccccccc}
0 & & & & 0 & & 0 & & \\
 & \searrow & & & \downarrow & & \downarrow & & \\
 & & I \oplus I & & I & = & I & & \\
 & & & \searrow & \downarrow & & \downarrow & & \\
0 & \to & I & \to & R' \times_R R'' & \to & R' & \to & 0 \\
 & & \| & & \downarrow & \searrow & \downarrow & & \\
0 & \to & I & \to & R'' & \to & R & \to & 0 \\
 & & & & \downarrow & & \downarrow & \searrow & \\
 & & & & 0 & & 0 & & 0
\end{array}$$

which defines an A-extension:

$$(R' \times_R R'', \zeta) : \qquad 0 \to I \oplus I \to R' \times_R R'' \xrightarrow{\zeta} R \to 0$$

We define

$$[R', \varphi] + [R'', \psi] := [\delta_*(R' \times_R R'', \zeta)]$$

where $\delta : I \oplus I \to I$ is the "sum homomorphism": $\delta(i \oplus j) = i + j$.

(I.1.3) PROPOSITION *Let $A \to R$ be a ring homomorphism and I an R-module. With the operations defined above $\mathrm{Ex}_A(R, I)$ is an R-module whose zero element is $[R\tilde{\oplus}I, p]$. This construction defines a covariant functor:*

$$\begin{array}{ccc} (R\text{-modules}) & \longrightarrow & (R\text{-modules}) \\ I & \longmapsto & \mathrm{Ex}_A(R, I) \\ (f : I \to J) & \longmapsto & (f_* : \mathrm{Ex}_A(R, I) \to \mathrm{Ex}_A(R, J)) \end{array}$$

Proof: Straightforward.

It is likewise straightforward to check that if $f : R \to S$ is a homomorphism of A-algebras and I is an S-module, then the operation of pullback induces an application:

$$f^* : \mathrm{Ex}_A(S, I) \to \mathrm{Ex}_A(R, I)$$

which is a homomorphism of S-modules.

We have the following useful result.

(I.1.4) PROPOSITION *Let A be a ring, $f : S \to R$ a homomorphism of A-algebras and let I be an R-module. Then there is an exact sequence of R-modules:*

$$\begin{array}{c} 0 \to \mathrm{Der}_S(R, I) \to \mathrm{Der}_A(R, I) \to \mathrm{Der}_A(S, I) \otimes_S R \xrightarrow{\rho} \\ \to \mathrm{Ex}_S(R, I) \xrightarrow{v} \mathrm{Ex}_A(R, I) \xrightarrow{f^*} \mathrm{Ex}_A(S, I) \otimes_S R \end{array}$$

Proof

v is the obvious application sending an S-extension to itself considered as an A-extension. An A-extension

$$0 \to I \to R' \xrightarrow{\varphi} R \to 0$$

is also an S-extension if and only if there exists $f' : S \to R$ such that the triangle

$$\begin{array}{ccc} R' & \to & R \\ & \nwarrow & \uparrow \\ & & S \end{array}$$

commutes, and this is equivalent to saying that $f^*(R', \varphi)$ is trivial. This proves the exactness in $\mathrm{Ex}_A(R, I)$.

The homomorphism ρ is defined by letting $\rho(d) = (R\tilde{\oplus}I, p)$ where the structure of S-algebra on $R\tilde{\oplus}I$ is given by the homomorphism

$$s \mapsto (f(s), d(s))$$

Clearly $v\rho = 0$. On the other hand for

$$\begin{array}{cccccccc} (R', \varphi): & 0 \to & I & \to & R' & \xrightarrow{\varphi} & R & \to 0 \\ & & & & \uparrow & & & \\ & & & & S & & & \end{array}$$

to define an element of $\ker(v)$ there must exist an isomorphism of A-algebras $R' \to R\tilde{\oplus}I$ inducing the identity on I and on R. Hence the composition $S \to R' \to R\tilde{\oplus}I$ is of the form

$$s \mapsto (f(s), d(s))$$

for some $d \in \mathrm{Der}_A(S, I)$: therefore the sequence is exact at $\mathrm{Ex}_S(R, I)$. To prove the exactness at $\mathrm{Der}_A(S, I)$ note that $\rho(d) = 0$ if and only if $p : R\tilde{\oplus}I \to R$ has a section as a homomorphism of S-algebras, if and only if there exists an A-derivation $R \to I$ whose restriction to S is d: this proves the assertion. The exactness at $\mathrm{Der}_S(R, I)$ and $\mathrm{Der}_A(R, I)$ is straightforward. *q.e.d.*

In the special case when we have a ring homomorphism $A \to R$ and we take $I = R$ the R-module $\mathrm{Ex}_A(R, R)$ is called the *first cotangent module* of R over A and it is denoted $T^1_{R/A}$. In case $A = \mathbf{k}$ we will write T^1_R instead of $T^1_{R/\mathbf{k}}$.

NOTES

1. The functor $\mathrm{Ex}_A(R, I)$ has been introduced for the first time in Grothendieck(1968) in the form presented here. See also [EGA], Ch. 0_{IV}, §18.

I.2. FORMAL SMOOTHNESS

The notion of "formal smoothness", introduced in [EGA], Ch. IV §17, is of crucial importance in deformation theory, and will therefore play a special role in what follows. It is closely related to the notion of "nonsingularity".

(I.2.1) DEFINITION *A ring homomorphism $f : R \to B$ is called* formally smooth, *and B is called a* formally smooth R-algebra, *if for every exact sequence:*

$$[I.2.1] \qquad 0 \to I \to A \xrightarrow{\eta} A' \to 0$$

where A and A' are local artinian R-algebras, each R-algebra homomorphism $B \to A'$ has a lifting $B \to A$; equivalently if the map:

$$[I.2.2] \qquad \mathrm{Hom}_{R-alg}(B, A) \to \mathrm{Hom}_{R-alg}(B, A')$$

is surjective.
f is called smooth *if it is formally smooth and e.f.t..*

It is easy to prove by induction that it suffices to check the above conditions only for the exact sequences [I.2.1] such that $I^2 = (0)$, i.e. for extensions of local artinian R-algebras.

If we modify the previous definition asking that the map [I.2.2] is bijective (instead of only being surjective) for all exact sequences [I.2.1], we obtain the notions of *formally etale* and *etale* homomorphism.

(I.2.2) PROPOSITION
(i) If B is a ring and $\Delta \subset B$ is a multiplicative system, $B \to \Delta^{-1}B$ is formally etale. In particular B is a formally etale B-algebra.
(ii) The composition of formally smooth (resp. formally etale) homomorphisms is formally smooth (resp. formally etale).
(iii) If $f : R \to B$ is formally smooth (resp. formally etale) and C is an R-algebra, then $C \to C \otimes_R B$ is formally smooth (resp. formally etale).
(iv) A finitely generated field extension $K \subset L$ is smooth if and only if L is separable over K.
(v) Let $R \xrightarrow{f} B \xrightarrow{g} C$ be ring homomorphisms, and assume that f is formally etale. Then gf is formally smooth (resp. formally etale) if and only if g is formally smooth (resp. formally etale).

Proof
(i) Given an exact sequence [I.2.1] and a commutative diagram

$$\begin{array}{ccc} B & \to & \Delta^{-1}B \\ \downarrow \varphi' & & \downarrow \varphi \\ A & \xrightarrow[p]{} & A' \end{array}$$

we must find $\tilde{\varphi} : \Delta^{-1}B \to A$ which makes it commutative. For every $s \in \Delta$ choose $a_s \in A$ such that $\varphi(s)^{-1} = p(a_s)$. Since

$$p(\varphi'(s)a_s) = \varphi(s)\varphi(s)^{-1} = 1_{A'}$$

we have

$$\varphi'(s)a_s = 1_A + i_s \qquad i_s \in I$$

for every $s \in \Delta$. Therefore

$$\varphi'(s)a_s(1_A - i_s) = 1_A$$

Hence $\varphi'(s) \in A$ is invertible. Now define $\tilde{\varphi}(r/s) = \varphi'(r)\varphi'(s)^{-1}$.

Noting that $\tilde{\varphi}$ is uniquely determined by φ' we get the assertion.

(ii) and (iii) are straightforward.

(iv) Assume first that $K \subset L$ is separable. By (ii) it suffices to consider the cases $L = K(X)$ and $L = K[X]/(f(X))$ where f is irreducible and $f'(x) \neq 0$. The first case is left to the reader (see remark (I.2.3)(i)).

In the second case consider an extension $\bar{A} = A/I$ of local artinian K-algebras, where $I \subset A$ is an ideal with $I^2 = (0)$. Let $\varphi : K[X]/(f(X)) \to \bar{A}$ be a homomorphism, sending $\bar{X} \mapsto \bar{\alpha}$. Choose arbitrarily $\alpha \in A$ such that $\bar{\alpha} = \alpha \bmod I$. It will suffice to find $e \in I$ such that

$$f(\alpha + e) = 0$$

We have $f(\alpha + e) = f(\alpha) + f'(\alpha)e$. Since $f'(\alpha)$ is a unit mod I it is also a unit in A, and therefore we can take $e = -f(\alpha)/f'(\alpha)$.

