Chapter III. Formal deformation theory

In this Chapter we develop the theory of "functors of Artin rings". The main result of this theory is a Theorem of Schlessinger giving necessary and sufficient conditions for a functor of Artin rings to have a semiuniversal or a universal formal element. We then apply the functorial machinery to the construction of formal semiuniversal, or universal, deformations, which is the final goal of formal deformation theory, and we explain the relation between formal and algebraic deformations. In the last part of the Chapter we study deformations of morphisms.

III.1. FUNCTORS OF ARTIN RINGS

A *functor of Artin rings* is a covariant functor

$$F : \mathcal{A}_\Lambda \longrightarrow \text{(sets)}$$

where $\Lambda \in \text{ob}(\hat{\mathcal{A}})$. Let A be in $\text{ob}(\mathcal{A})$. An element $\xi \in F(A)$ will be called an *infinitesimal deformation* of $\xi_0 \in F(\mathbf{k})$ if $\xi \mapsto \xi_0$ under the map $F(A) \to F(\mathbf{k})$; if $A = \mathbf{k}[\epsilon]$ then ξ is called a *first order deformation* of ξ_0.

Examples of functors of Artin rings are obtained by fixing an R in $\hat{\mathcal{A}}_\Lambda$ and letting:

$$h_{R/\Lambda}(A) = \text{Hom}_{\hat{\mathcal{A}}_\Lambda}(R, A) \qquad \text{for every } A \text{ in } \mathcal{A}_\Lambda$$

Such a functor is clearly nothing but the restriction to $\mathcal{A}_\Lambda$ of a representable functor on $\hat{\mathcal{A}}_\Lambda$. A functor of Artin rings isomorphic to $h_{R/\Lambda}$ for some R in $\hat{\mathcal{A}}_\Lambda$ is called *prorepresentable*. In case $\Lambda = \mathbf{k}$ we write h_R instead of $h_{R/\mathbf{k}}$.

Every *representable* functor $h_{R/\Lambda}$, R in $\mathcal{A}_\Lambda$, is a (trivial) example of prorepresentable functor.

Typically a prorepresentable functor of Artin rings arises as follows. One considers a scheme X and the restriction

$$\Phi : \mathcal{A} \longrightarrow \text{(sets)}$$

$$\Phi(A) = \text{Hom}(\text{Spec}(A), X)$$

of the representable functor

$$\text{Hom}(-, X) : \text{(schemes)}^\circ \longrightarrow \text{(sets)}$$

Then, for a fixed $\mathbf{k}$-rational point $x \in X$, one considers the subfunctor

$$F : \mathcal{A} \longrightarrow \text{(sets)}$$

of Φ defined as follows:

$$F(A) = \text{Hom}(\text{Spec}(A), X)_x = \{ \text{ morphisms Spec}(A) \to X \text{ whose image is } x\}$$

Then $F = h_R$, where $R = \hat{\mathcal{O}}_{X,x}$, so F is prorepresentable. In §III.3 we will consider some functors of Artin rings defined by deformation problems.

A prorepresentable functor $F = h_{R/\Lambda}$ has the following properties:

H_0) $F(\mathbf{k})$ consists of one element (the canonical quotient $R \to R/m = \mathbf{k}$)

Let

$$[III.1.1] \qquad \begin{array}{ccc} A' & & A'' \\ & \searrow \quad \swarrow & \\ & A & \end{array}$$

be a diagram in $\mathcal{A}_\Lambda$ and consider the natural map

$$\alpha : F(A' \times_A A'') \to F(A') \times_{F(A)} F(A'')$$

induced by the commutative diagram:

$$\begin{array}{ccc} F(A' \times_A A'') & \to & F(A'') \\ \downarrow & & \downarrow \\ F(A') & \to & F(A) \end{array}$$

Then

H_ℓ) (*left exactness*) For every diagram [III.1.1] α is bijective (straightforward to check).

H_f) $F(\mathbf{k}[\epsilon])$ has a structure of finite dimensional $\mathbf{k}$-vector space.
Infact

$$F(\mathbf{k}[\epsilon]) = \mathrm{Hom}_{\Lambda-alg}(R, \mathbf{k}[\epsilon]) = \mathrm{Der}_\Lambda(R, \mathbf{k}) = t_{R/\Lambda}$$

is the relative tangent space of R over Λ (here the Λ-algebra structure of $\mathbf{k}[\epsilon]$ is given by the composition $\Lambda \to \mathbf{k} \to \mathbf{k}[\epsilon]$, see (A.1.9)(vi)).

A property weaker than H_ℓ) satisfied by a prorepresentable functor F is the following:

H_ϵ) α is bijective if $A = \mathbf{k}$ and $A'' = \mathbf{k}[\epsilon]$.

