Chapter II. Infinitesimal deformations

In this Chapter we study several deformation problems from an elementary point of view. We will be especially concerned in *first order deformations* and *obstructions* and in giving them appropriate cohomological interpretations.

II.1. NONSINGULAR VARIETIES

Generalities on deformations

Let X be an algebraic scheme. A cartesian diagram of morphisms of schemes

$$\eta : \begin{array}{ccc} X & \to & \mathcal{X} \\ \downarrow & & \downarrow \\ \mathrm{Spec}(\mathbf{k}) & \xrightarrow{s} & S \end{array}$$

where $\mathcal{X}$ is flat over S is called a *family of deformations* of X parametrized by S, or over S; when $S = \mathrm{Spec}(A)$ with A in $\mathcal{A}^*$ and $s \in S$ is the closed point we have a *local family of deformations* (shortly a *local deformation*) of X over A. The deformation η will be also denoted (S, η) or (A, η) when $S = \mathrm{Spec}(A)$. The local deformation (A, η) is *infinitesimal* (resp. *first order*) if $A \in \mathrm{ob}(\mathcal{A})$ (resp. $A = \mathbf{k}[\epsilon]$). Given another deformation

$$\xi : \begin{array}{ccc} X & \to & \mathcal{Y} \\ \downarrow & & \downarrow \\ \mathrm{Spec}(\mathbf{k}) & \to & S \end{array}$$

of X over S, an isomorphism of η with ξ is an S-isomorphism $\phi : \mathcal{X} \to \mathcal{Y}$ inducing the identity on X, i.e. such that the following diagram is commutative:

$$\begin{array}{ccccc} & & X & & \\ & \swarrow & & \searrow & \\ \mathcal{X} & & \xrightarrow{\phi} & & \mathcal{Y} \\ & \searrow & & \swarrow & \\ & & S & & \end{array}$$

Observe that for every X and for every pointed scheme (S, s), with $s \in S$ a $\mathbf{k}$-rational point there always exists at least a family of deformation of X over S, namely the *product family*:

$$\begin{array}{ccc} X & \to & X \times S \\ \downarrow & & \downarrow \\ \mathrm{Spec}(\mathbf{k}) & \to & S \end{array}$$

A deformation of X over S is called *trivial* if it is isomorphic to the product family. The scheme X is called *rigid* if every infinitesimal deformation of X over A is trivial for every A in $\mathcal{A}$.

Given a deformation η of X over S as above and a morphism $(S', s') \to (S, s)$ of pointed schemes there is an induced a commutative diagram by base change

$$\begin{array}{ccc} X & \to & \mathcal{X} \times_S S' \\ \downarrow & & \downarrow \\ \mathrm{Spec}(\mathbf{k}) & \to & S' \end{array}$$

which is clearly a deformation of X over S'. This operation is functorial, in the sense that it commutes with composition of morphisms and the identity morphism does not change η. Moreover it carries isomorphic deformations to isomorphic ones.

An infinitesimal deformation η of X is called *locally trivial* if every point $x \in X$ has an affine open neighborhood $U_x \subset X$ such that $\mathcal{X}_{|U_x}$ is a trivial deformation of U_x.

(II.1.1) EXAMPLES (i) The quadric $Q \subset \mathbf{A}^3$ with equation $xy - t = 0$ defines, via the projection

$$\begin{array}{ccc} \mathbf{A}^3 & \to & \mathbf{A}^1 \\ (x, y, t) & \mapsto & t \end{array}$$

a flat family $Q \to \mathbf{A}^1$ whose fibres are conics. This family is not trivial since the fibre $Q(0)$ is singular, hence not isomorphic to the fibres $Q(t)$, $t \neq 0$, which are nonsingular.

(ii) Consider a rational ruled surface F_m, $m > 0$. The structural morphism $\pi : F_m \to \mathbb{P}^1$ defines a flat family whose fibres are all isomorphic to $\mathbb{P}^1$; but π is not a trivial family because $F_m \not\cong F_0$ (see Example (A.1.10)(iii)).

(iii) Let $0 \leq n < m$ be two distinct nonnegative integers having the same parity and let $k = \frac{1}{2}(m-n)$. Consider two copies of $\mathbf{A}^2 \times \mathbb{P}^1$ given as $\mathrm{Proj}(\mathbf{k}[t, z, \xi_0, \xi_1]) =: W$ and $\mathrm{Proj}(\mathbf{k}[t, z', \xi'_0, \xi'_1]) =: W'$ (here the rings are graded with respect to the variables ξ_i and ξ'_i). Letting $\xi = \xi_1/\xi_0$ and $\xi' = \xi'_1/\xi'_0$ consider the open subsets

$$\mathrm{Spec}(\mathbf{k}[t, z, \xi]) \subset W$$

$$\mathrm{Spec}(\mathbf{k}[t, z', \xi']) \subset W'$$

and glue them together along the open subsets $\mathrm{Spec}(\mathbf{k}[t, z, z^{-1}, \xi])$ and $\mathrm{Spec}(\mathbf{k}[t, z', z'^{-1}, \xi'])$ according to the following rules:

$$[II.1.1] \qquad z' = z^{-1}, \quad \xi' = z^m \xi + t z^k$$

This induces a gluing of W and W' along $\mathrm{Proj}(\mathbf{k}[t, z, z^{-1}, \xi_0, \xi_1])$ and $\mathrm{Proj}(\mathbf{k}[t, z', z'^{-1}, \xi'_0, \xi'_1])$; call the resulting scheme $\mathcal{W}$ and $f : \mathcal{W} \to \mathbf{A}^1 = \mathrm{Spec}(\mathbf{k}[t])$ the morphism induced by the projections. Then f is a flat morphism because it is locally a projection; moreover

$$\mathcal{W}(0) \cong F_m$$

Let $\mathcal{W}^\circ = f^{-1}(\mathbf{A}^1 \backslash \{0\})$ and $f^\circ : \mathcal{W}^\circ \to \mathbf{A}^1 \backslash \{0\}$ the restriction of f.

In $\mathbf{k}[t, t^{-1}, z, \xi]$ define

$$\zeta = \frac{z^k \xi - t}{t\xi}$$

and in $\mathbf{k}[t, t^{-1}, z', \xi']$

$$\zeta' = \frac{\xi'}{tz'^{m-k}\xi' + t^2}$$

It is straightforward to verify that the gluing [II.1.1] induces the relation

$$\zeta' = z^n \zeta$$

This means that we have an isomorphism

$$\mathcal{W}^\circ \cong F_n \times (\mathbf{A}^1 \backslash \{0\})$$

compatible with the projections to $\mathbf{A}^1 \backslash \{0\}$. Therefore the family f° is trivial, in particular all its fibres are isomorphic to F_n, but the family f is not trivial because $\mathcal{W}(0) \cong F_m$.

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Infinitesimal deformations of nonsingular affine schemes

We will start by considering infinitesimal deformations of affine schemes. We need the following

(II.1.2) LEMMA *Let Z_0 be a closed subscheme of a scheme Z, defined by a sheaf of nilpotent ideals $N \subset \mathcal{O}_Z$. If Z_0 is affine then Z is affine as well.*

Proof
Let $r \geq 2$ be the smallest integer such that $N^r = (0)$. Since we have a chain of inclusions

$$Z \supset V(N^{r-1}) \supset V(N^{r-2}) \supset \cdots \supset V(N) = Z_0$$

it suffices to prove the assertion in the case $r = 2$. In this case N is a coherent $\mathcal{O}_{Z_0}$-module, and therefore

$$H^1(Z, N) = H^1(Z_0, N) = 0$$

Let R_0 be the $\mathbf{k}$-algebra such that $Z_0 = \mathrm{Spec}(R_0)$. We have the exact sequence:

$$0 \to H^0(Z, N) \to H^0(Z, \mathcal{O}_Z) \to R_0 \to 0$$

Put $R = H^0(Z, \mathcal{O}_Z)$ and let $Z' = \mathrm{Spec}(R)$. We have a commutative diagram:

$$\begin{array}{ccc} Z & \xrightarrow{\theta} & Z' \\ & \nwarrow \quad \nearrow & \\ & Z_0 & \end{array}$$

The sheaf homomorphism $\theta^{-1}\mathcal{O}_{Z'} \to \mathcal{O}_Z$ is clearly injective and θ is a homeomorphism. It will therefore suffice to prove that $\theta^{-1}\mathcal{O}_{Z'} \to \mathcal{O}_Z$ is surjective.

Let $z \in Z$ and $f \in \Gamma(U, \mathcal{O}_Z)$ for some affine open neighborhood U of z. Let $f_0 = f_{|U \cap Z_0}$. It is possible to find $\varphi_0, \psi_0 \in R_0$ such that $f_0 = \frac{\varphi_0}{\psi_0}$, $\psi_0(z) \neq 0$ and $\psi_0 = 0$ on $Z_0 \backslash U$, because Z_0 is affine. Let $\psi \in R$ be such that $\psi_{|Z_0} = \psi_0$ (it exists by the surjectivity of $R \to R_0$). Then $\psi(z) \neq 0$ and $\psi = 0$ on $Z \backslash U$. There exists $n \gg 0$ such that $\psi^n f =: g \in R$ (it suffices to cover Z with affines). Then $f = \frac{g}{\psi^n} \in \theta^{-1}\mathcal{O}_{Z'}$. *q.e.d.*

Let B_0 be a **k**-algebra, and let $X_0 = \mathrm{Spec}(B_0)$. Consider an infinitesimal deformation of X_0 parametrized by $\mathrm{Spec}(A)$, where A is in $\mathcal{A}$. By definition this is a cartesian diagram

$$\begin{array}{ccc} X_0 & \to & \mathcal{X} \\ \downarrow & & \downarrow \\ \mathrm{Spec}(\mathbf{k}) & \to & \mathrm{Spec}(A) \end{array}$$

where $\mathcal{X}$ is a scheme flat over $\mathrm{Spec}(A)$. By Lemma (II.1.2) $\mathcal{X}$ is necessarily affine. Therefore, equivalently, we can talk about an *infinitesimal deformation of* B_0 *over* A as a cartesian diagram of **k**-algebras:

$$\begin{array}{ccc} B & \to & B_0 \\ \uparrow & & \uparrow \\ A & \to & \mathbf{k} \end{array} \qquad [II.1.2]$$

with $A \to B$ flat. Note that to give this diagram is the same as to give $A \to B$ flat and a **k**-isomorphism $B \otimes_A \mathbf{k} \to B_0$. We will sometimes abbreviate by calling $A \to B$ the deformation.

Given another deformation $A \to B'$ of B_0 over A, an *isomorphism* of $A \to B$ to $A \to B'$ is a homomorphism $\varphi : B \to B'$ of A-algebras inducing a commutative diagram:

$$\begin{array}{ccccc} & & B_0 & & \\ & \nearrow & & \nwarrow & \\ B & & \xrightarrow{\varphi} & & B' \\ & \nwarrow & & \nearrow & \\ & & A & & \end{array}$$

It follows from Lemma (A.2.3) that such a φ is an isomorphism.

An infinitesimal deformation of B_0 over A is trivial if it is isomorphic to the product deformation

$$\begin{array}{ccc} B_0 \otimes_{\mathbf{k}} A & \to & B_0 \\ \uparrow & & \uparrow \\ A & \to & \mathbf{k} \end{array}$$

The **k**-algebra B_0 is called *rigid* if $\mathrm{Spec}(B_0)$ is rigid.

(II.1.3) THEOREM *Every smooth* **k**-*algebra is rigid. In particular every affine nonsingular algebraic variety is rigid.*

Proof
Suppose $\mathbf{k} \to B_0$ is smooth, and suppose given a first order deformation of B_0:

$$\eta_0 : \begin{array}{ccc} B & \to & B_0 \\ \uparrow f & & \uparrow \\ \mathbf{k}[\epsilon] & \to & \mathbf{k} \end{array}$$

Consider the commutative diagram:

$$\begin{array}{ccc} B & \to & B_0 \\ \uparrow f & & \uparrow \\ \mathbf{k}[\epsilon] & \to & B_0[\epsilon] \end{array}$$

where $B_0[\epsilon] = B_0 \otimes \mathbf{k}[\epsilon]$. Since f is smooth (because flat with smooth fibre) and the right vertical morphism is a $\mathbf{k}[\epsilon]$-extension, there exists a $\mathbf{k}[\epsilon]$-homomorphism $\phi : B \to B_0[\epsilon]$ making the diagram

$$\begin{array}{ccc} B & \to & B_0 \\ \uparrow f & \searrow & \uparrow \\ \mathbf{k}[\epsilon] & \to & B_0[\epsilon] \end{array}$$

commutative. Therefore ϕ is an isomorphism of deformations and η_0 is trivial.

Consider more generally a deformation of B_0

$$\eta : \begin{array}{ccc} B & \to & B_0 \\ \uparrow f & & \uparrow \\ A & \to & \mathbf{k} \end{array}$$

parametrized by A in $\mathcal{A}$. To show that η is trivial we proceed by induction on $d = \dim_{\mathbf{k}}(A)$. The case $d = 2$ has been already proved; assume $d \geq 3$ and let

$$0 \to \mathbf{k} \to A \to A' \to 0$$

be a small extension. Consider the commutative diagram:

$$\begin{array}{cccc} B & \to & B \otimes_A A' & \cong B_0 \otimes_{\mathbf{k}} A' \\ \uparrow f & & \uparrow & \\ A & \to & B_0 \otimes_{\mathbf{k}} A & \end{array}$$

f is smooth, the upper right isomorphism is by the inductive hypothesis, and the right vertical homomorphism is an A-extension. By the smoothness of f we deduce the existence of an A-homomorphism $B \to B_0 \otimes_{\mathbf{k}} A$ which is an isomorphism of deformations. *q.e.d.*

(II.1.4) EXAMPLE Let $\lambda \in \mathbf{k}$ and $B_0 = \mathbf{k}[X,Y]/(Y^2 - X(X-1)(X-\lambda))$. If $\lambda \neq 0$ then B_0 is a smooth $\mathbf{k}$-algebra, being the coordinate ring of a nonsingular plane cubic curve. By Theorem (II.1.3) B_0 is rigid. On the other hand the elementary theory of elliptic curves (see Hartshorne(1977)) shows that the following flat family of affine curves

$$\begin{array}{c}\mathrm{Spec}\mathbf{k}[X,Y]/(Y^2 - X(X-1)(X-(\lambda+t))) \\ \downarrow \\ \mathrm{Spec}(\mathbf{k}[t])\end{array}$$

is not trivial around the origin $t = 0$ so that it defines a non-trivial (non-infinitesimal) deformation of B_0. This example shows that by studying infinitesimal deformations we are loosing something.

(II.1.5) LEMMA *Let B_0 be a* **k***-algebra, and*

$$e : 0 \to (\epsilon) \to \tilde{A} \to A \to 0$$

a small extension in $\mathcal{A}$. There is a 1-1 correspondence

$$\left\{\begin{array}{c}\textit{automorphisms of the trivial deformation } B_0 \otimes_{\mathbf{k}} \tilde{A} \\ \textit{inducing the identity on } B_0 \otimes_{\mathbf{k}} A\end{array}\right\} \leftrightarrow \mathrm{Der}_{\mathbf{k}}(B_0, B_0)$$

where the identity corresponds to the zero derivation, and the composition of automorphisms corresponds to the sum of derivations.

Proof

Every automorphism $\theta : B_0 \otimes_{\mathbf{k}} \tilde{A} \to B_0 \otimes_{\mathbf{k}} \tilde{A}$ must be $\tilde{A}$-linear and induce the identity mod ϵ. Therefore:

$$\theta(x) = x + \epsilon dx$$

where $d : B_0 \otimes_{\mathbf{k}} \tilde{A} \to B_0$ is a $\tilde{A}$-derivation (Lemma (I.1.1)). But

$$\mathrm{Der}_{\tilde{A}}(B_0 \otimes_{\mathbf{k}} \tilde{A}, B_0) = \mathrm{Hom}_{B_0 \otimes_{\mathbf{k}} \tilde{A}}(\Omega_{B_0 \otimes_{\mathbf{k}} \tilde{A}/\tilde{A}}, B_0) = \mathrm{Hom}_{B_0}(\Omega_{B_0/\mathbf{k}}, B_0) = \mathrm{Der}_{\mathbf{k}}(B_0, B_0)$$

Clearly the identity corresponds to the zero derivation. If we compose two automorphisms:

$$B_0 \otimes_{\mathbf{k}} \tilde{A} \xrightarrow{\theta} B_0 \otimes_{\mathbf{k}} \tilde{A} \xrightarrow{\sigma} B_0 \otimes_{\mathbf{k}} \tilde{A}$$

where $\theta(x) = x + \epsilon dx$, $\sigma(x) = x + \epsilon \delta x$, we obtain:

$$\sigma(\theta(x)) = \theta(x) + \epsilon\delta(\theta(x)) = x + \epsilon dx + \epsilon(\delta x + \epsilon\delta(dx)) = x + \epsilon(dx + \delta x)$$

q.e.d.

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First order deformations of nonsingular varieties

We will now apply (II.1.5) to deformations of any nonsingular algebraic variety.