Assume conversely that $K \subset L$ is smooth. Then $L = F[X]/J$ where F is a purely transcendental extension of K and J is a principal ideal. We have an exact sequence of finite dimensional L-vector spaces:

$$J/J^2 \to \Omega_{F[X]/K} \otimes L \to \Omega_{L/K} \to 0$$

where J/J^2 is 1-dimensional. By the first part of the proof F is smooth over K and by (A.1.3)(ii) the left map is injective because, by the smoothness of L over K, the surjection $F[X]/J^2 \to L$ splits. It follows that

$$\dim(\Omega_{L/K}) = \dim(\Omega_{F[X]/K} \otimes L) - 1 = \mathrm{trdeg}_K(F[X]) - 1 = \mathrm{trdeg}_K(F) = \mathrm{trdeg}_K(L)$$

From (A.1.1)(iii) it follows that $K \subset L$ is separable.

(v) "if" follows immediately from (ii); "only if" is left to the reader. *q.e.d.*

(I.2.3) REMARKS.

(i) Any polynomial algebra $R[X_1, X_2 \ldots]$ is trivially a formally smooth R-algebra. From (I.2.2)(i) it follows that a localization of a polynomial R-algebra is also a formally smooth R-algebra.

More precisely, a localization $P = S^{-1}R[X_1, X_2, \ldots]$ of a polynomial algebra over a ring R satisfies the following condition, stronger than formal smoothness:

For every extension of R-algebras:

$$0 \to I \to A \to A' \to 0$$

where A and A' are R-algebras and $I^2 = 0$ the map

$$\mathrm{Hom}_{R-alg}(P, A) \to \mathrm{Hom}_{R-alg}(P, A')$$

is surjective.

Every R-algebra B is a quotient of a formally smooth R-algebra, because it is a quotient of a polynomial R-algebra. From (I.2.2)(i) it follows that every e.f.t. R-algebra is a quotient of a smooth R-algebra.
This is trivial for polynomial rings, and in the general case it can be proved adapting the proof of (I.2.2)(i) in an obvious way.

(ii) if R is in $\hat{\mathcal{A}}$ then every formal power series ring $R[[X_1, X_2, \ldots]]$ is a formally smooth R-algebra, because local artinian R-algebras are complete.

More precisely a formal power series ring $R[[X_1, X_2, \ldots]]$ satisfies the following condition, stronger than formal smoothness over R:
For every extension:

$$0 \to I \to A \to A' \to 0$$

of complete local R-algebras the map

$$\mathrm{Hom}_{R-alg}(P, A) \to \mathrm{Hom}_{R-alg}(P, A')$$

is surjective.

The proof is straightforward and is left to the reader.

The following result characterizes an important class of formally smooth algebras.

(I.2.4) THEOREM *Let k be a field and let (B, m) be a noetherian local k-algebra with residue field K. Suppose that K is finitely generated and separable over k. Then the following are equivalent:*
(i) B is regular.
(ii) $\hat{B} \cong K[[X_1, \ldots, X_d]]$, where $d = \dim(B)$.
(iii) B is a formally smooth k-algebra.

Proof
(i) $\Leftrightarrow$ (ii) is standard (see Eisenbud(1995), prop. 10.16 and exercise 19.1).
(ii) $\Rightarrow$ (iii). It follows directly from the definition that B is formally smooth over k if and only if $\hat{B}$ is. Since $\hat{B}$ is formally smooth over K (remark (I.2.3)(ii)), and since K is smooth over k by (I.2.2)(iv), the conclusion follows by transitivity.
(iii) $\Rightarrow$ (i). Let $\{x_1, \ldots, x_d\}$ be a system of generators of m. Then, since B/m^2 is complete and K is separable over k, B/m^2 contains a coefficient field (Eisenbud(1995), theorem 7.8). Therefore there exists an isomorphism

$$v_1 : B/m^2 \cong K[X_1, \ldots, X_d]/M^2 \qquad M = (X_1, \ldots, X_d)$$

Let $v : B \to B/m^2 \xrightarrow{v_1} K[X_1, \dots, X_d]/M^2$. By the formal smoothness of B and by induction we can find a lifting of v:

$$v_n : B \to K[X_1, \dots, X_d]/M^{n+1}$$

for every $n \geq 2$. Consider the elements

$$v_n(x_1), \dots, v_n(x_d) \in M/M^{n+1}$$

Their classes generate M/M^2, hence they generate M/M^{n+1}, by Nakayama. Then we have:

$$K[X_1, \dots, X_d]/M^{n+1} = v_n(B) + (M/M^{n+1}) = v_n(B) + \sum_i v_n(x_i)\left[v_n(B) + (M/M^{n+1})\right] =$$

$$= v_n(B) + (M/M^{n+1})^2 = \cdots = v_n(B) + (M/M^{n+1})^{n+1} = v_n(B)$$

hence v_n is surjective. Since $m^{n+1} \subset \ker(v_n)$ we have:

$$\ell(B/m^{n+1}) \geq \ell(K[X_1, \dots, X_d]/M^{n+1}) = \binom{d+n}{d}$$

and this implies that $\dim(B) \geq d$. Since m is generated by d elements it follows that B is regular. *q.e.d.*

For the reader's convenience we include the proof of the following well known

(I.2.5) LEMMA *(i) A surjective endomorphism $f : A \to A$ of a noetherian ring is an isomorphism.*
(ii) Let A be a complete noetherian local ring and $\psi : A \to A$ an endomorphism inducing an isomorphism $\psi_1 : A/m_A^2 \to A/m_A^2$. Then ψ is an isomorphism.

Proof
(i) We have an ascending chain of ideals

$$\ker(f) \subseteq \ker(f^2) \subseteq \ker(f^3) \subseteq \cdots$$

Since A is noetherian we have $\ker(f^n) = \ker(f^{n+1}) = \ker(f^{n+2}) = \cdots$ for some n, and it suffices to prove that $\ker(f^n) = (0)$. After replacing f by f^n we may assume $\ker(f) = \ker(f^2)$. Let $a \in \ker(f)$; by assumption there exists $b \in A$ such that $a = f(b)$. Then $0 = f(a) = f^2(b)$ and therefore $b \in \ker(f^2) = \ker(f)$, i.e. $a = f(b) = 0$.

(ii) Let $gr(A) = A/m \oplus m/m^2 \oplus \cdots$ be the associated graded ring. Since $gr(A)$ is generated by m/m^2 over A/m the endomorphism $gr(\psi) : gr(A) \to gr(A)$ induced by ψ is surjective. It follows that also ψ is surjective. Infact given $a \in A$ the surjectivity of $gr(\psi)$ implies that there are $a_1, a_2, a_3, \dots, b_1, b_2, b_3, \dots \in A$ such that $a_i \in m^{i-1}$, $b_i \in m^i$, and

$$a = f(a_1) + b_1,\ b_1 = f(a_2) + b_2,\ b_2 = f(a_3) + b_3, \dots$$

We obtain a convergent power series $\bar{a} = a_1 + a_2 + a_3 + \cdots$ such that

$$a - \psi(a_1 + a_2 + \cdots + a_n) = b_n \in m^{n+1}$$

On the limit we therefore get $a = \psi(\bar{a})$. The conclusion is now a consequence of (i). *q.e.d.*

(I.2.6) PROPOSITION *Let $f : R \to B$ be a local homomorphism of noetherian local rings containing a field k isomorphic to their residue fields. Then the following conditions are equivalent:*
(i) f is formally smooth.
(ii) $\hat{B}$ is isomorphic to a formal power series ring over $\hat{R}$.
(iii) The homomorphism $\hat{f} : \hat{R} \to \hat{B}$ induced by f is formally smooth.

Proof
(i) $\Rightarrow$ (ii). Let $m \subset B$ and $n \subset R$ be the maximal ideals. Choose elements $x_1, \ldots, x_d \in \hat{B}$ inducing a k-basis of $\hat{B}/(\hat{m}^2 + \hat{f}(\hat{n}))$, and let $F = \hat{R}[[X_1, \ldots, X_d]]$, where $X_1, \ldots, X_d$ are indeterminates. Denote by $M \subset F$ the maximal ideal.