It is interesting to observe that if F is prorepresentable the structure of $\mathbf{k}$-vector space on $F(\mathbf{k}[\epsilon])$ can be reconstructed in a purely functorial way only using properties H_0) and H_ϵ), and without using the prorepresentability explicitly.

Indeed it is an easy exercise to check that the homomorphism:

$$\begin{array}{rccc} + : & \mathbf{k}[\epsilon] \times_{\mathbf{k}} \mathbf{k}[\epsilon] & \longrightarrow & \mathbf{k}[\epsilon] \\ & (a + b\epsilon, a + b'\epsilon) & \longmapsto & a + (b + b')\epsilon \end{array}$$

induces the operation of sum on $F(\mathbf{k}[\epsilon])$ as the composition

$$F(\mathbf{k}[\epsilon]) \times F(\mathbf{k}[\epsilon]) \to F(\mathbf{k}[\epsilon] \times_{\mathbf{k}} \mathbf{k}[\epsilon]) \xrightarrow{F(+)} F(\mathbf{k}[\epsilon])$$

where the first map is the inverse of α. Associativity is checked using H_ϵ).

The zero is the image of $F(\mathbf{k}) \to F(\mathbf{k}[\epsilon])$. The multiplication by a scalar $c \in \mathbf{k}$ is induced in $F(\mathbf{k}[\epsilon])$ by the morphism

$$\begin{array}{ccc} \mathbf{k}[\epsilon] & \longrightarrow & \mathbf{k}[\epsilon] \\ a + b\epsilon & \longmapsto & a + (cb)\epsilon \end{array}$$

We can therefore state the following

(III.1.1) LEMMA-DEFINITION *If F is a functor of Artin rings having properties H_0) and H_ϵ) then the set $F(\mathbf{k}[\epsilon])$ has a structure of $\mathbf{k}$-vector space in a functorial way. This vector space is called the* tangent space to the functor F, *and denoted t_F. If $F = h_{R/\Lambda}$ then $t_F = t_{R/\Lambda}$.*

If $f : F \to G$ is a morphism of functors of Artin rings then the induced map $t_F \to t_G$ is called the *differential* of f and it is denoted df. It is straightforward to check that if F and G satisfy H_0) and H_ϵ) then df is $\mathbf{k}$-linear.

Every functor of Artin rings F can be extended to a functor

$$\hat{F} : \hat{\mathcal{A}}_\Lambda \longrightarrow (\text{sets})$$

by letting, for every R in $\hat{\mathcal{A}}_\Lambda$:

$$\hat{F}(R) = \varprojlim F(R/m_R^{n+1})$$

and for every $\varphi : R \to S$:

$$\hat{F}(\varphi) : \hat{F}(R) \to \hat{F}(S)$$

to be the map induced by the maps $F(R/m_R^n) \to F(S/m_S^n)$, $n \geq 1$.

An element $\hat{u} \in \hat{F}(R)$ is called a *formal element* of F. By definition $\hat{u}$ can be represented as a system of elements $\{u_n \in F(R/m_R^{n+1})\}_{n\geq 0}$ such that for every $n \geq 1$ the map

$$F(R/m_R^{n+1}) \to F(R/m_R^n)$$

induced by the projection $R/m_R^{n+1} \to R/m_R^n$ sends

$$[III.1.2] \qquad u_n \longmapsto u_{n-1}$$

$\hat{u}$ is also called a *formal deformation* of u_0. If $f : F \to G$ is a morphism of functors of Artin rings then it can be extended in an obvious way to a morphism of functors $\hat{f} : \hat{F} \to \hat{G}$.

(III.1.2) LEMMA *Let R be in $\hat{\mathcal{A}}_\Lambda$. There is a 1-1 correspondence between $\hat{F}(R)$ and the set of morphisms of functors*

$$[III.1.3] \qquad h_{R/\Lambda} \longrightarrow F$$

Proof

To a formal element $\hat{u} \in \hat{F}(R)$ there is associated a morphism of functors [III.1.3] in the following way. Each $u_n \in F(R/m_R^{n+1})$ defines a morphism of functors $h_{(R/m^{n+1})/\Lambda} \to F$. The compatibility conditions [III.1.2] imply that the following diagram commutes:

$$\begin{array}{ccc} h_{(R/m^n)/\Lambda} & \to & h_{(R/m^{n+1})/\Lambda} \\ & \searrow & \downarrow \\ & & F \end{array}$$

for every n. Since for each A in $\mathcal{A}_\Lambda$

$$h_{(R/m^n)/\Lambda}(A) \to h_{(R/m^{n+1})/\Lambda}(A)$$

is a bijection for all $n \gg 0$ we may define

$$h_{R/\Lambda}(A) \to F(A)$$

as

$$\lim_{n\to\infty} [h_{(R/m^{n+1})/\Lambda}(A) \to F(A)]$$

Conversely each morphism [III.1.3] defines a formal element $\hat{u} \in \hat{F}(R)$, where $u_n \in F(R/m_R^{n+1})$ is the image of the canonical projection $R \to R/m_R^{n+1}$ via the map

$$h_{R/\Lambda}(R/m_R^{n+1}) \to F(R/m_R^{n+1})$$

q.e.d.