(II.1.6) PROPOSITION *Let X be a nonsingular algebraic variety. There is a 1-1 correspondence:*

$$\kappa : \frac{\left\{\text{first order deformations of } X\right\}}{\text{isomorphism}} \to H^1(X, T_X)$$

called the Kodaira-Spencer correspondence, *where* $T_X = Hom(\Omega^1_X, \mathcal{O}_X) = Der_{\mathbf{k}}(\mathcal{O}_X, \mathcal{O}_X)$, *such that* $\kappa(\xi) = 0$ *if and only if* ξ *is the trivial deformation class.*

Proof

Given a first order deformation

$$\begin{array}{ccc} X & \to & \mathcal{X} \\ \downarrow & & \downarrow \\ \mathrm{Spec}(\mathbf{k}) & \to & \mathrm{Spec}(\mathbf{k}[\epsilon]) \end{array}$$

choose an affine open cover $\mathcal{U} = \{U_i\}_{i \in I}$ of X such that $U_{ij} := U_i \cap U_j$ and $U_{ijk} := U_i \cap U_j \cap U_k$ are affine for every $i, j, k \in I$. For each index i we have an isomorphism of deformations:

$$\theta_i : U_i \times \mathrm{Spec}(\mathbf{k}[\epsilon]) \to \mathcal{X}_{|U_i}$$

by (II.1.3). Then for each $i, j \in I$

$$\theta_{ij} := \theta_i^{-1}\theta_j : U_{ij} \times \mathrm{Spec}(\mathbf{k}[\epsilon]) \to U_{ij} \times \mathrm{Spec}(\mathbf{k}[\epsilon])$$

is an automorphism of the trivial deformation $U_{ij} \times \mathrm{Spec}(\mathbf{k}[\epsilon])$. By Lemma (II.1.5) θ_{ij} corresponds to a $d_{ij} \in \Gamma(U_{ij}, T_X)$. Since on each U_{ijk} we have

$$[II.1.3] \qquad \theta_{ij}\theta_{jk}\theta_{ik}^{-1} = 1_{U_{ijk} \times \mathrm{Spec}(\mathbf{k}[\epsilon])}$$

it follows that

$$d_{ij} + d_{jk} - d_{ik} = 0$$

i.e. $\{d_{ij}\}$ is a Chech 1-cocycle and therefore defines an element of $H^1(X, T_X)$. It is easy to check that this element does not depend on the choice of the open cover $\mathcal{U}$. If we have another deformation

$$\begin{array}{ccc} X & \to & \mathcal{X}' \\ \downarrow & & \downarrow \\ \mathrm{Spec}(\mathbf{k}) & \to & \mathrm{Spec}(\mathbf{k}[\epsilon]) \end{array}$$

and $\Phi : \mathcal{X} \to \mathcal{X}'$ is an isomorphism of deformations then for each $i \in I$ there is induced an automorphism:

$$\alpha_i : U_i \times \mathrm{Spec}(\mathbf{k}[\epsilon]) \xrightarrow{\theta_i} \mathcal{X}_{|U_i} \xrightarrow{\Phi_{|U_i}} \mathcal{X}'_{|U_i} \xrightarrow{\theta_i'^{-1}} U_i \times \mathrm{Spec}(\mathbf{k}[\epsilon])$$

and therefore a corresponding $a_i \in \Gamma(U_i, T_X)$. We have $\theta'_i\alpha_i = \Phi_{|U_i}\theta_i$ and therefore

$$(\theta'_i\alpha_i)^{-1}(\theta'_j\alpha_j) = \theta_i^{-1}\Phi^{-1}_{|U_{ij}}\Phi_{|U_{ij}}\theta_j = \theta_i^{-1}\theta_j$$

thus

$$\alpha_i^{-1}\theta'_{ij}\alpha_j = \theta_{ij}$$

equivalently:

$$d'_{ij} + a_j - a_i = d_{ij}$$

namely $\{d_{ij}\}$ and $\{d'_{ij}\}$ are cohomologous, and therefore define the same element of $H^1(X, T_X)$.

Conversely, given $\theta \in H^1(X, T_X)$ we can represent it by a Chech 1-cocycle $\{d_{ij}\} \in \mathcal{Z}^1(\mathcal{U}, T_X)$ with respect to some affine open cover $\mathcal{U}$. To each d_{ij} we can associate an automorphism θ_{ij} of the trivial deformation $U_{ij} \times \mathrm{Spec}(\mathbf{k}[\epsilon])$ by Lemma (II.1.5). They satisfy the identities [II.1.3]. We can therefore use these automorphisms to patch the schemes $U_i \times \mathrm{Spec}(\mathbf{k}[\epsilon])$ by the well known procedure (see Hartshorne(1977), p. 69). We obtain a $\mathrm{Spec}(\mathbf{k}[\epsilon])$-scheme $\mathcal{X}$ which is immediately checked to define a first order deformation of X. The last assertion is easily proved. *q.e.d.*

For every first order deformation ξ the cohomology class $\kappa(\xi) \in H^1(X, T_X)$ is called the *Kodaira-Spencer class* of ξ.

(II.1.7) REMARK In Lemma (II.1.5) we did not assume B_0 to be nonsingular. Therefore the proof of the above Theorem applies to locally trivial deformations of an algebraic scheme and it shows that $H^1(X, T_X)$ *classifies isomorphism classes of first order locally trivial deformations of any algebraic scheme.*

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Higher order deformations - Obstructions

Let X be a nonsingular algebraic variety. Consider a small extension

$$e : 0 \to (\epsilon) \to \tilde{A} \to A \to 0$$

in $\mathcal{A}$ and let

$$\xi : \begin{array}{ccc} X & \to & \mathcal{X} \\ \downarrow & & \downarrow \\ \mathrm{Spec}(\mathbf{k}) & \to & \mathrm{Spec}(A) \end{array}$$

be an infinitesimal deformation of X. A *lifting of* ξ *to* $\tilde{A}$ is a deformation

$$\tilde{\xi} : \begin{array}{ccc} X & \to & \tilde{\mathcal{X}} \\ \downarrow & & \downarrow \\ \mathrm{Spec}(\mathbf{k}) & \to & \mathrm{Spec}(\tilde{A}) \end{array}$$

which induces ξ by pullback. It is important to know whether, given ξ and e, a lifting of ξ to $\tilde{A}$ exists, and how many are there. The following Proposition addresses this question.

(II.1.8) PROPOSITION *Given A in $\mathcal{A}$ and an infinitesimal deformation ξ of X over A:*
(i) To every small extension e of A there is associated an element $o_\xi(e) \in H^2(X, T_X)$, called the obstruction to lift ξ to $\tilde{A}$, *which is* 0 *if and only if a lifting of ξ to $\tilde{A}$ exists.*
(ii) If $o_\xi(e) = 0$ then there is a natural transitive action of $H^1(X, T_X)$ on the set of isomorphism classes of liftings of ξ to $\tilde{A}$.

Proof
Let $\mathcal{U} = \{U_i\}_{i\in I}$ be an affine open cover of X such that U_{ij} and U_{ijk} are affine for every $i, j, k \in I$. We have isomorphisms

$$\theta_i : U_i \times \mathrm{Spec}(A) \to \mathcal{X}_{|U_i}$$

and consequently $\theta_{ij} := \theta_i^{-1}\theta_j$ is an automorphism of the trivial deformation $U_{ij} \times \mathrm{Spec}(A)$. Moreover

$$[II.1.8] \qquad \theta_{ij}\theta_{jk} = \theta_{ik}$$

on $U_{ijk} \times \mathrm{Spec}(A)$. To give a lifting $\tilde{\xi}$ of ξ to $\tilde{A}$ it is necessary and sufficient to give a collection of automorphisms $\{\tilde{\theta}_{ij}\}$ of the trivial deformations $U_{ij} \times \mathrm{Spec}(\tilde{A})$ such that
(a) $\tilde{\theta}_{ij}\tilde{\theta}_{jk} = \tilde{\theta}_{ik}$
(b) $\tilde{\theta}_{ij}$ restricts to θ_{ij} on $U_{ij} \times \mathrm{Spec}(A)$

To establish the existence of the collection $\{\tilde{\theta}_{ij}\}$ let's choose arbitrarily automorphisms $\{\tilde{\theta}_{ij}\}$ satisfying the condition (b). Let

$$\tilde{\theta}_{ijk} = \tilde{\theta}_{ij}\tilde{\theta}_{jk}\tilde{\theta}_{ik}^{-1}$$

This is an automorphism of the trivial deformation $U_{ijk} \times \mathrm{Spec}(\tilde{A})$. Since by [II.1.8] it restricts on $U_{ijk} \times \mathrm{Spec}(A)$ to the identity, by Lemma (II.1.5) we can identify each $\tilde{\theta}_{ijk}$ with a $\tilde{d}_{ijk} \in \Gamma(U_{ijk}, T_X)$ and it is immediate to check that $\{\tilde{d}_{ijk}\} \in \mathcal{Z}^2(\mathcal{U}, T_X)$. If we choose different automorphisms $\{\Phi_{ij}\}$ of the trivial deformations $U_{ij} \times \mathrm{Spec}(\tilde{A})$ satisfying the analogous of condition (b) then

$$[II.1.5] \qquad \Phi_{ij} = \tilde{\theta}_{ij} + \epsilon d_{ij}$$

for some $d_{ij} \in \Gamma(U_{ij}, T_X)$, by Lemma (II.1.5). For each i, j, k the automorphism

$$\Phi_{ij}\Phi_{jk}\Phi_{ik}^{-1}$$

corresponds to the derivation

$$\delta_{ijk} = \tilde{d}_{ijk} + (d_{ij} + d_{jk} - d_{ik})$$

and therefore we see that the 2-cocycles $\{\tilde{d}_{ijk}\}$ and $\{\delta_{ijk}\}$ are cohomologous. Their cohomology class

$$o_\xi(e) \in H^2(X, T_X)$$

depends only on ξ and e and is 0 if and only if we can find a collection of automorphisms $\{\Phi_{ij}\}$ such that $\delta_{ijk} = 0$ for all $i,j,k \in I$. In such a case $\{\Phi_{ij}\}$ defines a lifting $\tilde{\xi}$ of ξ. This proves (i).

Assume that $o_\xi(e) = 0$, i.e. that the lifting $\tilde{\xi}$ of ξ exists. Then we can choose the collection $\{\tilde{\theta}_{ij}\}$ of automorphisms satisfying conditions (a) and (b) as above, in particular $\tilde{d}_{ijk} = 0$, all i,j,k. Any other choice of a lifting $\bar{\xi}$ of ξ to $\tilde{A}$ corresponds to a choice of automorphisms $\{\Phi_{ij}\}$ satisfying [II.1.5] and the analogous of condition (b). Therefore, for all i,j,k, we have

$$0 = \delta_{ijk} = d_{ij} + d_{jk} - d_{ik}$$

so that $\{d_{ij}\} \in \mathcal{Z}^1(\mathcal{U}, T_X)$ defines an element $\bar{d} \in H^1(X, T_X)$. As before one checks that this element only depends on the isomorphism class of $\bar{\xi}$; it follows in a straightforward way that the correspondence $(\tilde{\xi}, \bar{d}) \mapsto \bar{\xi}$ defines a transitive action of $H^1(X, T_X)$ on the set of isomorphism classes of liftings of ξ to $\tilde{A}$. This proves (ii). *q.e.d.*

The correspondence $e \mapsto o_\xi(e)$ defines a map

$$o_\xi : \mathrm{Ex}_{\mathbf{k}}(A, \mathbf{k}) \to H^2(X, T_X)$$

which can be easily seen to be $\mathbf{k}$-linear. The deformation ξ is called *unobstructed* if o_ξ is the zero map; otherwise ξ is called *obstructed*. X is *unobstructed* if every infinitesimal deformation of X is unobstructed; otherwise X is *obstructed*.

(II.1.9) COROLLARY *A nonsingular variety X is unobstructed if*

$$H^2(X, T_X) = 0$$

The proof is obvious.

(II.1.10) COROLLARY *A nonsingular variety X is rigid if and only if*

$$H^1(X, T_X) = 0$$

Proof

The hypothesis implies, by Proposition (II.1.6), that all first order deformations of X are trivial and, by Proposition (II.1.8), that every infinitesimal deformation of X over any A in $\mathcal{A}$ has at most one lifting to any small extension of A. These two facts together imply the conclusion. *q.e.d.*

(II.1.11) EXAMPLES (i) If X is a projective nonsingular curve of genus g then from the Riemann-Roch Theorem it follows that

$$h^1(X, T_X) = \begin{cases} 0 & \text{if } g = 0 \\ 1 & \text{if } g = 1 \\ 3g-3 & \text{if } g \geq 2 \end{cases}$$

and $H^2(X, T_X) = 0$. In particular projective nonsingular curves are unobstructed.

(ii) If X is a projective, irreducible and nonsingular surface X then

$$H^2(X, T_X) \cong H^0(X, \Omega^1_X \otimes K_X)^\vee$$

by Serre duality, and this rarely vanishes. For example a nonsingular surface of degree ≥ 5 in $\mathbb{P}^3$ satisfies $H^2(X, T_X) \neq 0$, but it is nevertheless unobstructed (see example (III.3.10)(iii)); therefore the sufficient condition of Corollary (II.1.9) is not necessary. In general a surface such that $H^2(X, T_X) \neq 0$ can be obstructed, but explicit examples are not elementary (see Kas(1967), Burns-Wahl(1974), Horikawa(1975)). We will not give such examples here; in §III.6 we will show how to construct examples of obstructed 3-folds.

(iii) The projective space $\mathbb{P}^n$ is rigid for every $n \geq 1$. In fact it follows immediately from the Euler sequence:

$$0 \to \mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}_{\mathbb{P}^n}(1)^{n+1} \to T_{\mathbb{P}^n} \to 0$$

that $H^1(\mathbb{P}^n, T_{\mathbb{P}^n}) = 0$. Similarly one shows that finite products $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ of projective spaces are rigid.

* * * * * *

The Kodaira-Spencer map - Examples

Let

[*II*.1.6]
$$\xi : \begin{array}{ccc} X & \to & \mathcal{X} \\ \downarrow & & \downarrow f \\ \mathrm{Spec}(\mathbf{k}) & \xrightarrow{s} & S \end{array}$$

be a family of deformations of a nonsingular variety X. By pulling back this family by morphisms $\mathrm{Spec}(\mathbf{k}[\epsilon]) \to S$ with image s and applying the Kodaira-Spencer correspondence (Proposition (II.1.6)) we define a linear map

$$\kappa_\xi : T_{S,s} \to H^1(X, T_X)$$

also denoted $\kappa_{f,s}$ or $\kappa_{\mathcal{X}/S,s}$, which is called the *Kodaira-Spencer map* of the family ξ.

(II.1.12) EXAMPLES (i) Let $m \geq 1$ and let $\pi : F_m \to \mathbb{P}^1$ be the structural morphism of the rational ruled surface F_m (see (A.1.10)(iii)). Then π is not a trivial family but has a trivial restriction around each closed point $s \in \mathbb{P}^1$, thus $\kappa_{\pi,s} = 0$.

(ii) Consider an unramified covering $\pi : X \to S$ of degree $n \geq 2$ where X and S are projective nonsingular and irreducible algebraic curves. All fibres of π over the closed points consist of n distinct points, hence they are all isomorphic. Moreover each such fibre is rigid and unobstructed as an abstract variety. In particular the

Kodaira-Spencer map is zero at each closed point $s \in S$. On the other hand $\pi^{-1}(U)$ is irreducible for each open subset $U \subset S$ and therefore the restriction $\pi_U : \pi^{-1}(U) \to U$ is a nontrivial family; this follows also from the fact that π does not have rational sections.

This example exhibits a phenomenon which is not detected by infinitesimal considerations: we can have a flat projective family of deformations all of whose geometric fibres are isomorphic but which is nevertheless non trivial over every Zariski open subset of the base. Note that this is different from what happens with the projections $F_m \to \mathbb{P}^1$, $m \geq 1$ of example (i), which are non trivial but have trivial restriction to a Zariski open neighborhood of every point of $\mathbb{P}^1$.

(iii) Let $0 \leq n < m$ be integers having the same parity, and let $k = \frac{1}{2}(m-n)$. Consider the smooth proper morphism $f : \mathcal{W} \to \mathbf{A}^1$ introduced in example (II.1.1.1)(iii), whose fibres are $\mathcal{W}(0) \cong F_m$, and $\mathcal{W}(t) \cong F_n$ for $t \neq 0$. Recall that the family f is given as the gluing of two copies of $\mathbf{A}^2 \times \mathbb{P}^1$, $W = \mathrm{Proj}(\mathbf{k}[t, z, \xi_0, \xi_1])$ and $W' = \mathrm{Proj}(\mathbf{k}[t, z', \xi'_0, \xi'_1])$, along $\mathrm{Proj}(\mathbf{k}[t, z, z^{-1}, \xi_0, \xi_1])$ and $\mathrm{Proj}(\mathbf{k}[t, z', z'^{-1}, \xi'_0, \xi'_1])$ according to the rules:

$$z' = z^{-1}, \quad \xi' = z^m \xi + t z^k$$

where $\xi = \xi_1/\xi_0$ and $\xi' = \xi'_1/\xi'_0$ are non-homogeneous coordinates on the corresponding copies of $\mathbb{P}^1$.

Let's compute the local Kodaira-Spencer map $\kappa_{f,0}$ of f at 0. The image $\kappa_{f,0}(\frac{d}{dt})$ is the element of $H^1(F_m, T_{F_m})$ corresponding to the first order deformation of F_m obtained by gluing $W_0 := \mathrm{Proj}(\mathbf{k}[\epsilon, z, \xi_0, \xi_1])$ and $W'_0 := \mathrm{Proj}(\mathbf{k}[\epsilon, z', \xi'_0, \xi'_1])$ along $\mathrm{Proj}(\mathbf{k}[\epsilon, z, z^{-1}, \xi_0, \xi_1])$ and $\mathrm{Proj}(\mathbf{k}[\epsilon, z', z'^{-1}, \xi'_0, \xi'_1])$ according to the rules

$$z' = z^{-1}, \quad \xi' = z^m \xi + \epsilon z^k$$

By definition we have that $\kappa_{f,0}(\frac{d}{dt})$ is the element of $H^1(\mathcal{U}, T_{F_m})$, where $\mathcal{U} = \{W_0, W'_0\}$, defined by the 1-cocycle corresponding to the vector field on $W_0 \cap W'_0$

$$\{z^k \frac{\partial}{\partial \xi}\}$$

According to (A.1.10)(iii) this element is non zero; therefore $\kappa_{f,0}$ is injective.

Similarly we can consider a smooth proper family $F : \mathcal{Y} \to \mathbf{A}^{m-1}$ defined as follows. $\mathcal{Y}$ is the gluing of

$$Y := \mathrm{Proj}(\mathbf{k}[t_1, \ldots, t_{m-1}, z, \xi_0, \xi_1])$$

and

$$Y' := \mathrm{Proj}(\mathbf{k}[t_1, \ldots, t_{m-1}, z', \xi'_0, \xi'_1])$$

along $\mathrm{Proj}(\mathbf{k}[t_1, \ldots, t_{m-1}, z, z^{-1}, \xi_0, \xi_1])$ and $\mathrm{Proj}(\mathbf{k}[t_1, \ldots, t_{m-1}, z', z'^{-1}, \xi'_0, \xi'_1])$ according to the rules:

$$z' = z^{-1}, \quad \xi' = z^m \xi + \sum_{j=1}^{m-1} t_j z^j$$

The morphism F is defined by the projections onto $\mathrm{Spec}(\mathbf{k}[t_1,\dots,t_{m-1}])$; the fibre of F over $\underline{0}$ is $\mathcal{Y}(\underline{0}) \cong F_m$. The computation we just did immediately implies that the local Kodaira-Spencer map

$$\kappa_{F,\underline{0}} : T_{\underline{0}}\mathbf{A}^{m-1} \to H^1(F_m, T_{F_m})$$

is an isomorphism.