The homomorphism

$$\begin{array}{cccc} u: & F & \to & \hat{B} \\ & X_i & \mapsto & x_i \end{array}$$

induces an isomorphism

$$u_1 : F/(M^2 + \hat{n}F) \to \hat{B}/(\hat{m}^2 + \hat{f}(\hat{n}))$$

By the formal smoothness of f the composition

$$v_1 : B \to \hat{B} \to \hat{B}/(\hat{m}^2 + \hat{f}(\hat{n})) \xrightarrow{u_1^{-1}} F/(M^2 + \hat{n}F)$$

can be lifted to an R-homomorphism

$$v_k : B \to F/M^k$$

for each $k \geq 2$. Therefore the sequence $\{v_k\}$ defines an $\hat{R}$-homomorphism

$$v : \hat{B} \to F$$

such that $vu : F \to F$ and $uv : \hat{B} \to \hat{B}$ induce isomorphisms $(vu)_1 : F/M^2 \to F/M^2$ and $(uv)_1 : \hat{B}/\hat{m}^2 \to \hat{B}/\hat{m}^2$ respectively. From Lemma (I.2.5) it follows that u and v are isomorphisms inverse of each other.
(ii) $\Rightarrow$ (iii) is obvious.
(iii) $\Rightarrow$ (i) is left to the reader. *q.e.d.*

(I.2.7) COROLLARY *Let $f : R \to B$ be a local homomorphism of noetherian local rings containing a field k isomorphic to their residue fields. Then the following conditions are equivalent:*

(i) f is formally etale.
(ii) The homomorphism $\hat{f}: \hat{R} \to \hat{B}$ induced by f is an isomorphism

Proof
left to the reader

(I.2.8) COROLLARY *Let R be in $\mathcal{A}^*$. The inclusion $f : R \to \hat{R}$ is formally etale.*

The proof is obvious.

* * * * * *

Next Theorem shows that when we have an e.f.t. ring homomorphism the defining condition of Definition (I.2.1) can be replaced by the more general condition (i) in the following statement.

(I.2.9) THEOREM *Let $f : R \to B$ be an e.f.t. ring homomorphism. Then the following conditions are equivalent:*
(i) For every extension of R-algebras:

$$[I.2.3] \qquad 0 \to I \to A \to A' \to 0$$

the map

$$\mathrm{Hom}_{R-alg}(B, A) \to \mathrm{Hom}_{R-alg}(B, A')$$

is surjective.
(ii) If $B = P/J$, where $P = S^{-1}R[X_1, \dots, X_d]$, $S \subset R[X_1, \dots, X_d]$ is a multiplicative system and $J \subset P$ is an ideal, the conormal sequence

$$0 \to J/J^2 \xrightarrow{\delta} \Omega_{P/R} \otimes_P B \to \Omega_{B/R} \to 0$$

is split exact. In particular J/J^2 and $\Omega_{B/R}$ are finitely generated projective B-modules.
(iii) B is a smooth R-algebra.
(iv) (Jacobian criterion of smoothness) If P and J are as in (ii) the map

$$(J/J^2) \otimes_B K(p) \xrightarrow{\delta \otimes_B K(p)} \Omega_{P/R} \otimes_P K(p) \qquad \text{where } K(p) = B_p/m_{B_p}$$

is injective for every prime ideal $p \subset B$.

Proof
(i) $\Rightarrow$ (ii). The hypothesis implies that the extension:

$$0 \to J/J^2 \to P/J^2 \to B \to 0$$

splits. Therefore the conormal sequence is split exact by (A.1.3)(iii) and it follows that J/J^2 and $\Omega_{B/R}$ are finitely generated projective because $\Omega_{P/R} \otimes_P B$ is free of finite rank.

(ii) $\Rightarrow$ (i). Consider an exact sequence [I.2.3] and a homomorphism of R-algebras $f' : B \to A'$. By Remark (I.2.3)(ii) there exists an R-homomorphism $g : P \to A$ making the following diagram commute:

$$\begin{array}{ccc} P & \to & B \\ \downarrow g & & \downarrow f' \\ A & \to & A' \end{array}$$

Since $g(J) \subset I$, we see that g factors through P/J^2, so that we have a commutative diagram:

$$\begin{array}{ccc} P/J^2 & \to & B \\ \downarrow \bar{g} & & \downarrow f' \\ A & \to & A' \end{array}$$

The hypothesis implies, via (A.1.3)(iii), that there is $h : B \to P/J^2$ a splitting of $P/J^2 \to B$. The composition $f = \bar{g}h : B \to A$ gives a lifting of f'.

(i) $\Rightarrow$ (iii) is obvious.

(iii) $\Rightarrow$ (iv). We may assume B and P local with residue field K. To prove that $\delta \otimes_B K$ is injective, it suffices to show that for every K-vector space V the map induced by δ:

$$\begin{array}{ccc} \mathrm{Hom}_K(\Omega_{P/R} \otimes_P K, V) & \to & \mathrm{Hom}_K((J/J^2) \otimes_B K, V) \\ \| & & \| \\ \mathrm{Der}_R(P, V) & & \mathrm{Hom}_B(J/J^2, V) \end{array}$$

is surjective. Consider a homomorphism $g : J/J^2 \to V$, and the associated pushout diagram (see §I.1 for the definition):

$$\begin{array}{ccccccccccc} \Lambda : & & 0 & \to & J/J^2 & \to & P/J^2 & \to & B & \to & 0 \\ & & & & \downarrow g & & \downarrow & & \| & & \\ g_*(\Lambda) : & & 0 & \to & V & \to & Q & \to & B & \to & 0 \end{array}$$

We can write $m_Q = V \oplus m'$, where $m' \subset Q$ is an ideal, because V is annihilated by m_Q. Therefore the previous diagram can be embedded in the following:

$$\begin{array}{ccccccccccc} & & & & & & P & & & & \\ & & & & & & \downarrow & \searrow & & & \\ \Lambda : & & 0 & \to & J/J^2 & \to & P/J^2 & \to & B & \to & 0 \\ & & & & \downarrow g & & \downarrow & & \| & & \\ g_*(\Lambda) : & & 0 & \to & V & \to & Q & \to & B & \to & 0 \\ & & & & \| & & \downarrow & & \downarrow \bar{v} & & \\ \eta : & & 0 & \to & V & \to & Q/m' & \to & K & \to & 0 \end{array}$$

where η is an extension of local artinian R-algebras. From the smoothness of B we deduce the existence of $v : B \to Q/m'$ lifting the projection $\bar{v} : B \to K$. Denoting by

$r : P \to B$ the natural map, and by $w : P \to P/J^2 \to Q \to Q/m'$ the composition, consider the homomorphism:

$$d = w - vr : P \to V$$

It is easy to show that this is an R-derivation, which induces g.

(iv) $\Rightarrow$ (ii). From Nakayama's Lemma it follows that $\ker(\delta)\otimes B_p = (0)$ and $\Omega_{B/R}\otimes_B B_p$ is free for all prime ideals $p \subset B$. Therefore $\ker(\delta) = (0)$, $\Omega_{B/R}$ is projective, δ has a splitting and J/J^2 is projective. *q.e.d.*

From now on we will freely replace the defining property for smooth homomorphisms given in Definition (I.2.1) by condition (i) of the Theorem. Here is a first example.

(I.2.10) PROPOSITION *Let K be a ring, P a smooth K-algebra and $B = P/J$ for an ideal $J \subset P$. If B is a smooth K-algebra the conormal sequence*

$$0 \to J/J^2 \to \Omega_{P/K} \otimes_P B \to \Omega_{B/K} \to 0$$

is split exact.

Proof

Since B is smooth the K-algebra extension

$$0 \to J/J^2 \to P/J^2 \to B \to 0$$

splits. The conclusion is therefore a consequence of (A.1.3)(iii). *q.e.d.*

(I.2.11) COROLLARY *Let P be a smooth* $\mathbf{k}$*-algebra and $B = P/J$ for an ideal $J \subset P$. Assume that B is reduced. Then in the conormal sequence*

$$J/J^2 \xrightarrow{\delta} \Omega_{P/\mathbf{k}} \otimes_P B \to \Omega_{B/\mathbf{k}} \to 0 \qquad [I.2.1]$$

$\ker(\delta)$ *is a torsion B-module.*

Proof

Since B is reduced there is a dense open subset $U \subset \mathrm{Spec}(B)$ such that B_p is a regular local ring for all $p \in U$. From Theorem (I.2.4) it follows that B_p is a smooth $\mathbf{k}$-algebra for all such p and, by Propositions (I.2.10) and (A.1.1)(ii) , the conormal sequence [I.2.1] localized at p is split exact. It follows that $\ker(\delta)_p = (0)$ for all $p \in U$ and the conclusion follows. *q.e.d.*

The next result explains the relation between smoothness and the relative cotangent sequence.

(I.2.12) THEOREM *Let $K \xrightarrow{f} R \xrightarrow{g} B$ be ring homomorphisms, with g smooth. Then the relative cotangent sequence:*

$$0 \to \Omega_{R/K} \otimes_R B \xrightarrow{\alpha} \Omega_{B/K} \to \Omega_{B/R} \to 0$$

is split exact.