(III.1.3) DEFINITION *If R is in $\hat{\mathcal{A}}_\Lambda$ and $\hat{u} \in \hat{F}(R)$, we call $(R, \hat{u})$ a* formal couple *for F. The differential $t_{R/\Lambda} \to t_F$ of the morphism $h_{R/\Lambda} \to F$ defined by $\hat{u}$ is called the* characteristic map *of $\hat{u}$ (or of the formal couple $(R, \hat{u})$) and is denoted $d\hat{u}$.*
If $(R, \hat{u})$ is such that the induced morphism [III.1.3] is an isomorphism, then F is prorepresentable, and we also say that F is prorepresented *by the formal couple $(R, \hat{u})$. In this case $\hat{u}$ is called* a universal formal element *for F, and $(R, \hat{u})$ is a* universal formal couple.

An ordinary couple (R, u) with R in $\mathcal{A}_\Lambda$ defines a special case of formal couple, in which $(R_n, u_n) = (R_{n+1}, u_{n+1})$ for all $n \gg 0$. A universal formal couple seldom exists; we will therefore need to introduce some weaker properties of a formal couple. They are based on the following definition.

(III.1.4) DEFINITION *Let $f : F \to G$ be a morphism of functors of Artin rings. f is called* smooth *if for every surjection $\mu : B \to A$ in $\mathcal{A}_\Lambda$ the natural map:*

$$[III.1.4] \qquad F(B) \to F(A) \times_{G(A)} G(B)$$

induced by the diagram:

$$\begin{array}{ccc} F(B) & \to & G(B) \\ \downarrow & & \downarrow \\ F(A) & \to & G(A) \end{array}$$

is surjective. The functor F is called smooth if the morphism from F to the constant functor

$$G(A) = \{\textit{one element}\} \qquad \textit{all } A \in \mathrm{ob}(\mathcal{A}_\Lambda)$$

is smooth; equivalently, if

$$F(\mu) : F(B) \to F(A)$$

is surjective for every surjection $\mu : B \to A$ in $\mathcal{A}_\Lambda$.

Note that, since every surjection in $\mathcal{A}_\Lambda$ factors as a finite sequence of small extensions (i.e. surjections with one-dimensional kernel), for f (or F) to be smooth it is necessary and sufficient that the defining condition is satisfied for small extensions in $\mathcal{A}_\Lambda$.

Next Proposition states some properties of the notion of smoothness of a morphism of functors of Artin rings.

(III.1.5) PROPOSITION

(i) Let $f : R \to S$ be a homomorphism in $\hat{\mathcal{A}}_\Lambda$. Then f is formally smooth if and only if the morphism of functors $h_f : h_{S/\Lambda} \to h_{R/\Lambda}$ induced by f is smooth.

(ii) If $f : F \to G$ is smooth then f is surjective, i.e. $F(B) \to G(B)$ is surjective for every B in $\mathcal{A}_\Lambda$. In particular the differential $df : t_F \to t_G$ is surjective.

(iii) If $f : F \to G$ is smooth, then F is smooth if and only if G is smooth.

(iv) If $f : F \to G$ is smooth then the induced morphism of functors $\hat{f} : \hat{F} \to \hat{G}$ is surjective, i.e. $\hat{F}(R) \to \hat{G}(R)$ is surjective for every R in $\hat{\mathcal{A}}_\Lambda$.

(v) If $F \to G$ and $G \to H$ are smooth morphisms of functors then the composition $F \to H$ is smooth.

(vi) If $F \to G$ and $H \to G$ are morphisms of functors and $F \to G$ is smooth, then $F \times_G H \to H$ is smooth.

Proof

(i) Let $B \to A$ be a surjection in $\mathcal{A}_\Lambda$ and let

$$\begin{array}{ccc} A & \leftarrow & S \\ \uparrow & & \uparrow f \\ B & \leftarrow & R \end{array}$$

be a commutative diagram of homomorphisms of Λ-algebras. The formal smoothness of f is equivalent to the existence, for each such diagram, of a morphism $S \to B$ such that the resulting diagram

$$\begin{array}{ccc} A & \leftarrow & S \\ \uparrow & \swarrow & \uparrow f \\ B & \leftarrow & R \end{array}$$

is commutative. This is another way to express the condition that the map

$$h_{S/\Lambda}(B) \longrightarrow h_{S/\Lambda}(A) \times_{h_{R/\Lambda}(A)} h_{R/\Lambda}(B)$$

is surjective, i.e. that h_f is smooth.

(ii) follows immediately from the definition.