NOTES

1. A straightforward generalization of the proof of Lemma (II.1.5) gives the following result:

(II.1.13) LEMMA *Let B_0 be a $\mathbf{k}$-algebra,*

$$e : 0 \to (\epsilon) \to \tilde{A} \to A \to 0$$

a small extension in $\mathcal{A}$, $\tilde{A} \to \tilde{B}$ a deformation of B_0 and $A \to B = \tilde{B} \otimes_{\tilde{A}} A$ the induced deformation of B_0 over A. Let $\sigma : B \to B$ be an automorphism of the deformation. If

$$\mathrm{Aut}_\sigma(\tilde{\mathrm{B}}) := \left\{\textit{automorphisms } \tau : \tilde{B} \to \tilde{B} \textit{ such that } \tau \otimes_{\tilde{A}} A = \sigma\right\} \neq \emptyset$$

then there is a faithful and transitive action

$$\mathrm{Der}_{\mathbf{k}}(B_0, B_0) \times \mathrm{Aut}_\sigma(\tilde{\mathrm{B}}) \to \mathrm{Aut}_\sigma(\tilde{\mathrm{B}})$$

defined by

$$(d, \tau) \mapsto \tau + \epsilon d$$

II.2. INVERTIBLE SHEAVES

Deformations of invertible sheaves on a fixed variety

Let X be an algebraic variety and L an invertible sheaf on X. An *infinitesimal deformation of* L over A, where A is in $\mathcal{A}$, is an invertible sheaf $\mathcal{L}$ on $X \times \mathrm{Spec}(A)$ such that $L = \mathcal{L}_{|X \times \mathrm{Spec}(\mathbf{k})}$. In case $A = \mathbf{k}[\epsilon]$ we speak of a *first order deformation* of L. Two deformations $\mathcal{L}$ and $\mathcal{L}'$ of L over A will be called *isomorphic* if there is an isomorphism $\mathcal{L} \cong \mathcal{L}'$ inducing the identity of their restrictions to the closed fibre $X \times \mathrm{Spec}(\mathbf{k})$.

Let

$$[II.2.1] \qquad e : 0 \to \epsilon\mathbf{k} \to \tilde{A} \to A \to 0$$

be an extension of local artinian $\mathbf{k}$-algebras, and let $\mathcal{L}$ be a deformation of L over A. A *lifting* of $\mathcal{L}$ to $\tilde{A}$ is a deformation $\tilde{\mathcal{L}}$ of L over $\tilde{A}$ whose restriction to $X \times \mathrm{Spec}(A)$ is isomorphic to $\mathcal{L}$.

(II.2.1) PROPOSITION *Let L be an invertible sheaf on an algebraic variety X. Then*
(i) there is a natural 1-1 correspondence:

$$\frac{\{\text{first order deformations of } L\}}{\text{isomorphism}} \leftrightarrow H^1(X, \mathcal{O}_X)$$

(ii) Given A in $\mathcal{A}$ and a deformation $\mathcal{L}$ of L over A there is a map

$$o_{\mathcal{L}} : \mathrm{Ex}_{\mathbf{k}}(A, \mathbf{k}) \to H^2(X, \mathcal{O}_X)$$

such that for every extension [II.2.1] we have $o_{\mathcal{L}}(e) = 0$ if and only if $\mathcal{L}$ has a lifting to $\tilde{A}$.

Proof
(i) Assume that L is given by a system of transition functions $\{f_{\alpha\beta}\}$ with respect to an open covering $\{U_\alpha\}$ of X, $f_{\alpha\beta} \in \Gamma(U_\alpha \cap U_\beta, \mathcal{O}_X^*)$. Then a first order deformation L_ϵ of L can be represented, in the same covering $\{U_\alpha\}$ of $X \times \mathrm{Spec}(\mathbf{k}[\epsilon])$, by transition functions:

$$\tilde{f}_{\alpha\beta} \in \Gamma(U_\alpha \cap U_\beta, O^*_{X \times \mathrm{Spec}(\mathbf{k}[\epsilon])})$$

such that

$$[II.2.2] \qquad \tilde{f}_{\alpha\beta}\tilde{f}_{\beta\gamma} = \tilde{f}_{\alpha\gamma}$$

and which restrict to the $f_{\alpha\beta}$'s modulo ϵ.

Since $\mathcal{O}^*_{X\times\mathrm{Spec}(\mathbf{k}[\epsilon])} = \mathcal{O}^*_X + \epsilon\mathcal{O}_X$ we can write

$$\tilde{f}_{\alpha\beta} = f_{\alpha\beta}(1+\epsilon a_{\alpha\beta}) \tag{II.2.3}$$

for suitable $a_{\alpha\beta}\in\Gamma(U_\alpha\cap U_\beta,\mathcal{O}_X)$. Identity [II.2.2] gives

$$a_{\alpha\beta}+a_{\beta\gamma}=a_{\alpha\gamma} \tag{II.2.4}$$

and therefore the system $\{a_{\alpha\beta}\}$ is a Chech 1-cocycle which defines an element of $H^1(X,\mathcal{O}_X)$. Conversely, given such a 1-cocycle we can define an invertible sheaf L_ϵ by the transition functions [II.2.3]. If we modify [II.2.4] by a coboundary

$$b_{\alpha\beta}=a_{\alpha\beta}+a_\beta-a_\alpha$$

where $a_\alpha\in\Gamma(U_\alpha,\mathcal{O}_X)$, the transition functions $\tilde{f}_{\alpha\beta}$ ar replaced by

$$\begin{gathered}\bar{f}_{\alpha\beta}=f_{\alpha\beta}[1+\epsilon(a_{\alpha\beta}+a_\beta-a_\alpha)]=\\ =f_{\alpha\beta}(1+\epsilon a_{\alpha\beta})(1+\epsilon a_\beta)(1-\epsilon a_\alpha)=\tilde{f}_{\alpha\beta}(1+\epsilon a_\beta)(1+\epsilon a_\alpha)^{-1}\end{gathered}$$

and the invertible sheaf L_ϵ is replaced by an invertible sheaf $\bar{L}_\epsilon$ which obviously defines an isomorphic deformation of L. It is clear that this correspondence is not affected by refining the open covering $\{U_\alpha\}$.

(ii) Let $[e]\in\mathrm{Ex}_{\mathbf{k}}(A,\mathbf{k})$ be given by the extension [II.2.1]. We have an exact sequence of sheaves of $\mathbf{k}$-algebras on X:

$$0\to\epsilon\mathcal{O}_X\to\mathcal{O}^*_{X\times\mathrm{Spec}(\tilde{A})}\to\mathcal{O}^*_{X\times\mathrm{Spec}(A)}\to 0$$

which induces an exact sequence of groups:

$$H^1(X\times\mathrm{Spec}(\tilde{A}),\mathcal{O}^*_{X\times\mathrm{Spec}(\tilde{A})})\to H^1(X\times\mathrm{Spec}(A),\mathcal{O}^*_{X\times\mathrm{Spec}(A)})\xrightarrow{\delta}H^2(X,\mathcal{O}_X)$$

The invertible sheaf $\mathcal{L}$ defines an element $[\mathcal{L}]\in H^1(X\times\mathrm{Spec}(A),\mathcal{O}^*_{X\times\mathrm{Spec}(A)})$ which can be lifted to $\tilde{A}$ if and only if $\delta([\mathcal{L}])=0$. Letting $o_{\mathcal{L}}(e)=\delta([\mathcal{L}])$ we define a map o_l having the required properties. *q.e.d.*

In case the Picard variety $\mathrm{Pic}(X)$ of X exists (e.g. when X is a projective nonsingular variety) the Proposition computes its tangent space at $[L]$.

* * * * * *

Deformations of pairs (X,L)

Let X be a nonsingular algebraic variety and let $D:\mathcal{O}_X\to\Omega^1_X$ be the canonical derivation. We can define a homomorphism of sheaves of abelian groups

$$\mathcal{O}^*_X\to\Omega^1_X$$

by the rule

$$u \mapsto \frac{Du}{u}$$

for all open sets $U \subset X$ and $u \in \Gamma(U, \mathcal{O}_X^*)$. We have an induced group homomorphism:

$$c : H^1(X, \mathcal{O}_X^*) \to H^1(X, \Omega_X^1)$$

To simplify the notation, given an invertible sheaf L on X we write $c(L)$ instead of $c([L])$. Since Ω_X^1 is locally free we have an identification

$$H^1(X, \Omega_X^1) = \mathrm{Ext}^1_{\mathcal{O}_X}(T_X, \mathcal{O}_X)$$

so that we can associate to $c(L)$ an extension

$$[II.2.5] \qquad 0 \to O_X \to \mathcal{E}_L \to T_X \to 0$$

called the *Atiyah extension* of L. The sheaf $\mathcal{E}_L$ is locally free of rank $\dim(X) + 1$ and

$$\mathcal{P}_L := \mathcal{E}_L^\vee \otimes_{\mathcal{O}_X} L$$

is called the *sheaf of first order principal parts* of L.

Let $\mathcal{U} = \{U_\alpha\}$ be an affine open covering of X such that L is represented by a system of transition functions $\{f_{\alpha\beta}\}$, $f_{\alpha\beta} \in \Gamma(U_{\alpha\beta}, \mathcal{O}_X^*)$. Then $c(L)$ is represented by the Chech 1-cocycle

$$\{\frac{Df_{\alpha\beta}}{f_{\alpha\beta}}\} \in \mathcal{Z}^1(\mathcal{U}, \Omega_X^1)$$

The sheaf $\mathcal{E}_{L|U_\alpha}$ is isomorphic to $\mathcal{O}_{U_\alpha} \oplus T_{X|U_\alpha}$. A section (a_α, d_α) of $\mathcal{O}_{U_\alpha} \oplus T_{X|U_\alpha}$ and a section (a_β, d_β) of $\mathcal{O}_{U_\beta} \oplus T_{X|U_\beta}$ are identified on $U_{\alpha\beta}$ if and only if $d_\alpha = d_\beta$ and $a_\beta - a_\alpha = \frac{d_\alpha(f_{\alpha\beta})}{f_{\alpha\beta}}$.

Let A be in $\mathcal{A}$. An *infinitesimal deformation of the pair* (X, L) over A consists of a pair $(\xi, \mathcal{L})$, where

$$\xi : \quad \begin{array}{ccc} X & \to & \mathcal{X} \\ \downarrow & & \downarrow \\ \mathrm{Spec}(\mathbf{k}) & \to & \mathrm{Spec}(A) \end{array}$$

is an infinitesimal deformation of X over A and $\mathcal{L}$ is invertible sheaf on $\mathcal{X}$ such that $L = \mathcal{L}_{|X}$. One can also say that $\mathcal{L}$ is a *deformation of* L *along* ξ. In case $A = \mathbf{k}[\epsilon]$ we speak of a *first order deformation* of (X, L). Two deformations $(\mathcal{X}, \mathcal{L})$ and $(\mathcal{X}', \mathcal{L}')$ of (X, L) over A will be called *isomorphic* if there is an isomorphism of deformations $f : \mathcal{X} \to \mathcal{X}'$ and an isomorphism $\mathcal{L} \to f^*\mathcal{L}'$ inducing the identity of their restrictions to the closed fibre X.

(II.2.2) PROPOSITION *Let (X, L) be a pair consisting of a nonsingular algebraic variety X and an invertible sheaf L on X. Then:*
(i) there is a canonical 1-1 correspondence

$$\frac{\{\text{1-st order deformations of } (X, L)\}}{\text{isomorphism}} \leftrightarrow H^1(X, \mathcal{E}_L)$$

(ii) The linear map

$$H^1(X,\mathcal{E}_L)\to H^1(X,T_X)$$

coming from the exact sequence [II.2.2] corresponds to the map

$$\frac{\{\text{1-st order deformations of } (X,L)\}}{\text{isomorphism}}\to\frac{\{\text{1-st order deformations of } X\}}{\text{isomorphism}}$$

obtained by associating to a first order deformation $(\xi,\mathcal{L})$ the deformation ξ.
(iii) Given a first order deformation ξ of X, there is a first order deformation of L along ξ if and only if

$$\kappa(\xi)\cdot c(L)=0$$

where the left hand side denotes cup product of cohomology classes (therefore it is an element of $H^2(X,\mathcal{O}_X)$).

Proof
Let $(\xi,\mathcal{L})$ be a first order deformation of (X,L), where

$$\xi:\quad\begin{array}{ccc} X & \to & \mathcal{X}\\ \downarrow & & \downarrow\\ \mathrm{Spec}(\mathbf{k}) & \to & \mathrm{Spec}(\mathbf{k}[\epsilon])\end{array}$$

Let $\mathcal{U}=\{U_\alpha\}$ be an affine open covering such that L is given by a system of transition functions $f_{\alpha\beta}\in\mathcal{Z}^1(\mathcal{U},\mathcal{O}_X^*)$ and $\kappa(\xi)\in H^1(X,T_X)$ is given by a Chech 1-cocycle $\{d_{\alpha\beta}\}\in\mathcal{Z}^1(\mathcal{U},T_X)$. Let $\theta_{\alpha\beta}=1+\epsilon d_{\alpha\beta}$ be the automorphism of $U_{\alpha\beta}\times\mathrm{Spec}(\mathbf{k}[\epsilon])$ corresponding to $d_{\alpha\beta}$.

The invertible sheaf $\mathcal{L}$ is given by a system of transition functions $\{F_{\alpha\beta}\}\in\mathcal{Z}^1(\mathcal{U},\mathcal{O}_{\mathcal{X}}^*)$ which reduces to $\{f_{\alpha\beta}\}$ mod ϵ. Therefore it can be represented on $U_{\alpha\beta}\times\mathrm{Spec}(\mathbf{k}[\epsilon])$ as

$$F_{\alpha\beta}=f_{\alpha\beta}(1+\epsilon a_{\alpha\beta})\qquad\qquad a_{\alpha\beta}\in\Gamma(U_{\alpha\beta},\mathcal{O}_X)$$

and the cocycle condition translates into

$$F_{\alpha\beta}\theta_{\alpha\beta}(F_{\beta\gamma})=F_{\alpha\gamma}$$

equivalently:

$$F_{\alpha\beta}(F_{\beta\gamma}+\epsilon d_{\alpha\beta}F_{\beta\gamma})=F_{\alpha\gamma}$$

which means:

$$f_{\alpha\beta}(1+\epsilon a_{\alpha\beta})[f_{\beta\gamma}(1+\epsilon a_{\beta\gamma})+\epsilon d_{\alpha\beta}(f_{\beta\gamma}(1+\epsilon a_{\beta\gamma}))]=f_{\alpha\gamma}(1+\epsilon a_{\alpha\gamma})$$

After dividing by $f_{\alpha\gamma}$ this identity translates into:

[II.2.6] $$a_{\alpha\beta}+a_{\beta\gamma}-a_{\alpha\gamma}+\frac{d_{\alpha\beta}f_{\beta\gamma}}{f_{\beta\gamma}}=0$$

This condition can be interpreted in several ways. It means that the data $\{(a_{\alpha\beta}, d_{\alpha\beta})\}$ define an element of $\mathcal{Z}^1(\mathcal{U}, \mathcal{E}_L)$, and that conversely such an element defines a first order deformation of (X, L). This proves (i) modulo verifying that this correspondence is independent from the choice of the covering $\mathcal{U}$ and of the cocycles representing (X, L); we leave this to the reader. It also proves (ii) because the linear map $H^1(X, \mathcal{E}_L) \to H^1(X, T_X)$ is induced by the projection

$$\{(a_{\alpha\beta}, d_{\alpha\beta})\} \mapsto \{d_{\alpha\beta}\}$$

Finally, observing that $\{\frac{d_{\alpha\beta} f_{\beta\gamma}}{f_{\beta\gamma}}\} \in \mathcal{Z}^2(\mathcal{U}, \mathcal{O}_X)$ represents $\kappa(\xi) \cdot c(L)$, the identity [II.2.6] expresses the condition that this 2-cocycle is a coboundary, and this proves (iii). *q.e.d.*

If X is a curve then $H^2(X, \mathcal{O}_X) = 0$ so that every line bundle can be deformed along any first order deformation of X. For surfaces this is not the case in general. For example if X is a K3-surface then $h^2(X, \mathcal{O}_X) = 1$ and

$$H^1(X, T_X) \xrightarrow{\cdot c(L)} H^2(X, \mathcal{O}_X)$$

can be shown to be surjective for every nontrivial line bundle L. This means that L deforms along a 19-dimensional subspace of $H^1(X, T_X)$, because $H^1(X, T_X) = 20$ (see example (III.3.16)(ii)).

NOTES

1. The coboundary maps δ_k in the cohomology sequence of the Atiyah extension [II.2.5] are induced by cup product with $c(L)$, since the extension is defined by $c(L)$. In particular (II.2.2)(iii) just says that L deforms along ξ if and only if $\kappa(\xi) \in \ker[\delta_1 : H^1(X, T_X) \to H^2(X, \mathcal{O}_X)]$, which is obvious in view of (i) and (ii).

2. The content of Proposition (II.2.2) is outlined in the Appendix to Chapter V of Zariski(1971) written by Mumford. See also Horikawa(1976b). This result is related with the notion of *deformation of a polarization* (see Popp(1977)).

II.3. CLOSED SUBSCHEMES

Infinitesimal deformations of a closed subscheme

Let Y and S be schemes with Y algebraic. A diagram of morphisms of schemes

$$\begin{array}{ccc} \mathcal{X} & \subset & Y \times S \\ \downarrow \pi & & \\ S & & \end{array}$$

where the vertical morphism is flat, and it is induced by the projection from $Y \times S$, is called a *(flat) family of closed subschemes of* Y parametrized by S. The family is called *trivial* if $\mathcal{X} = X \times S$ for some closed subscheme $X \subset Y$. If $s \in S$ is a $\mathbf{k}$-rational point then π is called a *family of deformations of* $\mathcal{X}(s)$ in Y, where $\mathcal{X}(s) \subset Y$ is the fibre of π over s.

If $X \subset Y$ is a closed embedding and $A \in \mathrm{ob}(\mathcal{A})$, *an infinitesimal deformation of* X *in* Y parametrized by $S = \mathrm{Spec}(A)$ (shortly, over A) is a family of closed subschemes of Y parametrized by S such that $X \subset Y$ is the fibre over the closed point of S. If $A = \mathbf{k}[\epsilon]$ we speak of a *first order deformation* of X in Y.

X is *rigid in* Y if every infinitesimal deformation of X in Y is trivial.