Proof

By Theorem (A.1.2) it suffices to prove that α is a split injection; this is equivalent to showing that, for any B-module M, the induced map:

$$\begin{array}{ccc} \mathrm{Hom}_B(\Omega_{B/K}, M) & \stackrel{\alpha^\vee}{\longrightarrow} & \mathrm{Hom}_B(\Omega_{R/K}\otimes_R B, M) \\ \| & & \| \\ \mathrm{Der}_K(B,M) & & \mathrm{Der}_K(R,M) \\ & & \\ D' & \mapsto & D'g \end{array}$$

is split surjective. Let $D : R \to M$ be a K-derivation and consider the commutative diagram:

$$\begin{array}{ccc} B & \stackrel{1_B}{\longrightarrow} & B \\ \uparrow g & & \uparrow \\ R & \stackrel{\gamma}{\longrightarrow} & B\tilde{\oplus}M \end{array}$$

where $\gamma(r) = (g(r), D(r))$, $r \in R$. By the smoothness of g we can find a homomorphism of R-algebras $\psi : B \to B\tilde{\oplus}M$ making the diagram

$$\begin{array}{ccc} B & \stackrel{1_B}{\longrightarrow} & B \\ \uparrow g & \searrow \psi & \uparrow \\ R & \underset{\gamma}{\longrightarrow} & B\tilde{\oplus}M \end{array}$$

commutative. The homomorphism ψ is necessarily of the form:

$$\psi(b) = (b, D'(b))$$

and $D' : B \to M$ is a K-derivation such that $D = D'g$. This proves the surjectivity of $\alpha^\vee$. Now take $M = \Omega_{R/K} \otimes_R B$ and $D = d_{R/K} \otimes g : R \to \Omega_{R/K} \otimes_R B$ and let

$$\alpha' : \Omega_{B/K} \to \Omega_{R/K} \otimes_R B$$

be the B-linear map corresponding to $D' : B \to \Omega_{R/K} \otimes_R B$. Then $\alpha'\alpha = 1_M$ and this proves that α is split injective. *q.e.d.*

(I.2.13) COROLLARY *Let* $K \stackrel{f}{\longrightarrow} R \stackrel{g}{\longrightarrow} B$ *be ring homomorphisms, with* g *etale. Then*

$$\Omega_{R/K} \otimes_R B \cong \Omega_{B/K}$$

is an isomorphism and

$$\Omega_{B/R} = (0)$$

Proof

By the relative cotangent sequence the two assertions are equivalent. We will prove the first. Keeping the notations of the proof of (I.2.12), the hypothesis that g is etale implies that the derivation D' is unique and consequently α is an isomorphism. *q.e.d.*

The following result follows easily from what we have seen so far.

(I.2.14) THEOREM *Let k be an algebraically closed field, and let B be an integral k-algebra of finite type and of dimension d. Then the following are equivalent:*
(i) B_p is smooth over k for each prime ideal $p \in \mathrm{Spec}(B)$.
(ii) B is a regular ring.
(iii) $\Omega_{B/k}$ is projective of rank d.
(iv) B is smooth over k.

Proof
(ii) $\Leftrightarrow$ (iii) is Corollary (A.1.6).
(i) $\Leftrightarrow$ (ii) follows from (I.2.4).
(iii) $\Leftrightarrow$ (iv) follows from (I.2.9). *q.e.d.*

* * * * * *

Etale neighborhoods

Let S be a scheme and $s \in S$ a point. An *etale neighborhood* of s is an etale morphism of pointed schemes $f : (T,t) \to (S,s)$ such that the following diagram commutes:

$$\begin{array}{ccc} & & T \\ & t \nearrow & \downarrow f \\ \mathrm{Spec}(\mathbf{k}(s)) & \xrightarrow{s} & S \end{array}$$

The definition implies that $\mathbf{k}(s) \cong \mathbf{k}(t)$, i.e. f induces a trivial extension of the residue fields at s and t; therefore $\hat{\mathcal{O}}_{S,s} \cong \hat{\mathcal{O}}_{T,t}$ by (I.2.7). Affine neighborhoods of s are particular etale neighborhoods.

Given two etale neighborhoods (T,t) and (U,u) of $s \in S$ a *morphism* $(T,t) \to (U,u)$ is given by a commutative diagram of pointed schemes:

$$\begin{array}{ccc} (T,t) & \to & (U,u) \\ \searrow & & \swarrow \\ & (S,s) & \end{array}$$

(I.2.15) LEMMA *Let $f : X \to Y$ be an etale morphism and $g : Y \to X$ a section of f. Then g is etale.*

Proof
Use (I.2.2)(v). *q.e.d.*

(I.2.16) PROPOSITION *Let S be a scheme. The etale neighborhoods of a given $s \in S$ form a filtered system of pointed schemes.*

Proof

Given two etale neighborhoods (S', s') and (S'', s''), they are dominated by a third, namely:

$$\begin{array}{ccc} S' \times_S S'' & \to & S'' \\ \downarrow & & \downarrow \\ S' & \to & S \end{array}$$

Now let $f_1, f_2 : (S'', s'') \to (S', s')$ be two morphisms between etale neighborhoods. Then there exists a third etale neighborhood (S''', s''') and a morphism $(S''', s''') \to (S'', s'')$ which equalizes them. Infact consider the diagram:

$$\begin{array}{ccc} S'' \times_S S' & \xrightarrow{pr'} & S' \\ \downarrow pr'' & & \downarrow \\ S'' & \to & S \end{array}$$

We can shrink S' and S'' so that S' is affine and S'' is connected. Then the graphs Γ_1 and Γ_2 of f_1 and f_2 are closed, because S' is affine, and open, because images of sections of the etale morphism pr'', which are etale. Therefore they are connected components of $S'' \times_S S'$. But $(s'', s') \in \Gamma_1 \cap \Gamma_2$ and therefore $\Gamma_1 = \Gamma_2$. It follows that $f_1 = f_2$ on $S'' = \Gamma_1 = \Gamma_2$. *q.e.d.*

(I.2.17) DEFINITION *Given a scheme S and a point $s \in S$ we define* the local ring of S in s in the etale topology *to be*

$$\tilde{\mathcal{O}}_{S,s} := \lim_{\to\ (S',s')} \mathcal{O}_{S',s'}$$

where the limit is taken for (S', s') varying through all the etale neighborhoods of s. The ring $\tilde{\mathcal{O}}_{S,s}$ is also called the henselization *of $\mathcal{O}_{S,s}$.* (Note that $\tilde{\mathcal{O}}_{S,s}$ is a local ring, because it is a limit of a filtering system of local rings and local homomorphisms).

A local ring A is called henselian *if for the closed point s of $S = \mathrm{Spec}(A)$ one has*

$$\tilde{A} := \tilde{\mathcal{O}}_{S,s} = \mathcal{O}_{S,s} = A$$

The henselization of an e.f.t. local **k***-algebra is called an* algebraic local ring.

Therefore the local ring in the etale topology of a point of an algebraic scheme is an algebraic local ring.

For a given scheme S and point $s \in S$ there is a canonical homomorphism $\mathcal{O}_{S,s} \to \tilde{\mathcal{O}}_{S,s}$ which is flat and induces an isomorphism of the completions

$$\hat{\mathcal{O}}_{S,s} \cong \widehat{\tilde{\mathcal{O}}_{S,s}}$$

because every $\mathcal{O}_{S,s} \to \mathcal{O}_{S',s'}$ does. Moreover

$$\mathcal{O}_{S,s} \subset \tilde{\mathcal{O}}_{S,s} \subset \hat{\mathcal{O}}_{S,s}$$

because $\mathcal{O}_{S,s} \to \tilde{\mathcal{O}}_{S,s}$ is faithfully flat and $\mathcal{O}_{S,s}$ is separated for the m-adic topology. In particular we see that $\mathcal{O}_{S,s} = \tilde{\mathcal{O}}_{S,s}$ if $\mathcal{O}_{S,s} = \hat{\mathcal{O}}_{S,s}$, i.e. *a local* **k**-*algebra in* $\hat{\mathcal{A}}$ *is henselian.*

(I.2.18) THEOREM (Nagata) *If A is a noetherian local ring then $\tilde{A}$ is noetherian.*

Proof

We have $A \subset \tilde{A} \subset \hat{A}$ and $\hat{A} = \hat{\tilde{A}}$. Moreover

$$\tilde{A} = \lim_{\rightarrow} A'$$

with A' local algebras etale over A and inducing trivial residue field extension. To prove that $\tilde{A}$ is noetherian it suffices to prove that every ascending chain of finitely generated ideals of $\tilde{A}$

$$\underline{a}_1 \subseteq \underline{a}_2 \subseteq \cdots \subseteq \underline{a}_n \subseteq$$

stabilizes. The chain $\{\underline{a}_n\hat{A}\}$ stabilizes because $\hat{A}$ is noetherian. Therefore it suffices to prove that if $\underline{a}, \underline{b} \subset \tilde{A}$ are finitely generated ideals such that $\underline{a}\hat{A} = \underline{b}\hat{A}$ then $\underline{a} = \underline{b}$. Since $\underline{a}$ and $\underline{b}$ are finitely generated one can find $A' \supset A$ as above and finitely generated ideals $\underline{a}', \underline{b}' \subset A'$ such that $\underline{a} = \underline{a}'\tilde{A}$, $\underline{b} = \underline{b}'\tilde{A}$. It follows that $\underline{a}'\hat{A} = \underline{b}'\hat{A}$. But since A' is noetherian it follows that $\underline{a}' = \underline{b}'$ and therefore $\underline{a} = \underline{b}$. *q.e.d.*

The following Proposition gives a geometrical characterization of the henselization.