(iii) Let $\mu : B \to A$ be a surjection in $\mathcal{A}_\Lambda$, and consider the diagram:

$$\begin{array}{ccccc} F(B) & \to & F(A) \times_{G(A)} G(B) & \to & G(B) \\ & & \downarrow & & \downarrow G(\mu) \\ & & F(A) & \underset{f(A)}{\longrightarrow} & G(A) \end{array}$$

Suppose G is smooth and let $\xi \in F(A)$. By the smoothness of G there exists $\eta \in G(B)$ such that $G(\mu)(\eta) = f(A)(\xi)$. By the smoothness of f there is $\zeta \in F(B)$ which is mapped to $(\xi, \eta) \in F(A) \times_{G(A)} G(B)$. It follows that $F(\mu)(\zeta) = \xi$ and F is smooth.

The converse is proved similarly using the surjectivity of $f(A)$.

(iv) Let $\hat{v} = \{v_n\} \in \hat{G}(R)$. Since f is smooth the map

$$F(R/m_R^2) \to F(\mathbf{k}) \times_{G(\mathbf{k})} G(R/m_R^2) = G(R/m_R^2)$$

is surjective. Therefore there exists $w_1 \in F(R/m_R^2)$ such that $f(R/m_R^2)(w_1) = v_1$. Let's assume that for some $n \geq 1$ there exist $w_i \in F(R/m_R^{i+1})$, $i = 1, \ldots, n$ such that:

$$f(R/m_R^{i+1})(w_i) = v_i$$

and $w_i \longmapsto w_{i-1}$ under $F(R/m_R^{i+1}) \longrightarrow F(R/m_R^i)$.

The surjectivity of the map:

$$F(R/m_R^{n+1}) \longrightarrow F(R/m_R^n) \times_{G(R/m^n)} G(R/m_R^{n+1})$$

implies that there exists $w_{n+1} \in F(R/m_R^{n+1})$ whose image is (w_n, v_{n+1}). By induction we conclude that there exists $\hat{w} \in \hat{F}(R)$ whose image under $\hat{f}$ is $\hat{v}$.

The proofs of (v) and (vi) are straightforward. *q.e.d.*

We now introduce the notions of "versality" and "semiuniversality", which are slightly weaker than universality.

(III.1.6) DEFINITION *Let F be a functor of Artin rings. A formal element $\hat{u} \in \hat{F}(R)$, for some R in $\hat{\mathcal{A}}_\Lambda$, is called* versal *if the morphism $h_{R/\Lambda} \to F$ defined by $\hat{u}$ is smooth; $\hat{u}$ is called* semiuniversal *if it is versal and moreover the differential $t_{R/\Lambda} \to t_F$ is bijective.*
We will correspondingly speak of a versal formal couple *(respectively* a semiuniversal formal couple*).*

The condition satisfied by the morphism $h_{R/\Lambda} \to F$ in case $\hat{u} \in \hat{F}(R)$ is semiuniversal in some sense corresponds to the notion of formally etale morphism in $\mathcal{A}^*$. It is clear from the definitions that:

$$\hat{u} \quad \text{universal} \quad \Rightarrow \quad \hat{u} \quad \text{semiuniversal} \quad \Rightarrow \quad \hat{u} \quad \text{versal}$$

but none of the inverse implications is true (see examples (III.4.6)). We can also describe these properties as follows.

Assume given $\mu : (B, \xi_B) \to (A, \xi_A)$, a surjection of couples for F (i.e. μ is surjective), and a homomorphism $\varphi : R \to A$ such that $\varphi(\hat{u}) = \xi_A$. Then $\hat{u}$ is *versal* if for every such data there is a lifting $\psi : R \to B$ of φ (i.e. $\mu\psi = \varphi$) such that $\psi(\hat{u}) = \xi_B$:

$$\begin{array}{ccc} & & B \\ & \psi \nearrow & \\ R & & \downarrow \mu \\ & \varphi \searrow & \\ & & A \end{array} \qquad\qquad \begin{array}{ccc} & & \xi_B \\ & \nearrow & \\ \hat{u} & & \downarrow \\ & \searrow & \\ & & \xi_A \end{array}$$

$\hat{u}$ is *universal* if moreover the lifting ψ is unique. $\hat{u}$ is *semiuniversal* if it is versal and moreover the lifting ψ is unique when $\mu : \mathbf{k}[\epsilon] \to \mathbf{k}$.

Let $(R, \hat{u})$ and $(S, \hat{v})$ be two formal couples for F. A *morphism of formal couples*

$$f : (R, \hat{u}) \to (S, \hat{v})$$

is a morphism $f : R \to S$ in $\hat{\mathcal{A}}$ such that $\hat{F}(f)(\hat{u}) = \hat{v}$. We will call f an *isomorphism of formal couples* if in addition $f : R \to S$ is an isomorphism.