(II.3.1) THEOREM *Let $X \subset Y$ be a closed embedding of schemes. There is a natural 1-1 correspondence*

$$\chi : \left\{\textit{first order deformations of } X \textit{ in } Y\right\} \to H^0(X, N_{X/Y})$$

Proof

Let $\mathcal{I} \subset \mathcal{O}_Y$ be the ideal sheaf of X. A first order deformation of X in Y, i.e. a flat family:

$$[II.3.1] \qquad \begin{array}{ccc} \mathcal{X} & \subset & Y \times \mathrm{Spec}(\mathbf{k}[\epsilon]) \\ \downarrow & & \\ \mathrm{Spec}(\mathbf{k}[\epsilon]) & & \end{array}$$

is defined by a sheaf $\mathcal{O}_{\mathcal{X}}$ of flat $\mathbf{k}[\epsilon]$-algebras, such that $\mathcal{O}_{\mathcal{X}} \otimes_{\mathbf{k}[\epsilon]} \mathbf{k} \cong \mathcal{O}_X$. The closed embedding $\mathcal{X} \subset Y \times \mathrm{Spec}(\mathbf{k}[\epsilon])$ is determined by a sheaf of ideals $\mathcal{I}_\epsilon \subset \mathcal{O}_Y[\epsilon] := \mathcal{O}_Y \otimes_{\mathbf{k}} \mathbf{k}[\epsilon]$ such that $\mathcal{O}_{\mathcal{X}} = \mathcal{O}_Y[\epsilon]/\mathcal{I}_\epsilon$. The above data are obtained by gluing together their restrictions to an affine open cover. On an affine open set $U = \mathrm{Spec}(P) \subset Y$, let $X \cap U = \mathrm{Spec}(B)$ where $B = P/J$ for an ideal $J = (f_1, \ldots, f_N) \subset P$.

Consider the exact sequence

$$0 \to \mathbf{R} \xrightarrow{v} P^N \xrightarrow{\mathbf{f}} J \to 0$$

where $\mathbf{R}$ is the module of relations among $f_1, \ldots, f_N$. Taking $\mathrm{Hom}_P(-, B)$ we obtain the exact sequence:

$$0 \to \mathrm{Hom}_B(J/J^2, B) \to \mathrm{Hom}_P(P^N, B) \xrightarrow{v^\vee} \mathrm{Hom}_P(\mathbf{R}, B)$$

which identifies $\mathrm{Hom}_B(J/J^2, B)$ with $\ker(v^\vee)$. An element of $\ker(v^\vee)$ can be represented as an N-tuple $\underline{h} = (h_1, \ldots, h_N)$ of elements of P which, interpreted as an element of $\mathrm{Hom}_P(P^N, B)$ by scalar product (i.e. $\underline{h}(p_1, \ldots, p_N) = \sum_j h_j p_j \bmod J$) must be zero on $\mathbf{R}$. Hence

$$\sum_j h_j r_j \in J \text{ for every } (r_1, \ldots, r_N) \in \mathbf{R}$$

This means that there exist $\Delta r_1, \ldots, \Delta r_N \in P$ such that

$$\sum_j h_j r_j = -\sum_j \Delta r_j f_j$$

or, equivalently, such that

$$(\underline{f} + \epsilon \underline{h})\ {}^t(\underline{r} + \epsilon \Delta \underline{r}) = 0$$

in $P \otimes_{\mathbf{k}} \mathbf{k}[\epsilon]$. Therefore from Corollary (A.2.10) it follows that $\underline{f} + \epsilon \underline{h}$ generates an ideal in $P \otimes_{\mathbf{k}} \mathbf{k}[\epsilon]$ which defines a first order deformation of $\mathrm{Spec}(B)$ in $\mathrm{Spec}(P)$ because every relation among $f_1, \ldots, f_N$ extends to a relation among $f_1 + \epsilon h_1, \ldots, f_N + \epsilon h_N$. Using the same argument backwards one sees that every first order deformation of $\mathrm{Spec}(B)$ in $\mathrm{Spec}(P)$ defines an element of $\mathrm{Hom}_B(J/J^2, B)$.

It follows that at the global level we have a canonical 1-1 correspondence between first order deformations of X in Y and $H^0(X, N_{X/Y})$. *q.e.d.*

(II.3.2) EXAMPLES

(i) If $X \subset \mathbb{P}^r$, $r \geq 1$, is the complete intersection of $r - n$ hypersurfaces $f_1, \ldots, f_{r-n}$ of degrees $d_1 \leq d_2 \leq \ldots \leq d_{r-n}$ respectively, we have a presentation

$$\bigwedge^2 [\oplus_j \mathcal{O}(-d_j)] \xrightarrow{\mathbf{f}} \oplus_j \mathcal{O}(-d_j) \to \mathcal{I}_X \to 0 \qquad [II.3.2]$$

Taking $\mathcal{H}om(-, \mathcal{O}_X)$ we obtain an exact sequence:

$$0 \to N_X \to \oplus_j \mathcal{O}_X(d_j) \xrightarrow{\check{\mathbf{f}}} \bigwedge^2 [\oplus_j \mathcal{O}_X(d_j)]$$

where $N_X = N_{X/\mathbb{P}^r}$. Since each nonzero entry of the matrix defining $\mathbf{f}$ is one of the f_j's, the map $\check{\mathbf{f}}$ is zero. Therefore

$$N_X \cong \oplus_j \mathcal{O}_X(d_j)$$

Equivalently one can remark that [II.3.2] induces a surjective homomorphism

$$\oplus_j \mathcal{O}_X(-d_j) \to \mathcal{I}_X/\mathcal{I}_X^2$$

of locally free sheaves of the same rank, which must therefore be an isomorphism. In particular if X is a hypersurface of degree d we have $N_X \cong \mathcal{O}_X(d)$ and therefore

$$h^0(X, N_X) = \binom{d+r}{r} - 1$$

confirming the fact that X can be deformed in $\mathbb{P}^r$ only inside the linear system of hypersurfaces of degree d, which is a projective space of dimension $\binom{d+r}{r} - 1$.

If X is a linear subspace, i.e. $d_1 = \cdots = d_{r-n} = 1$, then $N_X \cong \mathcal{O}_X(1)^{\oplus r-n}$ and therefore

$$h^0(X, N_X) = (r-n)(n+1)$$

as expected, since such linear subspaces X are parametrized by the grassmannian $G(n+1, r+1)$, which is nonsingular of dimension $(r-n)(n+1)$.

(ii) Let X be a Cartier divisor on a connected projective scheme Y. Consider the exact sequence

$$0 \to \mathcal{O}_Y \to \mathcal{O}_Y(X) \to N_{X/Y} \to 0$$

and the cohomology sequence:

$$[II.3.3] \qquad 0 \to H^0(\mathcal{O}_Y) \to H^0(\mathcal{O}_Y(X)) \xrightarrow{\chi} H^0(N_{X/Y}) \to H^1(\mathcal{O}_Y)$$

Classically the map χ was called the *characteristic map* of the linear system $|X|$. By definition $\mathrm{Im}(\chi) \cong H^0(\mathcal{O}_Y(X))/H^0(\mathcal{O}_Y)$ is naturally identified with the tangent space to the linear system $|X|$. We can verify that this is so by identifying $\mathrm{Im}(\chi)$ as a subvector space of first order deformations of X in Y, as follows.

Assume that X is defined by a system of local equations $\{f_i\}$, $f_i \in \Gamma(U_i, \mathcal{O}_Y)$ not 0-divisor, with respect to an affine cover $\{U_i\}$ of Y. We have $f_{ij} := f_i f_j^{-1} \in \Gamma(U_i \cap U_j, \mathcal{O}_Y^*)$ for all i, j, and $\{f_{ij}\}$ is a Cech 1-cocycle which defines the line bundle $\mathcal{O}_Y(X)$. A first order deformation of X in Y is a Cartier divisor $\mathcal{X} \subset Y \times \mathrm{Spec}(\mathbf{k}[\epsilon])$ which is determined by a system $\{F_i = f_i + \epsilon g_i\}$, $g_i \in \Gamma(U_i, \mathcal{O}_Y)$, such that there exist $F_{ij} = f_{ij} + \epsilon g_{ij} \in \Gamma(U_i \cap U_j, \mathcal{O}^*_{Y \times \mathrm{Spec}(\mathbf{k}[\epsilon])})$ (hence $g_{ij} \in \Gamma(U_i \cap U_j, \mathcal{O}_Y)$) satisfying $F_i = F_{ij} F_j$ for all i, j. Therefore on $U_i \cap U_j$ we have:

$$f_i + \epsilon g_i = (f_{ij} + \epsilon g_{ij})(f_j + \epsilon g_j)$$

which is equivalent to the identity:

$$g_i = f_{ij} g_j + g_{ij} f_j$$

This identity shows that the system $\{\bar{g}_i = g_i \bmod \mathcal{I}_X\}$ defines a section of $N_{X/Y}$, as expected. We also see that $\{\bar{g}_i\} \in \mathrm{Im}(\chi)$ if and only if $g_i = f_{ij} g_j$, in which case $F_{ij} = f_{ij}$, i.e. X deforms inside the linear system $|X|$, as asserted. From [II.3.3] we see that if $H^1(\mathcal{O}_Y) \neq (0)$ then there are first order deformations of X in Y which are not linearly equivalent to X, i.e. which are not contained in the linear system $|X|$.

* * * * * *

Obstructions

Let $X \subset Y$ be a closed embedding, A in $\mathcal{A}$ and let

$$\xi : \begin{array}{ccc} \mathcal{X} & \subset & Y \times \mathrm{Spec}(A) \\ \downarrow f & & \\ \mathrm{Spec}(A) & & \end{array}$$

be a deformation of X in Y over A. Let

$$e : 0 \to \mathbf{k} \to \tilde{A} \to A \to 0$$

be a small extension. A *lifting* of ξ to $\tilde{A}$ is a deformation of X in Y over $\tilde{A}$:

$$\tilde{\xi} : \begin{array}{ccc} \tilde{\mathcal{X}} & \subset & Y \times \mathrm{Spec}(\tilde{A}) \\ \downarrow f & & \\ \mathrm{Spec}(\tilde{A}) & & \end{array}$$

whose pullback to $\mathrm{Spec}(A)$ is ξ.

(II.3.3) PROPOSITION *Let $X \subset Y$ be a regular closed embedding. and*

$$\xi : \begin{array}{ccc} \mathcal{X} & \subset & Y \times \mathrm{Spec}(A) \\ \downarrow f & & \\ \mathrm{Spec}(A) & & \end{array}$$

where A is in $\mathcal{A}$, an infinitesimal deformation of X in Y. Then
(i) there is a natural map

$$o_{\xi/Y} : \mathrm{Ex}_{\mathbf{k}}(A, \mathbf{k}) \to H^1(X, N_{X/Y})$$

such that, for every extension

$$e : 0 \to \epsilon\mathbf{k} \to \tilde{A} \to A \to 0$$

we have $o_{\xi/Y}(e) = 0$ if and only if ξ has a lifting to $\tilde{A}$.
(ii) If $o_{\xi/Y}(e) = 0$ the set of liftings of ξ to $\tilde{A}$ is a principal homogeneous space under a natural action of $H^0(X, N_{X/Y})$.

Proof

Since X is regularly embedded in Y we can find an affine open cover $\mathcal{U} = \{U_i\}_{i\in I}$ of Y such that $X_i := X \cap U_i$ is a complete intersection in U_i for each i. Let $U_i = \mathrm{Spec}(P_i)$, $X_i = \mathrm{Spec}(P_i/\mathcal{I}_i)$ where $\mathcal{I}_i = (f_{i1}, \dots, f_{iN})$ with $\{f_{i1}, \dots, f_{iN}\}$ a regular sequence in P_i. We then have $\mathcal{X}_{|U_i} = \mathrm{Spec}(P_{iA}/\mathcal{I}_{iA})$ where

$$\mathcal{I}_{iA} = (F_{i1}, \dots, F_{iN}) \subset P_{iA} := P_i \otimes A$$

and $f_{i\alpha} = F_{i\alpha} \bmod m_A$, $\alpha = 1, \dots, N$. Choose arbitrarily $\tilde{F}_{i1}, \dots, \tilde{F}_{iN} \in P_{i\tilde{A}}$ such that $F_{i\alpha} = \tilde{F}_{i\alpha} \bmod \epsilon$. By example (A.2.11) $\{F_{i1}, \dots, F_{iN}\}$ and $\{\tilde{F}_{i1}, \dots, \tilde{F}_{iN}\}$ are regular sequences in P_{iA} and in $P_{i\tilde{A}}$ respectively; in particular, letting $\mathcal{I}_{i\tilde{A}} = (\tilde{F}_{i1}, \dots, \tilde{F}_{iN})$,

$$\tilde{\mathcal{X}}_i := \mathrm{Spec}(P_{i\tilde{A}}/\mathcal{I}_{i\tilde{A}}) \subset U_i \times \mathrm{Spec}(\tilde{A})$$

is a lifting of $\mathcal{X}_{|U_i} \subset U_i \times \mathrm{Spec}(A)$ to $\tilde{A}$. In order to find a lifting of $\mathcal{X}$ to $\tilde{A}$ we must be able to choose the $\tilde{F}_{i\alpha}$'s in such a way that

$$\tilde{\mathcal{X}}_{i|U_{ij}} = \tilde{\mathcal{X}}_{j|U_{ij}} \subset U_{ij} \times \mathrm{Spec}(\tilde{A}) \qquad [II.3.4]$$

for each $i, j \in I$. Letting $U_{ij} = \mathrm{Spec}(P_{ij})$ and viewing the $f_{i\alpha}$'s and $f_{j\alpha}$'s as elements of P_{ij} via the natural maps

$$\begin{array}{ccc} P_i & & P_j \\ & \searrow \quad \swarrow & \\ & P_{ij} & \end{array}$$

we have

$$\tilde{F}_{j\alpha} - \tilde{F}_{i\alpha} =: \epsilon h_{ij\alpha}$$

and $\underline{h}_{ij} := (h_{ij1}, \dots, h_{ijN}) \in \Gamma(U_{ij}, N_{X/Y})$, because $N_{X/Y}$ is locally free of rank N and is trivial on each U_i. By construction $\{\underline{h}_{ij}\} \in \mathcal{Z}^1(\mathcal{U}, N_{X/Y})$. The condition [II.3.4] means that we can choose the $\tilde{F}_{i\alpha}$'s so that $\underline{h}_{ij} = \underline{0}$ all i, j. A different choice of the $\tilde{F}_{i\alpha}$'s is of the form $\tilde{F}_{i\alpha} + \epsilon h_{i\alpha}$ and $\underline{h}_i := (h_{i1}, \dots, h_{iN}) \in \Gamma(U_i, N_{X/Y})$. Since we have

$$(\tilde{F}_{j\alpha} + \epsilon h_{j\alpha}) - (\tilde{F}_{i\alpha} + \epsilon h_{i\alpha}) = \epsilon(h_{ij\alpha} + h_{j\alpha} - h_{i\alpha}) \qquad [II.3.5]$$

we see that $\{\underline{h}_{ij}\}$ defines an element $o_{\xi/Y}(e) \in H^1(X, N_{X/Y})$ which is zero if and only if the $\tilde{\mathcal{X}}_i$'s satisfy condition [II.3.4] and define a lifting $\tilde{\mathcal{X}}$ of $\mathcal{X}$ to $\tilde{A}$.

(ii) If a lifting $\tilde{\mathcal{X}}$ exists then the $\tilde{\mathcal{X}}_i$'s satisfy condition [II.3.4] and $\underline{h}_{ij} = \underline{0}$ all i, j. Every other lifting $\bar{\mathcal{X}}$ is obtained by modifying the $\tilde{F}_{i\alpha}$'s to $\bar{F}_{i\alpha} = \tilde{F}_{i\alpha} + \epsilon h_{i\alpha}$ so that

$$\underline{0} = (\bar{h}_{ij1}, \dots, \bar{h}_{ijN}) = (\bar{F}_{j1} - \bar{F}_{i1}, \dots, \bar{F}_{jN} - \bar{F}_{iN})$$

which implies, by the identities [II.3.5], that $\underline{h}_i = \underline{h}_j$ on U_{ij} so that they define a section $\underline{h} \in H^0(X, N_{XY})$. The correspondence

$$(\tilde{\mathcal{X}}, \underline{h}) \mapsto \bar{\mathcal{X}}$$

gives the action. *q.e.d.*

It is easy to show that the map o_ξ is $\mathbf{k}$-linear. The element $o_{\xi/Y}(e) \in H^1(X, N_{X/Y})$ is called *the obstruction to lift* ξ *to* $\tilde{A}$; we call ξ *obstructed* if $o_{\xi/Y}(e) \neq 0$ for some $e \in \mathrm{Ex}_{\mathbf{k}}(A, \mathbf{k})$; otherwise it is *unobstructed*. X is said to be *unobstructed in* Y if all its infinitesimal deformations in Y are unobstructed; otherwise X is said to be *obstructed in* Y. Examples of obstructed closed subschemes are usually quite subtle, especially if one is interested in nonsingular obstructed subvarieties. In order to be able to describe such examples in a natural way it is necessary to know the existence of the Hilbert scheme of a projective scheme. We will give an example in §IV.6.

(II.3.4) COROLLARY *Let $j : X \subset Y$ be a regular closed embedding. Then*
(i) X is rigid in Y if and only if $H^0(X, N_{X/Y}) = 0$.
(ii) If $H^1(X, N_{X/Y}) = (0)$ then X is unobstructed in Y.

The proof is the same as for Corollaries (II.1.9) and (II.1.10).

(II.3.5) EXAMPLES (i) Let C be a projective nonsingular curve contained in a nonsingular surface S, and assume that C is *negatively embedded* in S, i.e. $\deg(\mathcal{O}_C(C)) < 0$. Then $H^0(C, \mathcal{O}_C(C)) = 0$ and therefore C is rigid in S.

This happens in particular when $C \cong \mathbb{P}^1$ is an exceptional curve of the first kind. Another example is when C has genus $g \geq 2$, $S = C \times C$ and C is identified to the diagonal $\Delta \subset S$. In this case $N_{C/S} = T_C$ which has degree $2 - 2g < 0$. Note that $H^1(T_C) \neq (0)$ but C is unobstructed in S, being rigid in S. This example shows that the sufficient condition of Corollary (II.3.4) is not necessary.