(I.2.19) PROPOSITION *Let A be a local ring, $S = \mathrm{Spec}(A)$, $s \in S$ the closed point. A is henselian if and only if every morphism $f : Z \to S$ such that there is a point $z \in Z$ with $f(z) = s$, $\mathbf{k}(s) = \mathbf{k}(z)$ and f etale in z, admits a section.*

Proof

Assume the condition satisfied. If $A \to A'$ is an etale homomorphism inducing an isomorphism of the residue fields then the induced morphism $f : \mathrm{Spec}(A') \to S$ admits a section, which defines an isomorphism $A' \cong A$; therefore A is henselian. Conversely assume A henselian and let $f : Z \to S$ be a morphism satisfying the stated conditions. Then f induces an isomorphism $A \cong \mathcal{O}_{Z,z}$ because A is henselian. The section is the composition

$$S = \mathrm{Spec}(A) \cong \mathrm{Spec}(\mathcal{O}_{Z,z}) \subset Z$$

q.e.d.

NOTES

1. Let $f : X \to Y$ be a smooth morphism of algebraic schemes. Prove that the relative cotangent sequence

$$0 \to \Omega^1_X \to f^*\Omega^1_Y \to \Omega^1_{X/Y} \to 0$$

and the relative tangent sequence

$$0 \to T_{X/Y} \to Hom(f^*\Omega^1_Y, \mathcal{O}_X) \to T_X \to 0$$

are both exact (*hint:* use (I.2.12)).

I.3. OBSTRUCTIONS

In this Section we investigate the notion of formal smoothness in the category $\mathcal{A}^*$ using the language of extensions. The results we prove are crucial for the understanding of obstructions in deformation theory. Our treatment is an expansion of Schlessinger(1973); for a more systematic treatment we refer to Fantechi-Manetti(1998).

Let $\Lambda \in \mathrm{ob}(\mathcal{A}^*)$ and $\mu : \Lambda \to R$ be in $\mathrm{ob}(\mathcal{A}^*_\Lambda)$. The *relative obstruction space* of R/Λ is

$$o(R/\Lambda) := \mathrm{Ex}_\Lambda(R, \mathbf{k})$$

If $\Lambda = \mathbf{k}$ then $o(R/\mathbf{k})$ is called the (absolute) *obstruction space* of R and simply denoted by $o(R)$. We say that R is *unobstructed* (resp. *obstructed*) *over* Λ if $o(R/\Lambda) = (0)$ (resp. if $o(R/\Lambda) \neq (0)$); R is said to be *unobstructed* (resp. *obstructed*) if $o(R) = (0)$ (resp. if $o(R) \neq (0)$). Given a homomorphism $f : R \to S$ in $\mathcal{A}^*_\Lambda$ we denote by

$$o(f/\Lambda) : o(S/\Lambda) \to o(R/\Lambda)$$

the linear map induced by pullback:

$$o(f/\Lambda)([\eta]) = [f^*\eta] \in \mathrm{Ex}_\Lambda(R, \mathbf{k})$$

for all $[\eta] \in \mathrm{Ex}_\Lambda(S, \mathbf{k})$. Since this definition is functorial we have a contravariant functor:

$$o(-/\Lambda) : \mathcal{A}^*_\Lambda \to (\text{vector spaces}/\mathbf{k})$$

When $\Lambda = \mathbf{k}$ we write $o(f)$ instead of $o(f/\mathbf{k})$. If μ is such that $o(\mu)$ is injective one simetimes says that R is *less obstructed* than Λ. By applying Proposition (I.1.4) we obtain an exact sequence for each $f : R \to S$ in $\mathcal{A}^*_\Lambda$:

$$[I.3.1] \qquad 0 \to t_{S/R} \to t_{S/\Lambda} \to t_{R/\Lambda} \to o(S/R) \to o(S/\Lambda) \xrightarrow{o(f/\Lambda)} o(R/\Lambda)$$

In case $\Lambda = \mathbf{k}$ and $f = \mu$ we obtain the exact sequence:

$$[I.3.2] \qquad 0 \to t_{R/\Lambda} \to t_R \to t_\Lambda \to o(R/\Lambda) \to o(R) \xrightarrow{o(\mu)} o(\Lambda)$$

which relates the absolute and the relative obstruction spaces.

The next result gives a description of $o(R/\Lambda)$ and an interpretation of formal smoothness of a Λ-algebra (R, m) in $\mathcal{A}^*_\Lambda$.

(I.3.1) PROPOSITION *Assume that Λ is in $\hat{\mathcal{A}}$.*
(i) Let (R,m) be in $\mathcal{A}^*_\Lambda$ and let $\chi : R \to \hat{R}$ be the natural homomorphism of R into its m-adic completion $\hat{R}$. Then the induced map:

$$o(\chi/\Lambda) : o(\hat{R}/\Lambda) \to o(R/\Lambda)$$

is an isomorphism.
(ii) For every (R,m) in $\mathcal{A}^*_\Lambda$ let $d = \dim_{\mathbf{k}}(t_{R/\Lambda})$ and let

$$\hat{R} = \Lambda[[X_1, \ldots, X_d]]/J$$

with $J \subset (\underline{X})^2$, be a presentation of the m-adic completion $\hat{R}$. Then there is a natural isomorphism:

$$o(R/\Lambda) \cong \left(J/(\underline{X})J\right)^\vee$$

In particular R is unobstructed over Λ if and only if it is a formally smooth Λ-algebra.

Proof
(i) Let

$$\eta : 0 \to \mathbf{k} \to S \to \hat{R} \to 0$$

be an extension; denote by m' the maximal ideal of S.

Claim: S is complete.
Let $\{f_n\} \subset S$ be a Cauchy sequence; then the image sequence $\{\bar{f}_n\}$ in $\hat{R}$ is Cauchy, hence it converges to a limit which we may assume to be zero, after possibly subtracting a constant sequence from $\{f_n\}$. We have $\bar{f}_n \in \hat{m}^{e(n)}$, with $\lim_n[e(n)] = \infty$. For every n we may find $g_n \in m'^{e(n)}$ lying above $\bar{f}_n$. The sequence $\{g_n\}$ in S is Cauchy and converges to zero, and $\{f_n - g_n\}$ is a Cauchy sequence in $\mathbf{k}$. Since $\mathbf{k}$ is complete as an S-module, because it is annihilated by the maximal ideal, $\{f_n - g_n\}$ converges to a limit $f \in \mathbf{k}$. This is also the limit of $\{f_n\}$ because

$$f_n - f = (f_n - g_n - f) + g_n$$

Therefore S is complete.

If $\chi^*(\eta/\Lambda)$ is trivial the section induces a homomorphism $g : R \to S$ which factors through $\hat{R}$ because S is complete. Hence η is trivial. This proves that $o(\chi/\Lambda)$ is injective.
Given a Λ-extension of R:

$$(S,\varphi) : \quad 0 \to \mathbf{k} \to S \to R \to 0$$

the map $\hat{\varphi} : \hat{S} \to \hat{R}$ is surjective and $\ker(\hat{\varphi}) = \hat{\mathbf{k}} = \mathbf{k}$. therefore $[\hat{S}, \hat{\varphi}] \in \mathrm{Ex}_\Lambda(\hat{R}, \mathbf{k})$ and $o(\chi/\Lambda)([\hat{S}, \hat{\varphi}]) = [S, \varphi]$: this means that $o(\chi/\Lambda)$ is also surjective.

(ii) R is a formally smooth Λ-algebra if and only if $\hat{R}$ is a power series ring over Λ, i.e. if and only if $J = (0)$. Therefore the last assertion follows from the fact that $J/(\underline{X})J = (0)$ if and only if $J = (0)$, by Nakayama's Lemma.

In order to prove the first assertion we may assume that R is in $\hat{\mathcal{A}}_\Lambda$, since $o(R/\Lambda) = o(\hat{R}/\Lambda)$ by the first part of the Proposition. Hence $R = \Lambda[[\underline{X}]]/J$ with $J \subset (\underline{X})^2$. The extension of R:

$$\Phi : \quad 0 \to J/(\underline{X})J \to \Lambda[[\underline{X}]]/(\underline{X})J \to R \to 0$$

induces by pushouts a homomorphism:

$$\begin{array}{rccccl} \alpha : & \left(J/(\underline{X})J\right)^\vee & \to & \mathrm{Ex}_\Lambda(R, \mathbf{k}) & = o(R) \\ & d & \longmapsto & [d_*\Phi] & \end{array}$$

Letting M be the maximal ideal of $\Lambda[[\underline{X}]]/(\underline{X})J$ we have $J/(\underline{X})J \subset M^2$. If $d \in (J/(\underline{X})J)^\vee$ is such that $[d_*\Phi] = 0$ then we have:

$$\begin{array}{lcccccccc} \Phi : & 0 \to & J/(\underline{X})J & \to & \Lambda[[\underline{X}]]/(\underline{X})J & \to & R & \to 0 \\ & & \downarrow d & & \downarrow h & & \| & \\ d_*\Phi : & 0 \to & \mathbf{k} & \to & A & \to & R & \to 0 \end{array}$$

with $d_*\Phi$ trivial. From the example (I.1.1)(ii) it follows that the generator ϵ of $\mathbf{k}$ in A is contained in $m_A \backslash m_A^2$. Since $h(J/(\underline{X})J) \subset m_A^2$ we deduce that $d = 0$. It follows that α is injective.