It is obvious that with this definition the formal couples for F and their morphisms form a category containing the category I_F of couples for F as a full subcategory.

(III.1.7) PROPOSITION *Let F be a functor of Artin rings. Then:*
(i) If $(R, \hat{u})$ and $(S, \hat{v})$ are universal formal couples for F there exists a unique isomorphism of formal couples $(R, \hat{u}) \cong (S, \hat{v})$.
(ii) If $(R, \hat{u})$ and $(S, \hat{v})$ are semiuniversal formal couples for F there exists an isomorphism of formal couples $(R, \hat{u}) \cong (S, \hat{v})$, which is not necessarily unique but the induced isomorphism $t_S \cong t_R$ is uniquely determined.
(iii) If $(R, \hat{u})$ is a semiuniversal formal couple for F and $(S, \hat{v})$ is versal, then there is an isomorphism, not necessarily unique:

$$\varphi : R[[X_1, \ldots, X_r]] \to S$$

for some $r \geq 0$, such that $\hat{F}(\varphi j)(\hat{u}) = \hat{v}$, where $j : R \subset R[[X_1, \ldots, X_r]]$ is the inclusion.

Proof
(i) By the universality of $(R, \hat{u})$ for every $n \geq 1$ there exists a unique $f_n \in h_{R/\Lambda}(S/m_S^{n+1})$ such that $f_n \mapsto v_n \in F(S/m_S^{n+1})$ under the isomorphism associated to $\hat{u}$. In this way we obtain

$$f = \lim_{\leftarrow}\{f_n\} : (R, \hat{u}) \to (S, \hat{v})$$

which is uniquely determined. Analogously we can construct a uniquely determined $g : (S, \hat{v}) \to (R, \hat{u})$. By universality the compositions

$$gf : (R, \hat{u}) \to (R, \hat{u})$$

and

$$fg : (S, \hat{v}) \to (S, \hat{v})$$

are the identity.

(ii) Using versality, and proceeding as above we can construct morphisms of formal couples

$$f : (R, \hat{u}) \to (S, \hat{v})$$

and

$$g : (S, \hat{v}) \to (R, \hat{u})$$

We obtain a commutative diagram:

$$\begin{array}{ccc} t_{R/\Lambda} & & \\ \uparrow\downarrow & \searrow & \\ t_{S/\Lambda} & \longrightarrow & t_F \end{array}$$

where the vertical arrows are the differentials of f and g, and the other arrows are the characteristic maps $d\hat{u}$ and $d\hat{v}$. From this diagram we deduce that $df = (d\hat{u})^{-1}d\hat{v}$ and $dg = (d\hat{v})^{-1}d\hat{u}$ are uniquely determined. Since

$$d(gf) = (dg)(df) = \text{identity of } t_{S/\Lambda}$$

and

$$d(fg) = (df)(dg) = \text{identity of } t_{R/\Lambda}$$

it follows that f and g are bijections inverse of each other.

(iii) By the versality of $(R, \hat{u})$ we can find a morphism of formal couples

$$f : (R, \hat{u}) \to (S, \hat{v})$$

We obtain a commutative diagram:

$$\begin{array}{ccc} t_{R/\Lambda} & & \\ \uparrow & \searrow & \\ t_{S/\Lambda} & \longrightarrow & t_F \end{array}$$

where $d\hat{u} : t_{R/\Lambda} \to t_F$ is bijective because $(R, \hat{u})$ is semiuniversal, and $d\hat{v} : t_{S/\Lambda} \to t_F$ is surjective because $(S, \hat{v})$ is versal. Hence $df : t_{S/\Lambda} \to t_{R/\Lambda}$ is surjective. This means that f induces an inclusion

$$t^{\vee}_{R/\Lambda} \subset t^{\vee}_{S/\Lambda}$$

Let $x_1, \ldots, x_r \in S$ be elements which induce a basis of $t^\vee_{S/R} = m_S/[m_S^2 + f(m_R)]$ and define:

$$\varphi : R[[X_1, \ldots, X_r]] \to S$$

by $\varphi(X_i) = x_i$, $i = 1, \ldots, r$. Let $j : R \subset R[[X_1, \ldots, X_r]]$ be the inclusion. Letting

$$\hat{w} = \hat{F}(j)(\hat{u}) \in \hat{F}(R[[X_1, \ldots, X_r]])$$

we have a commutative diagram of formal couples:

$$\begin{array}{ccc} (R[[X_1, \ldots, X_r]], \hat{w}) & \xrightarrow{\varphi} & (S, \hat{v}) \\ \uparrow j & \nearrow f & \\ (R, \hat{u}) & & \end{array}$$

such that $\hat{F}(\varphi j)(\hat{u}) = \hat{v}$. Since

$$\varphi_1 : R[[X_1, \ldots, X_r]]/M^2 \to S/m_S^2$$

is an isomorphism (here $M \subset R[[X_1, \ldots, X_r]]$ denotes the maximal ideal) φ is surjective. Let