(ii) Hypersurfaces in $\mathbb{P}^r$ are unobstructed. Infact, if $X \subset \mathbb{P}^r$ has degree d then

$$h^1(X, N_{X/\mathbb{P}^r}) = h^1(X, \mathcal{O}_X(d)) = 0$$

(iii) Let $Q \subset \mathbb{P}^3$ be a quadric cone with vertex v, and $L \subset Q$ a line. Then we have an inclusion

$$N_{L/Q} \subset N_{L/\mathbb{P}^3} = \mathcal{O}_L(1) \oplus \mathcal{O}_L(1)$$

whose cokernel is $\mathcal{O}_L(2)(-v)$ (see Note 2). It follows that $N_{L/Q} = \mathcal{O}_L(1)$, in particular it is locally trivial and $H^1(L, N_{L/Q}) = 0$, despite the fact that $L \subset Q$ is not a regular embedding (Note 2) and L is obstructed in Q (see §III.3).

* * * * * *

The characteristic map

Let $X \subset Y$ be a closed embedding of algebraic schemes, (S, s) a pointed scheme and let

$$\begin{array}{cccccc} & X & \subset & \mathcal{X} & \subset & Y \times S \\ \xi : & \downarrow & & \downarrow \pi & & \\ & \mathrm{Spec}(\mathbf{k}) & \xrightarrow{s} & S & & \end{array}$$

be a family of deformations of X in Y. By pulling back this family by morphisms $\mathrm{Spec}(\mathbf{k}[\epsilon]) \to S$ with image s and applying Theorem (II.3.1) we obtain a linear map

$$\chi_\xi : T_s S \to H^0(X, N_{X/Y})$$

called the *characteristic map* of the family ξ at s. Forgetting the embedding $\mathcal{X} \subset Y \times S$ one obtains a family deformations of $\mathcal{X}$ as an abstract scheme and, if X is nonsingular, one then has the Kodaira-Spencer map

$$\kappa_\xi : T_s S \to H^1(X, T_X)$$

(II.3.6) PROPOSITION *Let $X \subset Y$ be a closed embedding of nonsingular algebraic varieties and let*

$$\begin{array}{cccccc} & X & \subset & \mathcal{X} & \subset & Y \times S \\ \xi : & \downarrow & & \downarrow \pi & & \\ & \mathrm{Spec}(\mathbf{k}) & \xrightarrow{s} & S & & \end{array}$$

be a family of deformations of X in Y. Then we have a commutative diagram:

$$\begin{array}{ccccc} & & T_s S & & \\ & \swarrow \chi_\xi & & \searrow \kappa_\xi & \\ H^0(X, N_{X/Y}) & & \xrightarrow{\delta} & & H^1(X, T_X) \end{array}$$

where δ is the coboundary map coming from the exact sequence

$$0 \to T_X \to T_{Y|X} \to N_{X/Y} \to 0$$

Proof

It suffices to prove the assertion for a first order deformation of X in Y. Therefore we will assume that $S = \mathrm{Spec}(\mathbf{k}[\epsilon])$. Let $\chi(\xi) = \underline{h} \in H^0(X, N_{X/Y})$. Let $\mathcal{U} = \{U_i = \mathrm{Spec}(P_i)\}$ be an affine open cover of Y, and $X_i = X \cap U_i = \mathrm{Spec}(P_i/(f_{i1}, \ldots, f_{iN}))$. We have

$$\mathcal{X}_i := \mathcal{X}_{|U_i} = \mathrm{Spec}(P[\epsilon]/(f_{i1} + \epsilon h_{i1}, \ldots, f_{iN} + \epsilon h_{iN}))$$

Then $(h_{i1}, \ldots, h_{iN}) =: \underline{h}_i = \underline{h}_{|U_i} \in \Gamma(U_i, N_{X/Y})$. Since X_i is affine and nonsingular the abstract deformation $\mathcal{X}_i$ of X_i is trivial: thus there exist isomorphisms $\theta_i : X_i \times \mathrm{Spec}(\mathbf{k}[\epsilon]) \to \mathcal{X}_i$ and $\kappa(\xi) \in H^1(X, T_X)$ is defined by the 1-cocycle $\{d_{ij}\} \in \mathcal{Z}^1(\mathcal{U}, T_X)$ corresponding to the system of automorphisms

$$\theta_{ij} = \theta_i^{-1}\theta_j : X_{ij} \times \mathrm{Spec}(\mathbf{k}[\epsilon]) \to X_{ij} \times \mathrm{Spec}(\mathbf{k}[\epsilon])$$

where $X_{ij} = X \cap U_{ij}$. Let's compute $\delta(\underline{h})$. The isomorphism θ_i is given by an isomorphism of $\mathbf{k}[\epsilon]$-algebras

$$t_i : P_i[\epsilon]/(f_{i1} + \epsilon h_{i1}, \dots, f_{iN} + \epsilon h_{iN}) \to P_i[\epsilon]/(f_{i1}, \dots, f_{iN})$$

which is induced by a $\mathbf{k}[\epsilon]$-automorphism

$$T_i : P_i[\epsilon] \to P_i[\epsilon]$$

of the form $T_i(p + \epsilon q) = p + \epsilon(q + d_i(p))$ where $d_i \in \mathrm{Der}_{\mathbf{k}}(P_i, P_i) = \Gamma(U_i, T_Y)$ is such that $d_i(h_{i\alpha}) = -f_{i\alpha}$. We have $\{d_i\} \in \mathcal{C}^0(\mathcal{U}, T_Y)$ and $\delta(\underline{h})$ is defined by $\{(d_j - d_i)_{|X_i}\}$. Since $(d_j - d_i)_{|X_i} = d_{ij}$ we conclude that $\delta(\underline{h}) = \kappa(\xi)$. *q.e.d.*

A similar analysis can be made for the obstruction maps, as follows.

(II.3.7) PROPOSITION *Let $X \subset Y$ be a closed embedding of nonsingular algebraic varieties and*

$$\xi : \begin{array}{ccc} \mathcal{X} & \subset & Y \times \mathrm{Spec}(A) \\ \downarrow f & & \\ \mathrm{Spec}(A) & & \end{array}$$

where A is in $\mathcal{A}$, an infinitesimal deformation of X in Y. Then we have a commutative diagram

$$\begin{array}{ccccc} & & \mathrm{Ex}_{\mathbf{k}}(A, \mathbf{k}) & & \\ & \swarrow o_{\xi/Y} & & \searrow o_{\xi} & \\ H^1(X, N_{X/Y}) & & \xrightarrow{\delta_1} & & H^2(X, T_X) \end{array}$$

where δ_1 is the coboundary map defined by the exact sequence

$$0 \to T_X \to T_{Y|X} \to N_{X/Y} \to 0$$

The proof is similar to that of Proposition (II.3.6) and will be omitted.

* * * * * *

Deformations of morphisms with fixed domain and target

Let $f : X \to Y$ be a morphism of algebraic schemes. An *infinitesimal deformation of f (with fixed domain and target)* over A in $\mathcal{A}$ is a diagram

$$\begin{array}{ccccc} X \times S & & \xrightarrow{F} & & Y \times S \\ & \searrow & & \swarrow & \\ & & S & & \end{array}$$

where $S = \mathrm{Spec}(A)$, the morphisms to S are the projections, and f coincides with the restriction of F to the fibres over the closed point. If $A = \mathbf{k}[\epsilon]$ we have correspondingly a *first order deformation of f*.

Infinitesimal deformations of f can be interpreted as deformations of the graph of f so that the methods introduced in this section apply. Precisely:

(II.3.8) PROPOSITION *Let $f : X \to Y$ be a morphism of algebraic schemes, with X reduced and projective and Y nonsingular and quasiprojective. Then*
(i) there is a natural 1-1 correspondence

$$\left\{\text{first order deformations of } f\right\} \to H^0(X, f^*T_Y)$$

(ii) for every infinitesimal deformation

$$F : X \times S \to Y \times S$$

of f over $S = \mathrm{Spec}(A)$ there is a map

$$o_F : \mathrm{Ex}_{\mathbf{k}}(A, \mathbf{k}) \to H^1(X, f^*T_Y)$$

such that, for a given extension

$$e : 0 \to \mathbf{k} \to \tilde{A} \to A \to 0$$

we have $o_F(e) = 0$ if and only if F has a lifting to $\tilde{A}$. $o_F(e)$ is called the obstruction to lift F to $\tilde{A}$.

Proof
Let $j : \Gamma \subset X \times Y$ be the graph of f. We have natural 1-1 correspondences

$$\left\{\text{first order deformations of } f\right\} \leftrightarrow \left\{\text{first order deformations of } \Gamma \text{ in } X \times Y\right\} \leftrightarrow H^0(\Gamma, N_{\Gamma/X\times Y})$$

Since the projection $p : X \times Y \to X$ is smooth and the composition $pj : \Gamma \to X$ is an isomorphism, from Proposition (A.3.7) it follows that j is a regular embedding. Therefore applying Proposition (A.3.2) we obtain the exact sequence:

$$0 \to \mathcal{I}_\Gamma/\mathcal{I}_\Gamma^2 \to j^*\Omega^1_{X\times Y} \to \Omega^1_\Gamma \to 0$$

On the other hand we have the exact sequence:

$$0 \to f^*\Omega^1_Y \to j^*\Omega^1_{X\times Y} \to (pj)^*\Omega^1_X \to 0$$

obtained by restricting to Γ the sequence

$$0 \to q^*\Omega^1_Y \to \Omega^1_{X\times Y} \to p^*\Omega^1_X \to 0$$

(where $q : X \times Y \to Y$ is the second projection). Since $(pj)^*\Omega^1_X \cong \Omega^1_\Gamma$, comparing the two sequences we deduce that $f^*\Omega^1_Y \cong \mathcal{I}_\Gamma/\mathcal{I}_\Gamma^2$. Therefore

$$H^0(\Gamma, N_{\Gamma/X\times Y}) = \mathrm{Hom}(\mathcal{I}_\Gamma/\mathcal{I}_\Gamma^2, \mathcal{O}_\Gamma) \cong \mathrm{Hom}(f^*\Omega^1_Y, \mathcal{O}_\Gamma) = H^0(X, f^*T_Y)$$

and (i) follows.

Similarly $H^1(\Gamma, N_{\Gamma/X\times Y}) = H^1(X, f^*T_Y)$ and (ii) follows from Proposition (II.3.3). *q.e.d.*

(II.3.9) COROLLARY *If X is a nonsingular projective scheme, then the space of first order deformations of the identity $X \to X$ is $H^0(X, T_X)$.*

The proof is immediate. One can give in an obvious way the notion of *rigid morphism*. We leave to the reader the task of proving that, under the hypothesis of Proposition (II.3.7), $H^0(X, f^*T_Y) = 0$ implies that f is rigid.

(II.3.10) EXAMPLES (i) Let $f : X \to Y$ be a nonconstant morphism of projective nonsingular connected curves, with $g(Y) \geq 2$. Then $\deg(T_Y) < 0$ and therefore $h^0(X, f^*T_Y) = 0$. Thus f is rigid.

(ii) Let X be a projective irreducible and nonsingular curve, $f : X \to I\!P^r$ be a morphism and let $L = f^*\mathcal{O}_{I\!P^r}(1)$, $\deg(L) = n$. Then f is defined by a g^r_n, i.e. a linear series of degree n and dimension r on X. From the Euler sequence pulled back to X we have:

$$\chi(f^*T_{I\!P^r}) = (r+1)\chi(L) - \chi(\mathcal{O}_X) = \rho(g,r,n) + r(r+2)$$

where

$$\rho(g,r,n) := g - (r+1)(g-n+r)$$

is the *Brill-Noether number* and

$$r(r+2) = h^0(T_{I\!P^r}) = \dim[PGL(r+1)]$$

Assume that $r+1 = h^0(L)$, i.e. that f is defined by the complete linear series $|L|$, and consider the exact sequence

$$H^1(\mathcal{O}_X) \to H^1(L)^{r+1} \to H^1(f^*T_{I\!P^r}) \to 0$$

obtained from the Euler sequence. It dualizes as:

$$0 \to H^1(f^*T_{I\!P^r})^\vee \to H^0(L) \otimes H^0(\omega_X L^{-1}) \xrightarrow{\mu_0(L)} H^0(\omega_X)$$

where $\mu_0(L)$ is the natural multiplication map in cohomology, called the *Petri map*. Therefore we see that f is unobstructed if $\mu_0(L)$ is injective. A necessary condition for this to be true is that

$$(r+1)(g-n+r) = \dim[H^0(L) \otimes H^0(\omega_X L^{-1})] \leq h^0(\omega_X) = g$$

i.e. that $\rho(g,r,n) \geq 0$.

This necessary condition is not sufficient. The simplest example is given by a nonsingular complete intersection $X = Q \cap S \subset I\!P^3$ of a quadric cone Q and of a cubic surface S. Then X is a canonical curve of genus 4 and the projection from the vertex of the cone defines a complete g^1_3 $|L|$ such that $\omega_X L^{-1} \cong L$. In this case $\rho(4,1,3) = 0$ but $\dim[\ker(\mu_0(L))] = 1$.

NOTES

1. Prove the following Proposition:
If Y is a projective scheme and $C \subset Y$ is a projective integral l.c.i. curve, the normal sheaf $N_{C/Y}$ is torsion free. If C is nonsingular then $N_{C/Y}$ is locally free.

2. Consider a nonsingular curve $C \subset \mathbb{P}^3$ and a (possibly singular) surface $S \subset \mathbb{P}^3$ of degree n containing C. Prove that there is an exact sequence of locally free sheaves on C:

$$[II.3.6] \quad 0 \to \mathbf{a}^{-1} \otimes K_C(-n+4) \to N_{C/\mathbb{P}^3} \xrightarrow{\psi} \mathcal{O}_C(n) \to [\mathcal{O}_C/\mathbf{a}](n) \to 0$$

where $\mathbf{a} \subset \mathcal{O}_C$ is the ideal sheaf generated by the restriction to C of the partial derivatives

$$\frac{\partial F}{\partial X_0}, \ldots, \frac{\partial F}{\partial X_3}$$

where $F = 0$ is an equation of S.
(*Hint:* $\mathrm{Im}(\psi) = \mathcal{O}_C \otimes \mathrm{Im}(T_{\mathbb{P}^3|S} \to N_{S/\mathbb{P}^3})$).
In case S is nonsingular we obtain the sequence:

$$[II.3.7] \qquad 0 \to K_C(-n+4) \to N_C \to \mathcal{O}_C(n) \to 0$$

Deduce from [II.3.6] that if $\mathbf{a} \neq \mathcal{O}_C$ (i.e. if $C \cap \mathrm{Sing}(S) \neq \emptyset$) then C is not regularly embedded in S (yet the normal sheaf $N_{C/S}$ is locally free).

3. Consider a nonsingular curve $C \subset \mathbb{P}^3$ and a point $p \in \mathbb{P}^3 \backslash C$. Prove that there is an exact sequence

$$0 \to \mathcal{O}_C(1) \to N_{C/\mathbb{P}^3} \to \omega_C(3) \to 0$$

which is obtained as a special case of [II.3.6] by taking as S the cone projecting C from p. Deduce that

$$h^1(C, N_{C/\mathbb{P}^3}) \leq h^1(C, \mathcal{O}_C(1))$$

4. Let $X \subset \mathbb{P}^r$ be a nonsingular irreducible projective curve embedded by a non special linear series. Show that $H^1(X, N_{X/\mathbb{P}^r}) = (0)$ and therefore X is unobstructed in $\mathbb{P}^r$.

5. Let $X \subset \mathbb{P}^r$ be a nonsingular irreducible projective curve and let $L = \mathcal{O}_X(1) = \mathcal{O}_{\mathbb{P}^r}(1) \otimes \mathcal{O}_X$. Show that if the Petri map $\mu_0(L)$ (see example (II.3.10)(ii)) is injective then $H^1(X, N_{X/\mathbb{P}^r}) = (0)$ and X is unobstructed in $\mathbb{P}^r$. Deduce that canonical curves of genus $g \geq 3$ are unobstructed in $\mathbb{P}^{g-1}$.

6. Let $X \subset \mathbb{P}^r$ be a projective irreducible nonsingular curve of degree d and genus g. Prove that

$$\chi(N_{X/\mathbb{P}^r}) := h^0(X, N_{X/\mathbb{P}^r}) - h^1(X, N_{X/\mathbb{P}^r}) = (r+1)d + (r-3)(1-g)$$

II.4. AFFINE SCHEMES (I)

Let B_0 be a $\mathbf{k}$-algebra, and let $X_0 = \mathrm{Spec}(B_0)$. We continue the study of inifinitesimal deformations of X_0, equivalently of B_0, started in section II.1 in the nonsingular case.

(II.4.1) PROPOSITION *There is a natural 1-1 correspondence*

$$\frac{\left\{\textit{first order deformations of } X_0\right\}}{\textit{isomorphism}} \leftrightarrow T^1_{B_0}$$

where the class of trivial deformations corresponds to $0 \in T^1_{B_0}$.

Proof
A first order deformation of B_0 consists of a flat $\mathbf{k}[\epsilon]$-algebra B, plus a $\mathbf{k}$-isomorphism $B \otimes_{\mathbf{k}[\epsilon]} \mathbf{k} \cong B_0$. This set of data determines a $\mathbf{k}$-extension:

$$[II.4.1] \qquad \begin{array}{ccccccccc} 0 & \to & B_0 & \xrightarrow{j} & B & \to & B_0 & \to & 0 \\ & & \| & & & & & & \\ & & \epsilon B & & & & & & \end{array}$$

obtained after tensoring the exact sequence

$$0 \to (\epsilon) \to \mathbf{k}[\epsilon] \to \mathbf{k} \to 0$$

by $\otimes_{\mathbf{k}[\epsilon]} B$. Isomorphic deformations give rise to isomorphic extensions. Conversely, given a $\mathbf{k}$-extension [II.4.1], B has a structure of $\mathbf{k}[\epsilon]$-algebra given by

$$\epsilon \mapsto j(1)$$

B is $\mathbf{k}[\epsilon]$-flat by Lemma (A.2.8). *q.e.d.*

We will give some indications for the practical computation of $T^1_{B_0}$ when B_0 is e.f.t..

Let $B_0 = P/J$, where P is a smooth $\mathbf{k}$-algebra of the form

$$P = \Delta^{-1}\mathbf{k}[X_1, \ldots, X_d]$$

for some multiplicative system $\Delta \subset \mathbf{k}[X_1, \ldots, X_d]$, and $J \subset P$ is an ideal.