Conversely, given a Λ-extension (A, φ) of R by $\mathbf{k}$ it is possible to find a lifting:

$$\begin{array}{cccc} & \Lambda[[\underline{X}]] & & \\ & \downarrow \tilde{\varphi} & \searrow & \\ \varphi : & A & \to & R \end{array}$$

because A is complete (see proof of (I.3.1) and (I.2.3)(ii)). From the fact that $\ker(\varphi) = \mathbf{k}$ it follows that $\ker(\tilde{\varphi}) \supset (\underline{X})J$ and therefore we have a commutative diagram:

$$\begin{array}{lcccccccc} \Phi : & 0 \to & J/(\underline{X})J & \to & \Lambda[[\underline{X}]]/(\underline{X})J & \to & R & \to 0 \\ & & \downarrow d & & \downarrow \tilde{\varphi} & & \| & \\ (A, \varphi) : & 0 \to & \mathbf{k} & \to & A & \to & R & \to 0 \end{array}$$

in which d is the map induced by $\tilde{\varphi}$. It follows that $(A, \varphi) = d_*\Phi$; hence α is surjective. *q.e.d.*

(I.3.2) COROLLARY *For every R in $\mathcal{A}^*$ the following are true:*

(i) $\qquad \dim_{\mathbf{k}}[o(R)] < \infty$

(ii) $\qquad \dim_{\mathbf{k}}(t_R) \geq \dim(R) \geq \dim_{\mathbf{k}}(t_R) - \dim_{\mathbf{k}}[o(R)]$

(where $\dim(R)$ *means* Krull dimension *of* R*). In (ii) the first equality holds if and only if* R *is formally smooth; the second equality holds if and only if* $\hat{R} = \mathbf{k}[[X_1, \ldots, X_d]]/J$*, with* $J \subset (\underline{X})^2$ *and* J *generated by a regular sequence.*

Proof

We may assume that R is in $\hat{\mathcal{A}}$; hence $R = \hat{R} = \mathbf{k}[[X_1, \ldots, X_d]]/J$, with $J \subset (\underline{X})^2$ and $o(R) \cong (J/(\underline{X})J)^\vee$. Then (i) and (ii) follow from the fact that $\dim_{\mathbf{k}}[J/(\underline{X})J]$ is the number of elements of a minimal set of generators of J. *q.e.d.*

(I.3.3) REMARKS

The only formally smooth $\mathbf{k}$-algebra in $\mathcal{A}$ is $\mathbf{k}$ itself. By (I.3.1)(ii) this means that $o(\mathbf{k}) = (0)$ and that $o(A) \neq (0)$ for every $A \neq \mathbf{k}$ in $\mathcal{A}$. The following are some special cases.

If $A = \mathbf{k}\tilde{\oplus}V$, a trivial extension of $\mathbf{k}$ by a vector space V of dimension d, then $A \cong \mathbf{k}[X_1, \ldots, X_d]/(\underline{X})^2$, and $o(A) = [(\underline{X})^2/(\underline{X})^3]^\vee$.

If $A = \mathbf{k}[\underline{X}]/(\underline{X})^k$ then $o(A) = [(\underline{X})^k/(\underline{X})^{k+1}]^\vee$. In particular, if $A = \mathbf{k}[t]/(t^n)$, $n \geq 2$, then $o(A) = [(t^n)/(t^{n+1})]^\vee$ is 1-dimensional; from the proof of (I.3.1)(ii) it follows immediately that $o(A)$ is generated by the class of the extension:

$$0 \to (t^n)/(t^{n+1}) \to \mathbf{k}[t]/(t^{n+1}) \to \mathbf{k}[t]/(t^n) \to 0$$

We will need the following

(I.3.4) LEMMA

(i) Let $\mu : \Lambda \to R$ *be a homomorphism in* $\mathcal{A}^*$*. Given a small extension* $\eta : B \to A$ *in* $\mathcal{A}$ *and a homomorphism* $\varphi : R \to A$*, the condition* $\varphi^*(\eta) \in \ker(o(\mu))$ *is equivalent to the existence of a commutative diagram:*

$$[I.3.3] \qquad \begin{array}{ccc} A & \xleftarrow{\varphi} & R \\ \uparrow \eta & & \uparrow \mu \\ B & \xleftarrow[\tilde{\varphi}]{} & \Lambda \end{array}$$

Moreover $\varphi^*(\eta) = 0$ *if and only if there exists* $\varphi' : R \to B$ *such that the resulting diagram*

$$[I.3.4] \qquad \begin{array}{ccc} A & \xleftarrow{\varphi} & R \\ \uparrow \eta & \swarrow \varphi' & \uparrow \mu \\ B & \xleftarrow[\tilde{\varphi}]{} & \Lambda \end{array}$$

is commutative.

(ii) For every Λ *in* $\hat{\mathcal{A}}$ *and* $\mu : \Lambda \to R$ *in* $\mathcal{A}^*_\Lambda$ *there exists* A *in* $\mathcal{A}_\Lambda$ *and a homomorphism* $p : R \to A$ *such that* $o(p/\Lambda) : o(A/\Lambda) \to o(R/\Lambda)$ *is surjective.*

Proof

(i) is left to the reader.

(ii) We will show something more precise, namely that if $p_n : R \to R/m^{n+1}$ is the natural map, then

$$o(p_n/\Lambda) : o((R/m^{n+1})/\Lambda) \to o(R/\Lambda)$$

is surjective for all $n \gg 0$.

Since $o(p_n/\Lambda)$ factors through $o(\hat{R}/\Lambda)$, which is isomorphic to $o(R/\Lambda)$, we may assume that R is in $\hat{\mathcal{A}}_\Lambda$. Let's write:

$$R = \Lambda[[\underline{X}]]/J$$

where $J = (g_1, \ldots, g_s) \subset (\underline{X})^2$. Let $n \gg 0$ be such that $g_j \notin (\underline{X})^{n+2}$ for all $j = 1, \ldots, s$. Then we have:

$$\frac{R}{m^{n+1}} = \frac{\Lambda[[\underline{X}]]}{(J, (\underline{X})^{n+1})}$$

and therefore

$$o((R/m^{n+1})/\Lambda) = \Big[\frac{(J, (\underline{X})^{n+1})}{((\underline{X})J, (\underline{X})^{n+2})}\Big]^\vee$$

The map $o(p_n/\Lambda)$ is the transpose of

$$\imath_n : \frac{J}{(\underline{X})J} \to \Big[\frac{(J, (\underline{X})^{n+1})}{((\underline{X})J, (\underline{X})^{n+2})}\Big]$$

induced by the inclusion $J \subset (J, (\underline{X})^{n+1})$. From the hypothesis on n it follows that if $\gamma \in J \cap ((\underline{X})J, (\underline{X})^{n+2})$ then $\gamma \in (\underline{X})J$; this means that $\imath_n$ is injective, i.e. that $o(p_n/\Lambda)$ is surjective. *q.e.d.*

The following Theorem gives a characterization of formally smooth homomorphisms in $\mathcal{A}^*$.

(I.3.5) THEOREM *Let $\mu : \Lambda \to R$ be a homomorphism in $\mathcal{A}^*$. The following conditions are equivalent:*
(i) For every commutative diagram [I.3.3] with η a small extension in $\mathcal{A}^*$ there exists $\varphi' : R \to B$ such that diagram [I.3.4] is commutative.
(ii) μ is formally smooth.
(iii) $d\mu : t_R \to t_\Lambda$ is surjective and $o(\mu)$ is injective.
(iv) $o(R/\Lambda) = (0)$

Proof
$(i) \Rightarrow (ii)$ is trivial.

$(ii) \Rightarrow (iii)$ Let $v \in t_\Lambda$ be given as a $\mathbf{k}$-algebra homomorphism $\Lambda \to \mathbf{k}[\epsilon]$. The formal smoothness of μ implies the existence of a homomorphism $w : R \to \mathbf{k}[\epsilon]$ which makes the following diagram commutative:

$$\begin{array}{ccc} \mathbf{k} & \leftarrow & R \\ \uparrow & \swarrow & \uparrow \\ \mathbf{k}[\epsilon] & \leftarrow & \Lambda \end{array}$$

and this means that $d\mu(w) = v$. Therefore $d\mu$ is surjective.

Consider a commutative diagram of $\mathbf{k}$-algebra homomorphisms [I.3.3] with η a small extension in $\mathcal{A}$. Then $o(\varphi)([\eta]) \in \ker(o(\mu))$. By the formal smoothness of μ there exists $\varphi' : R \to B$ making [I.3.4] commutative: this implies that $o(\varphi)([\eta]) = 0$. Since, by Lemma (I.3.4), φ and η can be chosen so that $o(\varphi)([\eta])$ is an arbitrary element of $\ker(o(\mu))$, we deduce that $o(\mu)$ is injective.

(iii) $\Leftrightarrow$ *(iv)* follows from the exact sequence [I.3.2].