$$\psi_1 : S/m_S^2 \to R[[X_1, \ldots, X_r]]/M^2$$

be the inverse of φ_1. We have

$$F(\psi_1)(v_1) = w_1$$

hence, by the versality of $(S, \hat{v})$, it is possible to find a lifting of ψ_1:

$$\psi : S \to R[[X_1, \ldots, X_r]]$$

such that $\hat{F}(\psi)(\hat{v}) = \hat{w}$. Since by construction

$$d(\psi\varphi) = d\psi \ d\varphi = \text{identity of } t_{R[[\underline{X}]]}$$

it follows that $\psi\varphi$ is an automorphism of $R[[X_1, \ldots, X_r]]$. In particular we deduce that φ is injective. *q.e.d.*

The following is a useful remark.

(III.1.8) PROPOSITION *If a functor of Artin rings $F : \mathcal{A} \to$ (sets) satisfying H_0) has a semiuniversal element and $t_F = (0)$ then $F = h_{\mathbf{k}}$, the constant functor.*

Proof

The assumptions imply that there is a smooth morphism $h_{\mathbf{k}} \to F$. Since a smooth morphism of functors satisfying H_0) is surjective, the conclusion follows. *q.e.d.*

* * * * * *

For the rest of this Section we will only consider functors of Artin rings satisfying conditions H_0) and H_ϵ).

(III.1.9) DEFINITION *Let F be a functor of Artin rings. Suppose that $v(F)$ is a* **k***-vector space such that for every A in $\mathcal{A}_\Lambda$ and for every $\xi \in F(A)$ there is a* **k***-linear map*

$$\xi_v : \mathrm{Ex}_\Lambda(A, \mathbf{k}) \to v(F)$$

with the following property:
ker(ξ_v) consists of the isomorphism classes of extensions $(\tilde{A}, \varphi)$ such that

$$\xi \in \mathrm{Im}[F(\tilde{A}) \to F(A)]$$

Then $v(F)$ is called an obstruction space *for the functor F.*
If F has (0) *as an obstruction space then it is called* unobstructed.

It is an immediate consequence of the definition that an unobstructed functor is smooth.

If F is prorepresented by the formal couple $(R, \hat{u})$, then $o(R/\Lambda)$ is an obstruction space for F, as it follows from the functorial characterization of $o(R/\Lambda)$ given in §I.3. Conversely if $v(F)$ is an obstruction space for F, and F is prorepresented by the formal couple $(R, \hat{u})$, then it follows immediately from the definition that $v(F)$ is a relative obstruction space for R/Λ.

One can show that under relatively mild conditions a functor F has an obstruction space. We will not discuss this matter here, and we refer the interested reader to Fantechi-Manetti(1998).

The following are some basic properties of obstruction spaces.

(III.1.10) PROPOSITION
(i) If F has (0) *as an obstruction space, then it is smooth.*
(ii) Let $f : F \to G$ be a smooth morphism of functors of Artin rings. If $v(G)$ is an obstruction space for G then it is also an obstruction space for F.
(iii) Let $f : F \to G$ be a morphism of functors such that df is surjective and F is smooth. Then f and G are smooth.

Proof
(i) is obvious.
(ii) Let A be in $\mathcal{A}_\Lambda$ and $\xi \in F(A)$, and let

$$\xi_v := f(A)(\xi)_v : \mathrm{Ex}_\Lambda(A, \mathbf{k}) \to v(G)$$

the map defined by $f(A)(\xi) \in G(A)$. If $\tilde{A} \to A = \tilde{A}/(t)$ defines an element of $\ker(\xi_v)$ then, since $v(G)$ is an obstruction space for G, there is $\eta \in G(\tilde{A})$ such that $\eta \mapsto f(A)(\xi)$ under the map $G(\tilde{A}) \to G(A)$. From the smoothness of f it follows that the map

$$F(\tilde{A}) \to F(A) \times_{G(A)} G(\tilde{A})$$

is surjective, hence there is $\tilde{\xi} \in F(\tilde{A})$ which maps to the formal couple (ξ, η). It follows that $\tilde{\xi} \mapsto \xi$ under $F(\tilde{A}) \to F(A)$.

Conversely, if $\tilde{A}$ is such that there exists $\tilde{\xi} \in F(\tilde{A})$ such that

$$\begin{array}{ccc} F(\tilde{A}) & \to & F(A) \\ \tilde{\xi} & \mapsto & \xi \end{array}$$

then from the diagram:

$$\begin{array}{ccc} F(\tilde{A}) & \to & F(A) \\ \downarrow f(\tilde{A}) & & \downarrow f(A) \\ G(\tilde{A}) & \to & G(A) \end{array}$$

we see that $f(\tilde{A})(\tilde{\xi}) \mapsto f(A)(\xi)$ hence $f(A)(\xi) \in \mathrm{Im}[G(\tilde{A}) \to G(A)]$. Therefore the extension $\tilde{A} \to A$ defines an element of $\ker[f(A)(\xi)_v]$. This proves that ξ_v satisfies the conditions of definition (III.1.9).