Consider the exact sequence

$$0 \to \mathrm{Hom}(\Omega_{B_0/\mathbf{k}}, B_0) \to \mathrm{Hom}(\Omega_{P/\mathbf{k}} \otimes B_0, B_0) \xrightarrow{\delta^\vee} \mathrm{Hom}(J/J^2, B_0) \longrightarrow T^1_{B_0} \to 0$$

The module

$$\mathrm{Hom}(\Omega_{P/\mathbf{k}} \otimes B_0, B_0) = \mathrm{Der}_{\mathbf{k}}(P, B_0)$$

consists of all derivations D of the form

$$D(p) = \sum_{j=1}^{d} b_j \frac{\partial p}{\partial X_j}$$

for given $b_j \in B_0$, and

$$\delta^\vee(D)(\bar{f}) = D(f) = \sum_{j=1}^{d} b_j \frac{\partial f}{\partial X_j} \qquad f \in J$$

Assume that $J = (f_1, \ldots, f_n)$ and let

$$0 \to \mathbf{R} \xrightarrow{\iota} P^n \xrightarrow{j} J \to 0$$

be the corresponding presentation. We have the exact sequence:

$$0 \to \mathrm{Hom}(J/J^2, B_0) \xrightarrow{j^\vee} \mathrm{Hom}(B_0^n, B_0) \xrightarrow{\iota^\vee} \mathrm{Hom}(\mathbf{R}, B_0)$$

where $j^\vee$ identifies $\mathrm{Hom}(J/J^2, B_0)$ with the submodule of $\mathrm{Hom}(B_0^n, B_0)$ consisting of those homomorphisms which are 0 on $\mathbf{R}$. Identifying $\mathrm{Hom}(B_0^n, B_0) = B_0^n$, thereby viewing its elements as column vectors, we see that the condition for

$$\underline{q} = \begin{pmatrix} q_1 \\ \vdots \\ q_n \end{pmatrix} \in B_0^n$$

to be in $\mathrm{Hom}(J/J^2, B_0)$ is that ${}^t\underline{q} \cdot \underline{r} = 0$ for each $\underline{r} \in \mathbf{R}$ (where we are viewing $\mathbf{R}$ as consisting of column vectors as well). $j^\vee$ associates to a homomorphism

$$\varphi : \begin{array}{ccc} J/J^2 & \to & B_0 \\ \sum b_j \bar{f}_j & \mapsto & \sum b_j \varphi(\bar{f}_j) \end{array}$$

the column vector

$$\begin{pmatrix} \varphi(\bar{f}_1) \\ \vdots \\ \varphi(\bar{f}_n) \end{pmatrix}$$

Therefore $\mathrm{Im}(\delta^\vee) \subset B_0^n$ is generated by the column vectors corresponding to $\delta^\vee(\frac{\partial}{\partial X_1}), \ldots,$ $\delta^\vee(\frac{\partial}{\partial X_d})$, i.e. by the classes mod J of:

$$\begin{pmatrix} \frac{\partial f_1}{\partial X_1} \\ \vdots \\ \frac{\partial f_n}{\partial X_1} \end{pmatrix}, \ldots, \begin{pmatrix} \frac{\partial f_1}{\partial X_d} \\ \vdots \\ \frac{\partial f_n}{\partial X_d} \end{pmatrix}$$

If $J = (f_1, \ldots, f_n)$ is generated by a regular sequence then $\iota^\vee = 0$, equivalently $j^\vee$ is an isomorphism, and it follows that

$$T^1_{B_0} \cong \frac{B_0^n}{\begin{pmatrix} \frac{\partial f_1}{\partial X_1} & \cdots & \frac{\partial f_1}{\partial X_d} \\ \vdots & \vdots & \vdots \\ \frac{\partial f_n}{\partial X_1} & \cdots & \frac{\partial f_n}{\partial X_d} \end{pmatrix}}$$

In particular, if $B_0 = P/(f)$ then

$$T^1_{B_0} \cong \frac{B_0}{(\frac{\partial f}{\partial X_1}, \ldots, \frac{\partial f}{\partial X_d})} = \frac{P}{(f, \frac{\partial f}{\partial X_1}, \ldots, \frac{\partial f}{\partial X_d})}$$

It follows from this description that the hypersurface $V(f)$ is rigid if and only if it is nonsingular. In particular *a singular hypersurface is not rigid.* A similar remark holds for complete intersections.

(II.4.2) Let P be the local ring of a nonsingular algebraic surface X at a $\mathbf{k}$-rational point p, $m = (x, y)$ its maximal ideal, and $B_0 = P/(f)$ the local ring of a curve $C \subset X$ at p. Let's compute $T^1_{B_0}$ in some cases. We assume $\mathrm{char}(\mathbf{k}) = 0$ here.

a) *Node (ordinary double point)* - By definition $\hat{B}_0 \cong \mathbf{k}[[X, Y]]/(X^2 + Y^2)$. Then $f = x^2 + y^2 +$ higher order terms, and

$$T^1_{B_0} \cong \frac{B_0}{(x, y)} \cong \mathbf{k}$$

b) *Ordinary cusp* - In this case $\hat{B}_0 \cong \mathbf{k}[[X, Y]]/(X^2 + Y^3)$. Then $f = x^2 + y^2 +$ higher order terms, and

$$T^1_{B_0} \cong \frac{B_0}{(y, x^2)} \cong \mathbf{k}^2$$

c) *Tacnode* - We have in this case $\hat{B}_0 \cong \mathbf{k}[[X, Y]]/(Y(Y + X^2))$ and

$$T^1_{B_0} \cong \frac{B_0}{(x^2 + 2y, xy)} \cong \mathbf{k}^3$$

Conversely we have the following result:

(II.4.3) PROPOSITION *Let P be the local ring of a nonsingular algebraic surface X at a* **k***-rational point p, $m = (x,y)$ its maximal ideal, and $B_0 = P/(f)$ the local ring of a curve $C \subset X$ at p; let $t = \dim_{\mathbf{k}} T^1_{B_0}$. Then*
(a) $t = 0$ if and only if B_0 is regular (a DVR).
(b) $t = 1$ if and only if B_0 is the local ring of a node.
(c) $t = 2$ if and only if B_0 is the local ring of an ordinary cusp.

Proof
The 'if' implication follows from the above computations. We have

$$t = \dim_{\mathbf{k}} P/(f, f_x, f_y)$$

$f \in m_P^3$ immediately implies $t \geq 4$; then $f \in m_P^2$ and, after suitable choice of generators of m_P we may suppose $f = y^2 + x^n +$ higher order terms, $n \geq 2$ or $f = y(y+x^n)+$ higher order terms, $n \geq 2$. Now the conclusion follows easily. *q.e.d.*

(II.4.4) *The affine cone over* $\mathbb{P}^1 \times \mathbb{P}^2$ - Let

$$P = \mathbf{k}[X_0, X_1, X_2, Y_0, Y_1, Y_2]$$

$$J = (X_1Y_2 - X_2Y_1, X_2Y_0 - X_0Y_2, X_0Y_1 - X_1Y_0)$$

Then $B_0 = P/J$ is the coordinate ring of the affine cone over the Segre embedding $\mathbb{P}^1 \times \mathbb{P}^2 \subset \mathbb{P}^5$. We have the following presentation:

$$0 \to P^2 \xrightarrow{A} P^3 \to J \to 0$$

where

$$A = \begin{pmatrix} X_0 & Y_0 \\ X_1 & Y_1 \\ X_2 & Y_2 \end{pmatrix}$$

A direct computation shows that $\mathrm{Hom}(J/J^2, B_0)$ is generated by the following column vectors:

$$\begin{matrix} Y_1 & Y_2 & 0 & X_1 & X_2 & 0 \\ -Y_0 & 0 & Y_2 & -X_0 & 0 & X_2 \\ 0 & -Y_0 & -Y_1 & 0 & -X_0 & -X_1 \end{matrix}$$

Since these vectors are, up to permutation,

$$\delta^\vee(\frac{\partial}{\partial X_0}),\ \delta^\vee(\frac{\partial}{\partial X_1}),\ \delta^\vee(\frac{\partial}{\partial X_2}),\ \delta^\vee(\frac{\partial}{\partial Y_0}),\ \delta^\vee(\frac{\partial}{\partial Y_1}),\ \delta^\vee(\frac{\partial}{\partial Y_2})$$

we see that $T^1_{B_0} = 0$. This implies that B_0 is rigid (see Corollary (II.5.4)).

More generally one can prove that *the coordinate ring of the affine cone over the Segre embedding $\mathbb{P}^n \times \mathbb{P}^m \subset \mathbb{P}^{(n+1)(m+1)-1}$ is rigid whenever $n + m \geq 3$.* This has been computed for the first time in Grauert-Kerner(1964) in the case

$n = m \geq 2$; the general case is in Schlessinger(1973). We will give the proof in Corollary (II.5.11) below.

(II.4.5) Let $P = \mathbf{k}[X_1, X_2, X_3]_{(\underline{X})}$, $J = (X_2X_3, X_1X_3, X_1X_2)$. Then $B_0 = P/J$ is the local ring at the origin of the union of the coordinate axes in $\mathbf{A}^3$. We have the presentation

$$P^3 \xrightarrow{A} P^3 \to J \to 0$$

where

$$A = \begin{pmatrix} X_1 & X_1 & 0 \\ -X_2 & 0 & X_2 \\ 0 & -X_3 & -X_3 \end{pmatrix}$$

and the columns of A generate $\mathbf{R}$.

$\mathrm{Hom}(J/J^2, B_0)$ is generated by the following column vectors mod J:

$$\begin{matrix} X_2 & X_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & X_1 & X_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & X_1 & X_2 \end{matrix}$$

and $\mathrm{Im}(\delta^\vee)$ is generated by the column vectors mod J:

$$\begin{matrix} 0 & X_3 & X_2 \\ X_3 & 0 & X_1 \\ X_2 & X_1 & 0 \end{matrix}$$

It follows at once that

$$T^1_{B_0} = \mathrm{Hom}(J/J^2, B_0)/\mathrm{Im}(\delta^\vee) \cong \mathbf{k}^3$$

because there are 3 generators of $\mathrm{Hom}(J/J^2, B_0)$ which are linearly independent modulo the generators of $\mathrm{Im}(\delta^\vee)$, and all other elements of $\mathrm{Hom}(J/J^2, B_0)$ are in $\mathrm{Im}(\delta^\vee)$.

In a similar vein one can consider, for any $d \geq 3$

$$B_0 = \mathbf{k}[X_1, \ldots, X_d]_{(\underline{X})}/J$$

where

$$J = (\ldots, X_1X_2 \cdot \hat{X}_i \cdot X_d, \ldots)_{i=1,\ldots,d}$$

Then B_0 is the local ring at the origin of the union of the coordinate axes in $\mathbf{A}^d$. One computes easily, along the same lines of the case $d = 3$, that

$$T^1_{B_0} \cong \mathbf{k}^{d(d-2)}$$

NOTES

1. The main references for this and the following Section are Schlessinger(1964), Lichtenbaum-Schlessinger(1967), Artin(1976).

2. (Sheets(1977)) Let B be an e.f.t. local $\mathbf{k}$-algebra, $I \subset B$ an ideal generated by a regular sequence. Prove that if B/I is rigid then B is rigid as well.

3. Let P be the local ring of a nonsingular algebraic variety V of dimension $n \geq 2$ at a $\mathbf{k}$-rational point p and let $B_0 = P/(f)$ be the local ring of a hypersurface $X \subset V$ at p. We say that p is a *node* for X if $\hat{B}_0 \cong \mathbf{k}[[X_1, \ldots, x_n]]/(X_1^2 + \cdots + X_n^2)$, equivalently if we can choose generators $x_1, \ldots, x_n$ of the maximal ideal m_P so that $f = \sum x_i^2 +$ higher order terms. Prove that if p is a node for X then $T^1_{B_0} \cong \mathbf{k}$.

II.5. AFFINE SCHEMES (II)

The second cotangent module and obstructions

Assume that $B_0 = P/J$ for a smooth $\mathbf{k}$-algebra P, and an ideal $J \subset P$. Consider a presentation:

$$\eta : \quad 0 \to \mathbf{R} \xrightarrow{\iota} F \xrightarrow{j} J \to 0$$

where F is a finitely generated free P-module. Let $\lambda : \bigwedge^2 F \to F$ be defined by:

$$\lambda(x \wedge y) = (jx)y - (jy)x$$

and $\mathbf{R}^{tr} = \mathrm{Im}(\lambda)$. Obviously $\mathbf{R}^{tr} \subset \mathbf{R}$ and $\mathbf{R}^{tr} \subset JF$.

If $J = (f_1, \ldots, f_n)$ then $F = P^n$ and $\mathbf{R}$ is the module of relations among $f_1, \ldots, f_n$. $\mathbf{R}^{tr}$ is called the module of *trivial (or Koszul) relations*; it is generated by the relations of the form

$$\begin{array}{ccccccc} (0, & \ldots, & -f_j, & \ldots, & f_i, & \ldots, & 0) \\ & & i & & j & & \end{array}$$

Note that $\mathbf{R}/\mathbf{R}^{tr} = H_1(K_\bullet(f_1, \ldots, f_n))$, the first homology module of the Koszul complex associated to $f_1, \ldots, f_n$.

(II.5.1) LEMMA *The P-module $\mathbf{R}/\mathbf{R}^{tr}$ is annihilated by J and therefore it is a B_0-module in a natural way.*

Proof

Let $x \in \mathbf{R}$, $a \in J$. Let $y \in F$ be such that $j(y) = a$. Then

$$ax = j(y)x = j(y)x - j(x)y = \lambda(y \wedge x) \in \mathbf{R}^{tr}$$

q.e.d.

Since $\mathbf{R}^{tr} \subset JF$ the presentation η induces an exact sequence of B_0-modules:

$$[II.5.1] \qquad \mathbf{R}/\mathbf{R}^{tr} \xrightarrow{\bar{\iota}} F \otimes_P B_0 \xrightarrow{\bar{j}} J/J^2 \to 0$$

We define $T^2_{B_0}$ by the induced exact sequence:

$$0 \to \mathrm{Hom}_{B_0}(J/J^2, B_0) \to \mathrm{Hom}_{B_0}(F \otimes_P B_0, B_0) \to \mathrm{Hom}_{B_0}(\mathbf{R}/\mathbf{R}^{tr}, B_0) \to T^2_{B_0} \to 0$$

Obviously $T^2_{B_0}$ is a B_0-module of finite type. It is called the *second cotangent module of* B_0.

(II.5.2) LEMMA *For every e.f.t.* $\mathbf{k}$*-algebra* B_0 *the* B_0*-module* $T^2_{B_0}$ *is independent of the presentation* η.

Proof
Assume that $F \cong P^n$ and that $j : P^n \to J$ is defined by the system of generators $f_1, \dots, f_n$ of J. Let

$$0 \to \mathbf{S} \to P^m \to J \to 0$$

be another presentation of J, defined by the system of generators $g_1, \dots, g_m$. We may assume that $m \geq n$ and that $g_k = f_k$, $k = 1, \dots, n$. Let

$$g_k = \sum_s b_{ks} f_s \qquad k = n+1, \dots, m$$

for some $b_{ks} \in P$. Denote by $\alpha : P^n \to P^m$ the map

$$\alpha(a_1, \dots, a_n) = (a_1, \dots, a_n, 0, \dots, 0)$$

and by $\beta : P^m \to P^n$ the map

$$\beta(a_1, \dots, a_m) = (a_1 + \sum_{s=n+1}^{m} b_{1s} a_s, \dots, a_n + \sum_{s=n+1}^{m} b_{ns} a_s)$$

Evidently $\alpha(\mathbf{R}) \subset \mathbf{S}$ and $\alpha(\mathbf{R}^{tr}) \subset \mathbf{S}^{tr}$. It is easy to verify that $\beta(\mathbf{S}) \subset \mathbf{R}$ and $\beta(\mathbf{S}^{tr}) \subset \mathbf{R}^{tr}$. It follows that α and β induce homomorphisms

$$\begin{aligned} \beta^\star &: \mathrm{Hom}(\mathbf{R}/\mathbf{R}^{tr}, B_0) \to \mathrm{Hom}(\mathbf{S}/\mathbf{S}^{tr}, B_0) \\ \alpha^\star &: \mathrm{Hom}(\mathbf{S}/\mathbf{S}^{tr}, B_0) \to \mathrm{Hom}(\mathbf{R}/\mathbf{R}^{tr}, B_0) \end{aligned}$$

whence homomorphisms

$$\begin{aligned} \tilde{\beta} &: \mathrm{Hom}(\mathbf{R}/\mathbf{R}^{tr}, B_0)/\mathrm{Hom}(P^n, B_0) \to \mathrm{Hom}(\mathbf{S}/\mathbf{S}^{tr}, B_0)/\mathrm{Hom}(P^m, B_0) \\ \tilde{\alpha} &: \mathrm{Hom}(\mathbf{S}/\mathbf{S}^{tr}, B_0)/\mathrm{Hom}(P^m, B_0) \to \mathrm{Hom}(\mathbf{R}/\mathbf{R}^{tr}, B_0)/\mathrm{Hom}(P^n, B_0) \end{aligned}$$

Since

$$\alpha^\star \beta^\star = \text{identity of } \mathrm{Hom}(\mathbf{R}/\mathbf{R}^{tr}, B_0)$$

it follows that

$$\tilde{\alpha}\tilde{\beta} = \text{identity of } \mathrm{Hom}(\mathbf{R}/\mathbf{R}^{tr}, B_0)/\mathrm{Hom}(P^n, B_0)$$

We now prove that

$$[II.5.2] \qquad \tilde{\beta}\tilde{\alpha} = \text{identity of } \mathrm{Hom}(\mathbf{S}/\mathbf{S}^{tr}, B_0)/\mathrm{Hom}(P^m, B_0)$$

Let $g \in \mathrm{Hom}(\mathbf{S}/\mathbf{S}^{tr}, B_0)$ be induced by the homomorphism $G : \mathbf{S} \to B_0$. Then, if $(a_1, \ldots, a_m) \in \mathbf{S}$ and $(\bar{a}_1, \ldots, \bar{a}_m) \in \mathbf{S}/\mathbf{S}^{tr}$ is its class, we have:

$$(\beta^\star \alpha^\star)(g)(\bar{a}_1, \ldots, \bar{a}_m) = G(\alpha(\beta(a_1, \ldots, a_m))) =$$

$$G(a_1 + \textstyle\sum_{s=n+1}^m b_{1s}a_s, \ldots, a_n + \sum_{s=n+1}^m b_{ns}a_s, 0, \ldots, 0) =$$

$$G(a_1, \ldots, a_m) + G(\textstyle\sum_{s=n+1}^m b_{1s}a_s, \ldots, \sum_{s=n+1}^m b_{ns}a_s, -a_{n+1}, \ldots, -a_m) = (*)$$

Now note that

$$(\sum_{s=n+1}^m b_{1s}p_s, \ldots, \sum_{s=n+1}^m b_{ns}p_s, -p_{n+1}, \ldots, -p_m) \in \mathbf{S}$$

for every $(p_1, \ldots, p_m) \in P^m$. Therefore letting

$$\tau(p_1, \ldots, p_m) = G(\sum_{s=n+1}^m b_{1s}p_s, \ldots, \sum_{s=n+1}^m b_{ns}p_s, -p_{n+1}, \ldots, -p_m)$$

we define a homomorphism $\tau : P^m \to B_0$. It follows that

$$(*) = g(\bar{a}_1, \ldots, \bar{a}_m) + \tau(a_1, \ldots, a_m)$$

Hence

$$(\beta^\star \alpha^\star)(g) - g \in \mathrm{Im}[\mathrm{Hom}(P^m, B_0) \to \mathrm{Hom}(\mathbf{S}/\mathbf{S}^{tr}, B_0)]$$

or equivalently [II.5.2] holds. *q.e.d.*

From the definition it easily follows that $T^2_{B_0}$ localizes. Namely, for every multiplicative subset $\Delta \subset P$ we have:

$$\Delta^{-1}T^2_{B_0} = T^2_{\Delta^{-1}B_0}$$

It follows that for any scheme X we can define in an obvious way the *second cotangent sheaf* which we will denote by T^2_X. It satisfies

$$T^2_{X,x} = T^2_{\mathcal{O}_{X,x}}$$

(II.5.3) PROPOSITION *Assume that $B_0 = P/J$ for a smooth $\mathbf{k}$-algebra P. Then for every A in $\mathcal{A}$ and for every deformation ξ of B_0 over A there is a $\mathbf{k}$-linear map*

$$\xi_v : \mathrm{Ex}_{\mathbf{k}}(A, \mathbf{k}) \to T^2_{B_0}$$

whose kernel consists of the extensions

$$\eta : 0 \to \mathbf{k} \to \tilde{A} \to A \to 0$$

such that ξ has a lifting over $\tilde{A}$.
Moreover if $\xi_v(\eta) = 0$ then there is a natural transitive action of $T^1_{B_0}$ on the set of isomorphism classes of liftings of ξ over $\tilde{A}$.