(iii) $\Rightarrow$ *(i)*. Consider a diagram [I.3.3] with η a small extension in $\mathcal{A}^*$. Then

$$o(\varphi)([\eta]) \in \ker(o(\mu))$$

By assumption $o(\varphi)([\eta]) = 0$, and therefore there exists $\bar{\varphi} : R \to B$ such that $\eta\bar{\varphi} = \varphi$. It follows that $\tilde{\varphi} - \bar{\varphi}\mu : R \to \ker(\eta) = \mathbf{k}$ is a $\mathbf{k}$-derivation. By assumption there exists a $\mathbf{k}$-derivation $v : R \to \mathbf{k}$ such that $\tilde{\varphi} - \bar{\varphi}\mu = d\mu(v) = v\mu$.

Then $\varphi' := \bar{\varphi} + v : R \to B$ is a $\mathbf{k}$-homomorphism which obviously satisfies $\eta\varphi' = \varphi$. Moreover

$$\varphi'\mu = (\bar{\varphi} + v)\mu = \bar{\varphi}\mu + v\mu = \tilde{\varphi}$$

and therefore φ' makes [I.3.4] commutative. *q.e.d.*

In the special case $\Lambda = \mathbf{k}$ we obtain that a $\mathbf{k}$-algebra R in $\mathcal{A}^*$ is unobstructed if and only if R is formally smooth, a result already proven in (I.3.1). More generally the Theorem says that μ is formally smooth if and only if $d\mu$ is surjective and R is less obstructed than Λ. The following Corollary is immediate.

(I.3.6) COROLLARY
(i) Let $\mu : \Lambda \to R$ be a homomorphism in $\mathcal{A}^*$ such that $d\mu$ is surjective and R is formally smooth. Then Λ and μ are formally smooth.
(ii) A homomorphim $\mu : \Lambda \to R$ in $\mathcal{A}^*$ is formally etale if and only if $d\mu$ is an isomorphism and $o(\mu)$ is injective. This happens in particular if $d\mu$ is an isomorphism and R is formally smooth.
(iii) If $R \in \mathrm{ob}(\mathcal{A}^*)$ then the natural homomorphism $R \to \hat{R}$ is formally etale.

In practise it is seldom possible to compute the obstruction map $o(\mu)$ explicitly for a given $\mu : \Lambda \to R$ in $\mathcal{A}^*$. But for the purpose of studying the formal smoothness of μ all that counts is to have informations about $\ker[o(\mu)]$. This can be achieved somehow indirectly by means of the following simple result, which turns out to be very effective in practise.

(I.3.7) PROPOSITION *Let $\mu : \Lambda \to R$ be a morphism in $\mathcal{A}^*$. Assume that there exists a $\mathbf{k}$-vector space $v(R/\Lambda)$ such that for every homomorphism $\varphi : R \to A$ in $\mathcal{A}^*_\Lambda$ with A in $\mathcal{A}_\Lambda$ there is a $\mathbf{k}$-linear map*

$$\varphi_v : \mathrm{Ex}_\Lambda(A, \mathbf{k}) \to v(R/\Lambda)$$

satisfying

$$\ker(\varphi_v) = \ker[o(\varphi/\Lambda)]$$

where

$$\begin{array}{ccc} \mathrm{Ex}_\Lambda(A,\mathbf{k}) & \xrightarrow{\varphi_v} & v(R/\Lambda) \\ & \searrow o(\varphi/\Lambda) & \\ & \mathrm{Ex}_\Lambda(R,\mathbf{k}) = o(R/\Lambda) & \end{array}$$

Then there is a natural **k**-*linear inclusion*

$$o(R/\Lambda) \subset v(R/\Lambda)$$

Proof

Choosing φ such that $o(\varphi/\Lambda)$ surjects onto $o(R/\Lambda)$ (Proposition (I.3.4)) we obtain an inclusion $o(R/\Lambda) \subset v(R/\Lambda)$ as asserted. *q.e.d.*

In practise this Proposition will be applied as follows. Given φ, to give an element of $\mathrm{Ex}_\Lambda(A,\mathbf{k})$ is the same as to give a commutative diagram [I.3.3]. Assume that to each such diagram one associates in a linear way an element of $v(R/\Lambda)$ which vanishes if and only if there is an extension $\varphi' : R \to B$ making [I.3.4] commutative. Then the Proposition applies.

Taking $\Lambda = \mathbf{k}$ we get the following absolute version of the Proposition, where for the last assertion we apply (I.3.2)(ii).

(I.3.8) COROLLARY *Let R be in $\mathcal{A}^*$. Assume that there exists a* **k**-*vector space $v(R)$ such that for every morphism $\varphi : R \to A$ in $\mathcal{A}^*$ with A in $\mathcal{A}$ there is a* **k**-*linear map*

$$\varphi_v : o(A) \to v(R)$$

satisfying

$$\ker(\varphi_v) = \ker[o(\varphi)]$$

Then there is a natural **k**-*linear inclusion*

$$o(R) \subset v(R)$$

If $v(R)$ is finite dimensional then

$$\dim_{\mathbf{k}}(t_R) \geq \dim(R) \geq \dim_{\mathbf{k}}(t_R) - \dim_{\mathbf{k}}[v(R)]$$

A **k**-vector space $v(R)$ satisfying the conditions of (I.3.8) will sometimes be called *an obstruction space for R.*

I.4. EXTENSIONS OF SCHEMES

Let $X \to S$ be a morphism of schemes. An *extension* of X/S is a closed immersion $X \subset X'$, where X' is an S-scheme, defined by a sheaf of ideals $\mathcal{I} \subset \mathcal{O}_{X'}$ such that $\mathcal{I}^2 = 0$. It follows that $\mathcal{I}$ is, in a natural way, a sheaf of $\mathcal{O}_X$-modules, which coincides with the conormal sheaf of $X \subset X'$. To give an extension $X \subset X'$ of X/S is equivalent to giving an exact sequence on X:

$$\mathcal{E}: \quad 0 \to \mathcal{I} \to \mathcal{O}_{X'} \xrightarrow{\varphi} \mathcal{O}_X \to 0$$

where $\mathcal{I}$ is an $\mathcal{O}_X$-module, φ is a homomorphism of $\mathcal{O}_S$-algebras and $\mathcal{I}^2 = 0$ in $\mathcal{O}_{X'}$; we call $\mathcal{E}$ *an extension of* X/S *by* $\mathcal{I}$ or *with kernel* $\mathcal{I}$. Two such extensions $\mathcal{O}_{X'}$ and $\mathcal{O}_{X''}$ are called *isomorphic* if there is an $\mathcal{O}_S$-homomorphism $\alpha : \mathcal{O}_{X'} \to \mathcal{O}_{X''}$ inducing the identity on both $\mathcal{I}$ and $\mathcal{O}_X$. It follows that α must necessarily be an S-isomorphism.

We denote by $\text{Ex}(X/S, \mathcal{I})$ the set of isomorphism classes of extensions of X/S with kernel $\mathcal{I}$. In case $\text{Spec}(B) \to \text{Spec}(A)$ is a morphism of affine schemes and $\mathcal{I} = \tilde{M}$ we have an obvious identification:

$$\text{Ex}_A(B, M) = \text{Ex}(X/S, \mathcal{I})$$

If $S = \text{Spec}(A)$ is affine we will sometimes write $\text{Ex}_A(X, \mathcal{I})$ instead of $\text{Ex}(X/\text{Spec}(A), \mathcal{I})$. Exactly as in the affine case one proves that $\text{Ex}(X/S, \mathcal{I})$ is a $\Gamma(X, \mathcal{O}_X)$-module with identity element the class of the extension:

$$0 \to \mathcal{I} \to \mathcal{O}_X \tilde{\oplus} \mathcal{I} \to \mathcal{O}_X \to 0$$

where $\mathcal{O}_X \tilde{\oplus} \mathcal{I}$ is defined as in the affine case (see section I.1). The correspondence

$$\mathcal{I} \mapsto \text{Ex}(X/S, \mathcal{I})$$

defines a covariant functor from $\mathcal{O}_X$-modules to $\Gamma(X, \mathcal{O}_X)$-modules.

In deformation theory the case $\mathcal{I} = \mathcal{O}_X$ is the most important one, being related to first order deformations. If more generally $\mathcal{I}$ is a locally free sheaf we get the notions of *ribbon, carpet* etc. (see Bayer-Eisenbud(1995)).