(iii) is left to the reader. *q.e.d.*

(III.1.11) COROLLARY *Let $F : \mathcal{A} \to$ (sets) be a functor of Artin rings, and suppose that $(R, \hat{u})$ is a versal formal couple for F. If F has a finite dimensional obstruction space $v(F)$ then*

$$\dim_{\mathbf{k}}(t_R) \geq \dim(R) \geq \dim_{\mathbf{k}}(t_R) - \dim_{\mathbf{k}}[v(F)]$$

Proof

From the definition of versal formal couple and from Proposition (III.1.10) we deduce that $v(F)$ is an obstruction space for h_R, hence for R. The conclusion follows from (I.3.8). *q.e.d.*

We refer the reader to Proposition (III.2.4) for a result about obstruction spaces which is often used in application.

III.2. THE THEOREM OF SCHLESSINGER

In this Section we will prove a well known theorem of Schlessinger which gives necessary and sufficient conditions, easy to verify in practice, for the existence of a (semi)universal element for a functor of Artin rings. Before stating the theorem we want to make some introductory remarks, which can be useful in what follows. Let's fix $\Lambda \in \mathrm{ob}(\hat{\mathcal{A}})$ throughout this section. We start with a characterization of prorepresentable functors:

(III.2.1) PROPOSITION *Let $F : \mathcal{A}_\Lambda \to$ (sets) be a functor of Artin rings satisfying condition H_0. Then F is prorepresentable if and only if it is left exact and has finite dimensional tangent space, i.e. it has properties H_ℓ and H_f.*

Proof
The "only if" implication is obvious (see also §III.1). So let's assume that F satisfies H_ℓ and H_f. Then, by Proposition (A.4.9), the category I_F of couples for F is cofiltered and

$$F = \lim_{\to\,(X,\xi)} h_X$$

Let $(A_i, \xi_i) \in \mathrm{ob}(I_F)$ and consider all the subrings of A_i images of morphisms $(A_j, \xi_j) \to (A_i, \xi_i)$. By the descending chain condition there is $(\bar{A}_i, \bar{\xi}_i)$ such that

$$\bar{A}_i = \bigcap \mathrm{Im}[A_j \to A_i]$$

By construction the couples $(\bar{A}_i, \bar{\xi}_i)$ form a full subcategory of I_F in which all maps are surjective. Moreover the corresponding category of representable functors $h_{\bar{A}_i/\Lambda}$ is clearly cofinal in $(I_F)^\circ$, and therefore

$$F = \lim_{\to} h_{\bar{A}_i/\Lambda}$$

Therefore replacing I_F by this subcategory we can assume that all homomorphisms are surjective, and therefore we have:

$$F = \bigcup h_{A_i/\Lambda}$$

Moreover, since H_f holds, $F(\mathbf{k}[\epsilon]) = h_{A_i}(\mathbf{k}[\epsilon])$ when $i \gg 0$. We can discard all those A_i for which this is not true and we get again a full subcategory. Therefore

$$F = \lim_{\to} h_{A_i/\Lambda}$$

and $F(\mathbf{k}[\epsilon]) = h_{A_i/\Lambda}(\mathbf{k}[\epsilon])$ for all i. Then for each i we can find a surjection

$$\Lambda[[X_1,\dots,X_r]] \to A_i$$

with $r = \dim(F(\mathbf{k}[\epsilon]))$, and these surjections are compatible, i.e. the diagrams

$$\begin{array}{ccc} \Lambda[[X_1,\dots,X_r]] & \to & A_i \\ & \searrow & \downarrow \\ & & A_j \end{array}$$

are commutative. Define $B_1 = \Lambda[[X_1,\dots,X_r]]/(\underline{X})^2 = A_i/m_i^2$ for all i. Then fix $\nu \geq 2$, and set $A_{i,\nu} = A_i/m_i^{\nu+1}$. All $A_{i,\nu}$'s are quotients of $\Lambda[[X_1,\dots,X_r]]/(\underline{X})^{\nu+1}$ and form a projective system:

$$\cdots \to A_{j,\nu} \to A_{i,\nu} \to \cdots$$

Let

$$B_\nu = \lim_{\leftarrow} A_{i,\nu} = A_{i,\nu} \quad i \gg 0$$

Then by construction

$$\cdots B_{\nu+1} \to B_\nu \to \cdots$$

form a projective system and

$$F = \lim_{\to} h_{B_\nu/\Lambda}$$

Then

$$\hat{B} := \lim_{\leftarrow} B_\nu$$

is in $\hat{\mathcal{A}}_\Lambda$ and prorepresents F. *q.e.d.*

Unfortunately this characterization of prorepresentable functors is not very useful in practise because given homomorphisms