Proof
Let A be an object of $\mathcal{A}$ and let

$$\xi: \begin{array}{ccc} B & \to & B_0 \\ \uparrow & & \uparrow \\ A & \to & \mathbf{k} \end{array}$$

be an infinitesimal deformation of B_0 over A.

We must associate to ξ a $\mathbf{k}$-linear map

$$\xi_v : \mathrm{Ex}_{\mathbf{k}}(A, \mathbf{k}) \to T^2_{B_0}$$

satisfying the conditions of the statement. Let $B_0 = P/J$ for a smooth $\mathbf{k}$-algebra P and an ideal $J = (f_1, \ldots, f_n) \subset P$. We have an exact sequence:

$$0 \to \mathbf{R} \to P^n \xrightarrow{\mathbf{f}} J \to 0$$

Then, by the smoothness of P we have

$$B = (P \otimes_{\mathbf{k}} A)/(F_1, \ldots, F_n)$$

where $f_j = F_j \pmod{m_A}$, $j = 1, \ldots, n$. The flatness of B over A implies that for every $\underline{r} = (r_1, \ldots, r_n) \in \mathbf{R}$ there exist $R_1, \ldots, R_n \in P \otimes_{\mathbf{k}} A$ such that $r_j = R_j$ (mod. m_A), $j = 1, \ldots, n$, and $\sum_j R_j F_j = 0$.

Let $[\gamma] \in \mathrm{Ex}_{\mathbf{k}}(A, \mathbf{k})$ be represented by an extension

$$\gamma: \quad 0 \to \epsilon\mathbf{k} \to \tilde{A} \xrightarrow{\phi} A \to 0$$

Choose $\tilde{F}_1, \ldots, \tilde{F}_n, \tilde{R}_1, \ldots, \tilde{R}_n \in P \otimes_{\mathbf{k}} \tilde{A}$ liftings of $F_1, \ldots, F_n, R_1, \ldots, R_n$ respectively; then

$$\sum_j \tilde{R}_j \tilde{F}_j \in \ker[P \otimes_{\mathbf{k}} \tilde{A} \to P \otimes_{\mathbf{k}} A] = \epsilon P \cong P$$

It is easy to check that a different choice of $\tilde{R}_1, \ldots, \tilde{R}_n$ or of $\tilde{F}_1, \ldots, \tilde{F}_n$ modifies $\sum_j \tilde{R}_j \tilde{F}_j$ by an element of J or by one of the form $\sum_j q_j r_j$, where $q_j \in P$, respectively. Therefore sending

[*II*.5.3] $$\underline{r} \mapsto \sum_j \tilde{R}_j \tilde{F}_j$$

defines an element of

$$\mathrm{coker}[\mathrm{Hom}(B_0^n, B_0) \to \mathrm{Hom}(\mathbf{R}, B_0)]$$

Moreover, since if $\underline{r} = \underline{r}_{ij} = (0, \ldots, f_j, \ldots, -f_i, \ldots, 0)$ we can take

$$(\tilde{R}_1, \ldots, \tilde{R}_n) = (0, \ldots, \tilde{F}_j, \ldots, -\tilde{F}_i, \ldots, 0)$$

and we get $\sum_j \tilde{R}_j \tilde{F}_j = 0$, it follows that [II.5.3] is zero on $\mathrm{Hom}(\mathbf{R}^{tr}, B_0)$. Therefore the n-tuple of liftings $(\underline{\tilde{F}}) = (\tilde{F}_1, \ldots, \tilde{F}_n)$ defines an element $\xi_v(\gamma)$ of

$$\mathrm{coker}[\mathrm{Hom}(B_0^n, B_0) \to \mathrm{Hom}(\mathbf{R}/\mathbf{R}^{tr}, B_0)] = T^2_{B_0}$$

Let's prove that the map $\gamma \mapsto \xi_v(\gamma)$ is $\mathbf{k}$-linear.
Let $[\zeta] \in \mathrm{Ex}_{\mathbf{k}}(A, \mathbf{k})$ be another element defined by the extension:

$$\zeta : \quad 0 \to \epsilon\mathbf{k} \to A' \to A \to 0$$

and let $(\underline{F}') = (F'_1, \ldots, F'_n)$, $F'_j \in P \otimes_{\mathbf{k}} A'$ be the corresponding lifting, which defines $\xi_v(\zeta)$. Then $\xi_v(\gamma) + \xi_v(\zeta)$ is defined by

$$\underline{r} \mapsto \sum_j \tilde{R}_j \tilde{F}_j + \sum_j R'_j F'_j$$

where $R'_1, \ldots, R'_n \in P \otimes_{\mathbf{k}} A'$ are liftings of $R_1, \ldots, R_n$. Consider the diagram:

$$\begin{array}{cccccccc} & 0 \to & \mathbf{k} \oplus \mathbf{k} & \to & \tilde{A} \times_A A' & \to & A & \to 0 \\ & & \downarrow \delta & & \downarrow \sigma & & \| & \\ \gamma + \zeta : & 0 \to & \mathbf{k} & \to & C & \to & A & \to 0 \end{array}$$

Then $\xi_v(\gamma + \zeta)$ is defined by

$$\underline{r} \mapsto \sum_j \Psi_j \Phi_j$$

where $\Psi_1, \ldots, \Psi_n, \Phi_1, \ldots, \Phi_n \in P \otimes_{\mathbf{k}} C$ are liftings of $R_1, \ldots, R_n, F_1, \ldots, F_n$.

Since

$$P \otimes_{\mathbf{k}} (\tilde{A} \times_A A') \cong (P \otimes_{\mathbf{k}} \tilde{A}) \times_A (P \otimes_{\mathbf{k}} A')$$

letting $\rho : P \otimes_{\mathbf{k}} (\tilde{A} \times_A A') \to P \otimes_{\mathbf{k}} C$ be the homomorphism induced by σ, we may assume that

$$\Phi_j = \rho(\tilde{F}_j, F'_j); \quad \Psi_j = \rho(\tilde{R}_j, R'_j)$$

Then:

$$\sum_j \Psi_j \Phi_j =$$

$$= \sum_j \rho(\tilde{R}_j, R'_j)\rho(\tilde{F}_j, F'_j) = \rho[\sum_j (\tilde{R}_j, R'_j)(\tilde{F}_j, F'_j)] =$$

$$= \rho(\sum_j \tilde{R}_j \tilde{F}_j, \sum_j R'_j, F'_j) = \delta(\sum_j \tilde{R}_j \tilde{F}_j, \sum_j R'_j, F'_j) =$$

$$= \sum_j \tilde{R}_j \tilde{F}_j + \sum_j R'_j, F'_j$$

This proves that $\xi_v(\gamma+\zeta) = \xi_v(\gamma)+\xi_v(\zeta)$. A similar argument shows that $\xi_v(\lambda\gamma) = \lambda\xi_v(\gamma)$, $\lambda \in \mathbf{k}$.

Now assume that $[\gamma] \in \mathrm{Ex}_{\mathbf{k}}(A, \mathbf{k})$ is such that there exists an infinitesimal deformation

$$\tilde{\xi} : \begin{array}{ccc} \tilde{B} & \to & B_0 \\ \uparrow & & \uparrow \\ \tilde{A} & \to & \mathbf{k} \end{array}$$

such that

$$\begin{array}{ccc} \mathrm{Def}_{B_0}(\tilde{A}) & \to & \mathrm{Def}_{B_0}(A) \\ & & \\ \tilde{\xi} & \mapsto & \xi \end{array}$$

It follows that there exist liftings $\tilde{F}_j \in P \otimes_{\mathbf{k}} \tilde{A}$ of the F_j's such that every $\underline{r} \in \mathbf{R}$ has a lifting $\underline{\tilde{R}} \in (P \otimes_{\mathbf{k}} \tilde{A})^N$ such that $\sum_j \tilde{R}_j \tilde{F}_j = 0$. This means that $\xi_v(\gamma) = 0$.

Conversely, assume that $\xi_v(\gamma) = 0$, and let $\tilde{F}_1, \ldots, \tilde{F}_n \in P \otimes_{\mathbf{k}} \tilde{A}$ be arbitrary liftings of $F_1, \ldots, F_n$. Then there exists $(h_1, \ldots, h_n) \in P^n$ such that for every choice of a lifting $\underline{\tilde{R}} \in (P \otimes_{\mathbf{k}} \tilde{A})^n$ of a relation $\underline{r} \in \mathbf{R}$ we have:

$$\sum_j \tilde{R}_j \tilde{F}_j = -\epsilon \sum_j r_j h_j = -\sum_j \tilde{R}_j h_j$$

This means that the ideal $(\tilde{F}_1+\epsilon h_1, \ldots, \tilde{F}_n+\epsilon h_n) \subset P\otimes_{\mathbf{k}}\tilde{A}$ defines a flat deformation of B_0 over $\tilde{A}$ lifting the deformation $B = (P \otimes_{\mathbf{k}} A)/(F_1, \ldots, F_n)$.

Any other choice of a lifting of the deformation ξ over $\tilde{A}$ is of the form

$$(\tilde{F}_1 + \epsilon(h_1 + k_1), \ldots, \tilde{F}_n + \epsilon(h_n + k_n))$$

where $\underline{k} = (k_1, \ldots, k_n) \in B_0^n$ satisfy $\sum_j r_j k_j = 0$ for every relation $\underline{r} \in \mathbf{R}$. Therefore $\underline{k} \in \mathrm{Hom}(J/J^2, B_0)$. It is straightforward to verify that if $\underline{k} \in \mathrm{Im}(\delta^\vee)$ then $\underline{\tilde{F}} + \epsilon\underline{h}$ and $\underline{\tilde{F}} + \epsilon(\underline{h} + \underline{k})$ define isomorphic liftings of ξ over $\tilde{A}$. This means that we have an action of $T^1_{B_0}$ on the set of liftings of ξ over $\tilde{A}$. By construction it follows that this action is transitive. *q.e.d.*

(II.5.4) COROLLARY *(i) If B_0 is an e.f.t. local complete intersection* $\mathbf{k}$-*algebra then it is unobstructed. In particular hypersurface singularities are unobstructed.*
(ii) If $T^1_{B_0} = 0$ then B_0 is rigid.

Proof
(i) If J is generated by a regular sequence then $\mathbf{R} = \mathbf{R}^{tr}$ and therefore $T^2_{B_0} = (0)$.
(ii) Left to the reader. *q.e.d.*

(II.5.5) PROPOSITION *With the same notations as in Proposition [II.5.3] we have:*
(i) If the locus where B_0 is a l.c.i. is dense in $\mathrm{Spec}(B_0)$ *(e.g. B_0 is reduced) then*

$$T^2_{B_0} \cong \mathrm{Ext}^1_{B_0}(J/J^2, B_0)$$

(ii) Under the additional hypothesis that $\mathrm{Spec}(B_0)$ *is reduced and has depth at least 2 along the locus where it is not a l.c.i. (e.g.* $\mathrm{Spec}(B_0)$ *is normal of dimension* ≥ 2*) there is an isomorphism*

$$T^2_{B_0} \cong \mathrm{Ext}^2(\Omega_{B_0/\mathbf{k}}, B_0)$$

Proof
(i) From the exact sequence [II.5.1] we deduce the following commutative diagram with exact rows and columns:

$$\begin{array}{ccccccccc} & & & & \mathrm{Hom}(\ker(\bar\imath), B_0) & & & & \\ & & & & \uparrow & & & & \\ 0 \to \mathrm{Hom}(J/J^2, B_0) \to & \mathrm{Hom}(F \otimes_P B_0, B_0) & \xrightarrow{\bar{j}^\vee} & \mathrm{Hom}(\mathbf{R}/\mathbf{R}^{tr}, B_0) & & \to T^2_{B_0} \to 0 \\ \| & \| & & \cup & & \cup \\ 0 \to \mathrm{Hom}(J/J^2, B_0) \to & \mathrm{Hom}(F \otimes_P B_0, B_0) & \to & \mathrm{Hom}(\mathrm{Im}(\bar\imath), B_0) & & \to \mathrm{Ext}^1_{B_0}(J/J^2, B_0) \to 0 \end{array}$$

Since the exact sequence η localizes we see that $\ker(\bar\imath)$ is supported on the locus where B_0 is not a l.c.i.; in particular $\ker(\bar\imath)$ is torsion. Therefore we have $\mathrm{Hom}(\ker(\bar\imath), B_0) = (0)$ and the conclusion follows.

(ii) Consider the conormal sequence

$$J/J^2 \xrightarrow{\delta} \Omega_{P/\mathbf{k}} \otimes B_0 \to \Omega_{B_0/\mathbf{k}} \to 0$$

Since $\mathrm{Spec}(B_0)$ is reduced $\ker(\delta)$ is supported in the locus where $\mathrm{Spec}(B_0) \subset \mathrm{Spec}(P)$ is not a regular embedding (Proposition (A.3.2)), and this locus coincides with the locus where $\mathrm{Spec}(B_0)$ is not a l.c.i. (Proposition (A.3.7)). From the assumption about the depth of $\mathrm{Spec}(B_0)$ it follows that $\mathrm{Hom}(\ker(\delta), B_0) = \mathrm{Ext}^1(\ker(\delta), B_0) = (0)$. Using this fact and recalling that $\mathrm{Ext}^i(\Omega_{P/\mathbf{k}} \otimes B_0, B_0) = (0)$, $i > 0$, we obtain:

$$\mathrm{Ext}^2(\Omega_{B_0/\mathbf{k}}, B_0) \cong \mathrm{Ext}^1(\mathrm{Im}(\delta), B_0) \cong \mathrm{Ext}^1(J/J^2, B_0)$$

q.e.d.

(II.5.6) EXAMPLE (Schlessinger(1964)) *An obstructed affine curve* - Let s be an indeterminate and let $B_0 = \mathbf{k}[s^7, s^8, s^9, s^{10}] \subset \mathbf{k}[s]$ be the coordinate ring of the affine rational curve $C \subset \mathbf{A}^4 = \mathrm{Spec}(\mathbf{k}[x, y, z, w])$ having parametric equations:

$$x = s^7, \; y = s^8, \; z = s^9, \; w = s^{10}$$

Write $P = \mathbf{k}[x, y, z, w]$ and

$$B_0 = P/I$$

for an ideal $I \subset P$. One can check, using for example a computer algebra package, that I is generated by the six 2×2 minors of the following matrix:

$$\begin{pmatrix} x & y & z & w^2 \\ y & z & w & x^3 \end{pmatrix}$$

i.e. by the following polynomials:

$$f_1 = y^2 - xz; \qquad f_2 = xw - yz; \qquad f_3 = z^2 - yw;$$

$$f_4 = x^4 - w^2 y; \quad f_5 = x^3 y - zw^2; \quad f_6 = w^3 - x^3 z$$

This ideal is prime of height 3. Consider a presentation:

$$0 \to \mathbf{R} \to F \to I \to 0$$

where $F \cong P^6$ with generators say $e_1, \ldots, e_6$, so that $e_i \mapsto f_i$. To describe $\mathbf{R}$ one can use the beginning of the free resolution of I given by the Eagon-Northcott complex (see Eisenbud(1995)). One obtains a set of generators for $\mathbf{R}$ given by the rows of the following matrix:

$$\begin{array}{lcccccc}
R_1: & z & y & x & 0 & 0 & 0 \\
R_2: & w^2 & 0 & 0 & y & -x & 0 \\
R_3: & 0 & 0 & w^2 & 0 & z & y \\
R_4: & 0 & -w^2 & 0 & z & 0 & x \\
R_5: & w & z & y & 0 & 0 & 0 \\
R_6: & x^3 & 0 & 0 & z & -y & 0 \\
R_7: & 0 & 0 & x^3 & 0 & w & z \\
R_8: & 0 & -x^3 & 0 & w & 0 & y
\end{array}$$

Here each row gives the coefficients a_i of the linear combination $\sum_i a_i e_i \in \mathbf{R}$. We then have an exact sequence

$$0 \to \bar{\mathbf{R}} \to \bar{F} \to I/I^2 \to 0$$

where $\bar{F} = F/IF$ and $\bar{\mathbf{R}} = \mathbf{R}/(IF \cap \mathbf{R})$. Reducing mod I the above relations one gets the following set of generators of $\bar{\mathbf{R}}$ as elements of $\bar{F}$:

$$[II.5.3] \qquad \begin{array}{lcccccc}
r_1: & s^9 & s^8 & s^7 & 0 & 0 & 0 \\
r_2: & s^{20} & 0 & 0 & s^8 & -s^7 & 0 \\
r_3: & 0 & 0 & s^{20} & 0 & s^9 & s^8 \\
r_4: & 0 & -s^{20} & 0 & s^9 & 0 & s^7 \\
r_5: & s^{10} & s^9 & s^8 & 0 & 0 & 0 \\
r_6: & s^{21} & 0 & 0 & s^9 & -s^8 & 0 \\
r_7: & 0 & 0 & s^{21} & 0 & s^{10} & s^9 \\
r_8: & 0 & -s^{21} & 0 & s^{10} & 0 & s^8
\end{array}$$

Since B_0 is reduced we have

$$T^2_{B_0} = \mathrm{Ext}^1_{B_0}(I/I^2, B_0) = \mathrm{Hom}(\bar{\mathbf{R}}, B_0)/\mathrm{Hom}(\bar{F}, B_0)$$

Representing an element $h \in \mathrm{Hom}(\bar{\mathbf{R}}, B_0)$ as the 8-tuple $(h(r_1), \ldots, h(r_8)) \in B_0^8$ we see that the submodule $\mathrm{Hom}(\bar{F}, B_0)$ is generated by the columns of [II.5.3]. Therefore to prove that B_0 is obstructed it will suffice to produce a first order deformation $\xi \in \mathrm{Def}_{B_0}(\mathbf{k}[\epsilon])$ whose obstruction to lift to $\mathbf{k}[t]/(t^3)$ is represented by an $h : \bar{\mathbf{R}} \to B_0$ not in the submodule generated by the columns of [II.5.3]. We define ξ by the ideal

$$(f_1 + \Delta f_1, \ldots, f_6 + \Delta f_6) \subset \mathbf{k}[\epsilon, x, y, z, w]$$

where

$$\Delta f := (\Delta f_1, \ldots, \Delta f_6) = (0, 0, 0, zw, w^2, -x^3)$$

this defines a deformation because $R_j \cdot \Delta f \in I$ for all $j = 1, \ldots, 8$. More precisely:

$$(R_1 \cdot \Delta f, \ldots, R_8 \cdot \Delta f) = (0, -wf_2, -f_5, f_4 - wf_3, 0, wf_3, f_6, -f_5)$$

Therefore we see that the obstruction to lift ξ to second order is defined by the homomorphism $h : \bar{\mathbf{R}} \to B_0$ represented by

$$(0, 0, -t^{20}, t^{19}, 0, 0, -t^{21}, -t^{20})$$

Now it is immediate to check that this vector is not in $\mathrm{Hom}(\bar{F}, B_0)$ and therefore the deformation ξ cannot be lifted: thus B_0 is obstructed.