(I.4.1) PROPOSITION *Let $A \to B$ be an e.f.t. ring homomorphism and let $B = P/J$ where P is a smooth A-algebra. Then for every B-module M we have an exact sequence:*

$$[I.4.1] \qquad \text{Der}_A(P, M) \to \text{Hom}_B(J/J^2, M) \to \text{Ex}_A(B, M) \to 0$$

If $A \to B$ is a smooth homomorphism then $\mathrm{Ex}_A(B,M)=0$ *for every B-module M.*

Proof

We have a natural surjective homomorphism

$$\begin{array}{ccc} \mathrm{Hom}_B(J/J^2,M) & \to & \mathrm{Ex}_A(B,M) \\ \lambda & \mapsto & \lambda_*(\eta) \end{array}$$

where

$$\eta : 0 \to J/J^2 \to P/J^2 \to B \to 0$$

The surjectivity follows from the fact that η is versal. The extension $\lambda_*(\eta)$ is trivial if and only if we have a commutative diagram

$$\begin{array}{ccccccc} 0\to & J/J^2 & \to & P/J^2 & \to & B & \to 0 \\ & \downarrow\lambda & & \downarrow & & \| & \\ 0\to & M & \to & B\tilde{\oplus}M & \to & B & \to 0 \end{array}$$

if and only if λ extends to an A-derivation $\bar{D} : P/J^2 \to M$, equivalently to an A-derivation $D : P \to M$. The last assertion is immediate (see Theorem (I.2.9)). *q.e.d.*

(I.4.2) COROLLARY *If $A \to B$ is an e.f.t. ring homomorphism and M is a finitely generated B-module then* $\mathrm{Ex}_A(B,M)$ *is a finitely generated B-module. In particular $T^1_{B/A}$ is a finitely generated B-module and we have an exact sequence:*

$$[I.4.2] \qquad \mathrm{Hom}_B(\Omega_{P/A}\otimes_P B, B) \to \mathrm{Hom}_B(J/J^2,B) \to T^1_{B/A} \to 0$$

if $B = P/J$ for a smooth A-algebra P and an ideal $J \subset P$.

Proof

It is a direct consequence of the exact sequence [I.4.2]. *q.e.d.*

Using the fact that the exact sequence [I.4.2] localizes, it is immediate to check that the cotangent module localizes. More specifically it is straightforward to show that given a morphism of finite type of schemes $f : X \to S$ one can define a quasi-coherent sheaf $T^1_{X/S}$ on X with the following properties. If $U = \mathrm{Spec}(A)$ is an affine open subset of S and $V = \mathrm{Spec}(B)$ is an affine open subset of $f^{-1}(U)$, then

$$\Gamma(V, T^1_{X/S}) = T^1_{B/A}$$

It follows from the properties of the cotangent modules that $T^1_{X/S}$ is coherent if $X \to S$ is of finite type. $T^1_{X/S}$ is called the *first cotangent sheaf* of X/S. We will write T^1_X if $S = \mathrm{Spec}(\mathbf{k})$.

T^1_X is supported on the singular locus of X. More generally $T^1_{X/S}$ is supported on the locus where X is not smooth over S. If we have a closed embedding $X \subset Y$ with Y nonsingular, then we have an exact sequence of coherent sheaves on X:

$$[I.4.3] \qquad 0 \to T_X \to T_{Y|X} \to N_{X/Y} \to T^1_X \to 0$$

which globalizes the exact sequence [I.4.2].

For every scheme S and morphism of S-schemes $f : X \to Y$ we have an exact sequence of sheaves

$$[I.4.4] \quad 0 \to \mathrm{T}_{X/Y} \to T_{X/S} \to Hom(f^*\Omega^1_{Y/S}, \mathcal{O}_X) \to T^1_{X/Y} \to T^1_{X/S} \to f^*T^1_{Y/S}$$

which globalizes the exact sequence of Proposition (I.1.4). When $S = \mathrm{Spec}(\mathbf{k})$ and f is a closed embedding of algebraic schemes, with Y nonsingular, we obtain [I.4.3] as a special case of [I.4.4] (note that $\mathrm{T}_{X/Y} = 0$ and $N_{X/Y} = T^1_{X/Y}$ in this case, as it follows from [I.4.2] and (I.1.4)). The following is a basic result:

(I.4.3) THEOREM *Let S be an algebraic scheme, X a reduced algebraic S-scheme and $\mathcal{I}$ a coherent locally free sheaf on X. Then there is a canonical identification*

$$\mathrm{Ex}(X/S, \mathcal{I}) = \mathrm{Ext}^1_{\mathcal{O}_X}(\Omega^1_{X/S}, \mathcal{I})$$

which to the isomorphism class of an extension of X/S:

$$\mathcal{E} : 0 \to \mathcal{I} \to \mathcal{O}_{X'} \to \mathcal{O}_X \to 0$$

associates the isomorphism class of the relative conormal sequence of $X \subset X'$:

$$c_{\mathcal{E}} : 0 \to \mathcal{I} \xrightarrow{\delta} (\Omega^1_{X'/S})_{|X} \to \Omega^1_{X/S} \to 0$$

(which is exact also on the left).

Proof

Suppose given an extension $\mathcal{E}$. Since $\mathcal{I}$ is locally free in order to show that $c_{\mathcal{E}}$ is exact on the left it suffices to prove that $\ker(\delta)$ is torsion, equivalently that $c_{\mathcal{E}}$ is exact near every general closed point x of any irreducible component of X. Since X is reduced it is nonsingular at x: it follows from (I.4.1) that there is an affine open neighborhood U of x such that $\mathcal{E}_{|U}$ is trivial. From Theorem (A.1.3) we deduce that the relative conormal sequence of $\mathcal{E}_{|U}$ is split exact; since it coincides with the restriction of $c_{X'}$ to U we see that $\delta_{|U}$ is injective; this shows that $\ker(\delta)$ is torsion and $c_{\mathcal{E}}$ is exact. Since clearly isomorphic extensions have isomorphic relative cotangent sequences we have a well defined map

$$c_{-} : \mathrm{Ex}(X/S, \mathcal{I}) \to \mathrm{Ext}^1_{\mathcal{O}_X}(\Omega^1_{X/S}, \mathcal{I})$$

Let now

$$\eta : 0 \to \mathcal{I} \to \mathcal{A} \xrightarrow{p} \Omega^1_{X/S} \to 0$$

define an element of $\mathrm{Ext}^1_{\mathcal{O}_X}(\Omega^1_{X/S}, \mathcal{I})$. Letting $d : \mathcal{O}_X \to \Omega^1_{X/S}$ be the canonical derivation, consider the sheaf of $\mathcal{O}_S$ algebras $\mathcal{O} = \mathcal{A} \times_{\Omega^1_{X/S}} \mathcal{O}_X$: over an open subset $U \subset X$ we have $\Gamma(U, \mathcal{O}) = \{(a, f) : p(a) = d(f)\}$ and the multiplication rule is

$$(a, f)(a', f') = (fa' + f'a, ff')$$

Then we have an exact commutative diagram

$$\begin{array}{ccccccc} 0 \to & \mathcal{I} & \to & \mathcal{O} & \to & \mathcal{O}_X & \to 0 \\ & \| & & \downarrow \bar{d} & & \downarrow d & \\ 0 \to & \mathcal{I} & \to & \mathcal{A} & \to & \Omega^1_{X/S} & \to 0 \end{array}$$

where one immediately checks that the projection $\bar{d}$ is an $\mathcal{O}_S$-derivation and therefore it must factor as

$$\mathcal{O} \to \Omega^1_{\mathcal{O}/\mathcal{O}_S} \otimes_{\mathcal{O}} \mathcal{O}_X \to \mathcal{A}$$

and we have an exact commutative diagram

$$[I.4.5] \qquad \begin{array}{ccccccc} 0 \to & \mathcal{I} & \to & \mathcal{O} & \to & \mathcal{O}_X & \to 0 \\ & \| & & \downarrow & & \downarrow d & \\ & \mathcal{I} & \to & \Omega^1_{\mathcal{O}/\mathcal{O}_S} \otimes_{\mathcal{O}} \mathcal{O}_X & \to & \Omega^1_{X/S} & \to 0 \\ & \| & & \downarrow & & \| & \\ 0 \to & \mathcal{I} & \to & \mathcal{A} & \to & \Omega^1_{X/S} & \to 0 \end{array}$$

which implies $\Omega^1_{\mathcal{O}/\mathcal{O}_S} \otimes_{\mathcal{O}} \mathcal{O}_X \cong \mathcal{A}$. Therefore, letting e_η be the extension given by the firt row of [I.4.5], we see that $c_{e_\eta} = \eta$. Similarly one shows that $e_{c_\mathcal{E}} = \mathcal{E}$ for any $[\mathcal{E}] \in \mathrm{Ex}(X/S, \mathcal{I})$. Therefore c_- and e_- are inverse of each other and the conclusion follows. *q.e.d.*

(I.4.4) COROLLARY *Let $X \to S$ be a morphism of finite type of algebraic schemes with X reduced. Then there is a canonical isomorphism of coherent sheaves on X:*

$$T^1_{X/S} \cong Ext^1_{\mathcal{O}_X}(\Omega^1_{X/S}, \mathcal{O}_X)$$

Proof

Just apply the previous Theorem using the fact that both members localize. *q.e.d.*

NOTES

1. A closer analysis of the proof of Theorem (I.4.3) shows that, without assuming X reduced, we only have inclusions

$$\mathrm{Ext}^1_{\mathcal{O}_X}(\Omega^1_{X/S}, \mathcal{I}) \subset \mathrm{Ex}(X/S, \mathcal{I})$$

and

$$Ext^1_{\mathcal{O}_X}(\Omega^1_{X/S}, \mathcal{O}_X) \subset T^1_{X/S}$$

2. The topics of this section originate from Grothendieck(1968). The proof of Theorem (I.4.3) has been taken from Gonzales(2004).