* * * * * *

Comparison with deformations of the nonsingular locus

Under certain conditions it is possible to compare the deformations of an affine scheme with the deformations of the open subscheme of its nonsingular points. We will need a preliminary Lemma.

II.5(7) LEMMA *Let X be an affine scheme, $Z \subset X$ a closed subscheme and G a coherent sheaf on X. Let $G^\vee = \mathcal{H}om(G, \mathcal{O}_X)$. If $depth_Z(\mathcal{O}_X) \geq 2$ then $depth_Z(G^\vee) \geq 2$ and therefore*

$$H^0(X, G^\vee) \cong H^0(X \backslash Z, G^\vee)$$

Proof

Consider a presentation

$$0 \to \mathbf{R} \to F \to G \to 0$$

where F is a free $\mathcal{O}_X$-module. Then we obtain an exact sequence

$$[II.5.4] \qquad 0 \to G^\vee \to F^\vee \to Q \to 0$$

where $Q \subset \mathbf{R}^\vee$. Since $F^\vee$ is free we have $\mathrm{depth}_Z(F^\vee) = \mathrm{depth}_Z(\mathcal{O}_X) \geq 2$ and therefore $H^0_Z(F^\vee) = 0 = H^1_Z(F^\vee)$ (Grothendieck(1967), Theorem 3.8, p. 44); it follows that $H^0_Z(G^\vee) = 0$. Similarly one proves that $H^0_Z(\mathbf{R}^\vee) = 0$, and therefore $H^0_Z(Q) = 0$. From the sequence of local cohomology associated to [II.5.4] we obtain $H^1_Z(G^\vee) = 0$ and therefore $\mathrm{depth}_Z(G^\vee) \geq 2$ by (Grothendieck(1967), Theorem 3.8, p. 44). The last assertion follows from the exact sequence

$$0 \to H^0(X, G^\vee) \to H^0(X\backslash Z, G^\vee) \to H^1_X(G^\vee)$$

(see Hartshorne(1977), p. 212). *q.e.d.*

Consider an affine scheme $X = \mathrm{Spec}(B)$ where $B = P/J$ for a smooth $\mathbf{k}$-algebra P. Let $Z = \mathrm{Sing}(X)$ be the singular locus of X and $U = X\backslash Z$.

Let $Y = \mathrm{Spec}(P)$ and consider the exact sequence

$$[II.5.5] \qquad 0 \to T_X \to T_{Y|X} \to N_X \to T^1_X \to 0$$

where $N_X = N_{X/Y}$. Since T^1_X is supported on Z, by restricting to U we get the exact sequence:

$$[II.5.6] \qquad 0 \to T_U \to T_{Y|U} \to N_{X|U} \to 0$$

(II.5.8) PROPOSITION *(i) If $\mathrm{depth}_Z(\mathcal{O}_X) \geq 2$ (e.g. X is normal of dimension ≥ 2) we have an exact sequence*

$$0 \to T^1_B \to H^1(U, T_U) \to H^1(U, T_{Y|U})$$

(ii) If $\mathrm{depth}_Z(\mathcal{O}_X) \geq 3$ then

$$T^1_B \cong H^1(U, T_U)$$

Proof

(i) We have the local cohomology exact sequences (see Hartshorne(1977), p. 212):

$$0 \to H^0(X, N_X) \to H^0(U, N_{X|U}) \to H^1_Z(N_X)$$

$$0 \to H^0(X, T_{Y|X}) \to H^0(U, T_{Y|U}) \to H^1_Z(T_{Y|X})$$

If $\mathrm{depth}_Z(\mathcal{O}_X) \geq 2$ then from Lemma (II.5.7) we deduce that $\mathrm{depth}_Z(N_X) \geq 2$ and $\mathrm{depth}_Z(T_{Y|X}) \geq 2$. Therefore we have $H^1_Z(N_X) = 0 = H^1_Z(T_{Y|X})$ (Grothendieck(1967), Theorem 3.8, p. 44) and

$$H^0(X, N_X) \cong H^0(U, N_{X|U}), \quad H^0(X, T_{Y|X}) \cong H^0(U, T_{Y|U})$$

Comparing the exact cohomology sequences of [II.5.5] and [II.5.6] we get an exact and commutative diagram:

$$\begin{array}{ccccccc}
H^0(X, T_{Y|X}) & \to & H^0(X, N_X) & \to & T^1_B & \to & 0 \\
\| & & \| & & \cap & & \\
H^0(U, T_{Y|U}) & \to & H^0(U, N_{X|U}) & \to & H^1(U, T_U) & \to & H^1(U, T_{Y|U})
\end{array}$$

which proves (i).

If $\mathrm{depth}_Z(\mathcal{O}_X) \geq 3$ then X is normal and $H^1_Z(T_{Y|X}) = 0 = H^2_Z(T_{Y|X})$ because $T_{Y|X}$ is locally free; from the local cohomology exact sequence we get

$$H^1(U, T_{Y|U}) \cong H^1(X, T_{Y|X}) = 0$$

because X is affine. Using (i) we deduce (ii). *q.e.d.*

The above Proposition can be applied to prove the rigidity of a large class of cones over projective varieties. We will need the following well known Lemmas, which we include for the reader's convenience.

(II.5.9) LEMMA *Let $W \subset \mathbb{P}^r$ be a projective nonsingular variety, CW the affine cone over W, $v \in CW$ the vertex, $U = CW \backslash \{v\}$ and $p : U \to W$ the projection. If G is a coherent sheaf on CW such that $G_{|U} = p^*F$ for some coherent $F \neq (0)$ on W, then the following conditions are equivalent:*
(i) $\mathrm{depth}_v(G) \geq d$ for some $d \geq 2$
(ii) $H^0(CW, G) = \oplus_{\nu \in \mathbb{Z}} H^0(W, F(\nu))$ and $H^k(W, F(\nu)) = 0$ for all $1 \leq k \leq d-2$ and $\nu \in \mathbb{Z}$

Proof
We will use the equivalence

$$\mathrm{depth}_v(G) \geq d \quad \Leftrightarrow \quad H^k_v(G) = 0, \quad k < d$$

(Grothendieck(1967), Theorem 3.8, p. 44). We have an exact local cohomology sequence:

$$0 \to H^0_v(G) \to H^0(CW, G) \to H^0(U, G_{|U}) \to H^1_v(G) \to 0$$

and isomorphisms:

$$H^{k-1}(U, G_{|U}) \cong H^k_v(G) \qquad k \geq 2$$

Since $G_{|U} = p^*F$ with $F \neq (0)$ we have $\mathrm{depth}_v(G) \geq 1$, thus $H^0_v(G) = 0$. On the other hand, since $p_*G_{|U} = p_*p^*F = \oplus_{\nu \in \mathbb{Z}} F(\nu)$, we have $H^0(U, G_{|U}) = \oplus_{\nu \in \mathbb{Z}} H^0(W, F(\nu))$. Now the conclusion follows. *q.e.d.*

(II.5.10) LEMMA *Let $\underline{0} = [0, \ldots, 0, 1] \in \mathbb{P}^{r+1}$, $V = \mathbb{P}^{r+1} \backslash \{\underline{0}\}$ and let $\pi : V \to \mathbb{P}^r$ be the projection. Then*

$$T_{V/\mathbb{P}^r} = \pi^* \mathcal{O}(1)$$

Proof
It is an immediate consequence of the commutative exact diagram

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \uparrow & & \uparrow & & \\
0 \to & T_{V/\mathbb{P}^r} & \to & T_V & \to & \pi^* T_{\mathbb{P}^r} & \to 0 \\
 & \| & & \uparrow & & \uparrow & \\
0 \to & \mathcal{O}_V(1) & \to & \mathcal{O}_V(1)^{r+2} & \to & \mathcal{O}_V(1)^{r+1} & \to 0 \\
 & & & \uparrow & & \uparrow & \\
 & & & \mathcal{O} & & \mathcal{O} & \\
 & & & \uparrow & & \uparrow & \\
 & & & 0 & & 0 &
\end{array}$$

where the vertical sequences are restrictions of Euler sequences. *q.e.d.*

(II.5.11) COROLLARY *Let $W \subset \mathbb{P}^r$ be a projective nonsingular variety of dimension ≥ 2. Assume that*
(i) $H^0(\mathbb{P}^r, \mathcal{O}(\nu)) \to H^0(W, \mathcal{O}_W(\nu))$ surjective for all $\nu \in \mathbb{Z}$ (W is projectively normal)
(ii) $H^1(W, \mathcal{O}_W(\nu)) = 0$ for all $\nu \in \mathbb{Z}$.
(iii) $H^1(W, T_W(\nu)) = 0$ for all $\nu \in \mathbb{Z}$.
Then the affine cone CW over W is rigid.

Proof
CW has dimension ≥ 3 and hypothesis (i) implies that it is normal (Hartshorne(1977), p. 126). Hypothesis (i) and (ii) imply that $\text{depth}_v(\mathcal{O}_{CW}) \geq 3$, by Lemma [II.5.9]. Therefore by (II.5.8)(ii) it suffices to show that $H^1(U, T_U) = 0$ where $U = CW \setminus \{v\}$. Let $p : U \to W$ be the projection. We have

$$H^1(U, \mathcal{O}_U) = \oplus_{\nu \in \mathbb{Z}} H^1(W, \mathcal{O}_W(\nu)) = 0$$

$[II.5.7]$

$$H^1(U, p^*T_W) = \oplus_{\nu \in \mathbb{Z}} H^1(W, T_W(\nu)) = 0$$

by conditions (ii) and (iii). The relative tangent sequence of p takes the following form:

$$[II.5.8] \qquad 0 \to \mathcal{O}_U \to T_U \to p^*T_W \to 0$$

Infact it follows from Lemma (II.5.10) and from Proposition (A.1.1)(i) that we have $T_{\pi^{-1}(W)/W} = \pi^*\mathcal{O}_W(1)$ and therefore, since $U = \pi^{-1}(W) \setminus W$, we have $T_{U/W} = p^*\mathcal{O}_W(1)$ and this is clearly equal to $\mathcal{O}_U$. The conclusion follows from [II.5.7] and from the cohomology sequence of [II.5.8]. *q.e.d.*

(II.5.12) COROLLARY *(i) The affine cone over $\mathbb{P}^n \times \mathbb{P}^m$ in its Segre embedding is rigid for every n, m such that $n + m \geq 3$*
(ii) The affine cone over any Veronese embedding of $\mathbb{P}^n$, $n \geq 2$, is rigid.
(iii) If $W \subset \mathbb{P}^r$ is a projective nonsingular variety of dimension ≥ 2, such that $h^1(W, \mathcal{O}_W) = 0 = h^1(W, T_W)$ then the affine cone over the m-th Veronese embedding $W^{(m)}$ of W is rigid for every $m \gg 0$.

Proof
(i) and (ii) are easy computations. (iii) follows from Serre's vanishing Theorem (Hartshorne(1977), Th. III.5.2). *q.e.d.*

The affine cone over the quadric $\mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^3$ is not rigid because it is a hypersurface; it does not satisfy both conditions (ii) and (iii) of Corollary (II.5.11). The affine cone over a rational normal curve $\Gamma_r \subset \mathbb{P}^r$, $r \geq 2$, is not rigid either (for $r = 2$ it is a singular quadric surface in $\mathbf{A}^3$; for $r \geq 3$ see Mumford(1973), Pinkham(1974), Pinkham(1974b)).

* * * * * *

Quotient singularities

The analysis of the previous subsection can be applied to the study of deformations of a class of affine singular schemes obtained as quotients of nonsingular ones by the action of a finite group.

Let $Y = \mathrm{Spec}(P)$ be an affine nonsingular algebraic variety on which a finite group G acts. Let $X = Y/G$ be the quotient scheme, $q : Y \to X$ the projection. Assume that the action is free outside a G-invariant closed subscheme $W \subset Y$. Set $V = Y \backslash W$.

(II.5.13) PROPOSITION *Assume that* $\mathrm{depth}_W(\mathcal{O}_Y) \geq 2$. *Then*

$$T_X \cong (q_* T_Y)^G$$

where G *acts on* $T_Y = \mathrm{Der}_{\mathbf{k}}(P,P)^{\sim}$ *by* $D \mapsto D^g := gDg^{-1}$ *for all* $D \in \mathrm{Der}_{\mathbf{k}}(P,P)$ *and* $g \in G$.

Proof

Consider the exact sequence of coherent sheaves on Y:

$$0 \to T_{Y/X} \to T_Y \to q^* T_X \to T^1_{Y/X}$$

$\Omega^1_{Y/X}$ is supported on W since q is etale outside W; then we have $T_{Y/X} = 0$. Similarly $T^1_{Y/X}$ is supported on W so that from the above exact sequence restricted to V we deduce an isomorphism

$$H^0(V, T_Y) \cong H^0(V, q^* T_X)$$

Then by Lemma (II.5.7) we deduce that

$$H^0(Y, T_Y) \cong H^0(Y, q^* T_X)$$

Note that, letting $A = P^G$ the ring of invariant elements, we have $X = \mathrm{Spec}(A)$ and the above isomorphism is equivalent to an isomorphism

$$\mathrm{Der}_{\mathbf{k}}(P,P) \cong \mathrm{Der}_{\mathbf{k}}(A,P)$$

Therefore it will suffice to show that

$$\mathrm{Der}_{\mathbf{k}}(A,A) \cong \mathrm{Der}_{\mathbf{k}}(A,P)^G$$

So let $D \in \mathrm{Der}_{\mathbf{k}}(A,P)$ be such that $D = gDg^{-1}$ for all $g \in G$. Then for every $a \in A$ we have

$$D(a) = g(D(g^{-1}a)) = g(D(a))$$

so $D(a) \in A$ and therefore $\mathrm{Der}_{\mathbf{k}}(A,P)^G \subset \mathrm{Der}_{\mathbf{k}}(A,A)$. Conversely if $D \in \mathrm{Der}_{\mathbf{k}}(A,A)$ then it defines a $\mathbf{k}$-derivation of A in P which is clearly G-invariant and we also have $\mathrm{Der}_{\mathbf{k}}(A,A) \subset \mathrm{Der}_{\mathbf{k}}(A,P)^G$. *q.e.d.*

(II.5.14) COROLLARY *Let n be the order of G. Under the same assumptions of (II.5.13) if char($\mathbf{k}$) does not divide n then T_X is a direct summand of q_*T_Y.*

Proof
Define a homomorphism $q_*T_Y \to T_X = (q_*T_Y)^G$ by

$$D \mapsto \frac{1}{n}\sum_{g\in G} D^g$$

This defines the splitting. *q.e.d.*

(II.5.15) THEOREM *In the above situation, if the action is free outside a G-invariant closed subscheme W of codimension ≥ 3, and char($\mathbf{k}$) does not divide the order of G then $X = Y/G$ is rigid.*

Proof
Let $Z = q(W)$ where $q : Y \to X$ is the projection, $V = Y\backslash W$, $U = X\backslash Z = V/G$. We have $\mathrm{depth}_Z\mathcal{O}_X \geq 2$ because X is normal, being the quotient of a nonsingular variety by a finite group (for this elementary fact see e.g. Serre(1959), p. 58). Therefore

$$[II.5.9] \qquad T^1_X \subset H^1(U, T_U) \cong H^1(U, (q_*T_Y)^G)$$

where the inclusion follows from (II.5.8)(i) and the isomorphism is Proposition (II.5.13). We also have an exact sequence

$$\begin{array}{ccccc} H^1(Y, T_Y) & \to & H^1(V, T_Y) & \to & H^2_W(T_Y) \\ & & \| & & \\ & & H^1(U, q_*T_Y) & & \end{array}$$

where the left vector space is 0 because Y is affine and the right one is 0 because of the depth assumption on W. It follows that $H^1(U, q_*T_Y) = 0$ and therefore $H^1(U, (q_*T_Y)^G) = 0$ as well because it is a direct summand of it by Corollary (II.5.14). The conclusion now follows from [II.5.9]. *q.e.d.*

In the Theorem the hypothesis on the codimension of W cannot be removed. Infact all rational two-dimensional singularities are quotient singularities and they are hypersurfaces (see Badescu(2001)).