$$\begin{array}{ccc} A' & & A'' \\ & \searrow \swarrow & \\ & A & \end{array}$$

in $\mathcal{A}_\Lambda$ we have

$$\mathrm{Spec}(A' \times_A A'') = \mathrm{Spec}(A') \bigcup_{\mathrm{Spec}(A)} \mathrm{Spec}(A'')$$

and this is hard to visualize. That's why left exactness of a functor of Artin rings F is hard to check. On the other hand, if at least one of the above homomorphisms is surjective then $\mathrm{Spec}(A' \times_A A'')$ is easier to describe. The Theorem of Schlessinger reduces the prorepresentability to the verification of the condition of left exactness only in cases when at least one of the above maps is surjective. An analogous condition is given for the existence of a semiuniversal element. The result is the following.

(III.2.2) THEOREM (Schlessinger) *Let $F : \mathcal{A}_\Lambda \to$ (sets) be a functor of Artin rings satisfying condition H_0). Let $A' \to A$ and $A'' \to A$ be homomorphisms in $\mathcal{A}_\Lambda$ and let*

$$[III.2.1] \qquad \alpha : F(A' \times_A A'') \longrightarrow F(A') \times_{F(A)} F(A'')$$

be the natural map. Then
(i) F has a semiuniversal element if and only if it satisfies the following conditions:
$\bar{H}$) *if $A'' \to A$ is a small extension then the map [III.2.1] is surjective.*
H_ϵ) *If $A = \mathbf{k}$ and $A'' = \mathbf{k}[\epsilon]$ then the map [III.2.1] is bijective.*
H_f) $\dim_{\mathbf{k}}(t_F) < \infty$
(ii) F has a universal element if and only if it satisfies the following additional condition:
H) *the natural map*

$$F(A' \times_A A') \longrightarrow F(A') \times_{F(A)} F(A')$$

is bijective for every small extension $A' \to A$ in $\mathcal{A}_\Lambda$.

(III.2.3) REMARK The meaning of the conditions of the Theorem can be explained as follows. Consider a small extension in $\mathcal{A}_\Lambda$:

$$0 \to (t) \to A' \xrightarrow{\mu} A \to 0$$

Assume that $F \cong h_{R/\Lambda}$ is prorepresentable. Then two Λ-homomorphisms $f, g : R \to A'$ have the same image in $\mathrm{Hom}_{\Lambda-alg}(R, A)$ if and only if there exists a Λ-derivation $d : R \to \mathbf{k}$ (which is uniquely determined) such that

$$g(r) = f(r) + d(r)t$$

equivalently if and only if g and f differ by an element of

$$\mathrm{Der}_\Lambda(R, \mathbf{k}) = t_{R/\Lambda}$$

Therefore the fibres of

$$F(\mu) : \mathrm{Hom}_{\Lambda-alg}(R, A') \to \mathrm{Hom}_{\Lambda-alg}(R, A)$$

that are non empty are principal homogeneous spaces under the above action of $t_{R/\Lambda} = t_F$.

Assume now that F is just a functor of Artin rings having properties H_0 and H_ϵ, so that it has tangent space t_F. We can define an action of t_F on $F(A')$ by means of the composition

$$\tau : t_F \times F(A') \xrightarrow{\alpha^{-1}} F(\mathbf{k}[\epsilon] \times_{\mathbf{k}} A') \xrightarrow{F(b)} F(A')$$

where α^{-1} exists by property H_ϵ) and $F(b)$ is induced by the morphism

$$b: \quad \mathbf{k}[\epsilon]\times_{\mathbf{k}} A' \longrightarrow A'$$

$$(x+y\epsilon, a') \longmapsto a'+yt$$

The action τ maps the fibres of $F(\mu)$ into themselves. Indeed the isomorphism

$$\gamma: \quad \mathbf{k}[\epsilon]\times_{\mathbf{k}} A' \longrightarrow A'\times_A A'$$

$$(x+y\epsilon, a') \longmapsto (a'+yt, a')$$

induces a map

$$\beta: t_F\times F(A') \xrightarrow{\alpha^{-1}} F(\mathbf{k}[\epsilon]\times_{\mathbf{k}} A') \xrightarrow{F(\gamma)} F(A'\times_A A') \longrightarrow F(A')\times_{F(A)} F(A')$$

which coincides with

$$t_F\times F(A') \longrightarrow F(A')\times_{F(A)} F(A')$$

$$(v,\xi) \longmapsto (\tau(v,\xi),\xi)$$

In case F is prorepresentable we have just given another description of the action of t_F on the fibres of $F(\mu)$ introduced before. In general the map β is neither injective (i.e. in general the action τ is not free on the fibres of $F(\mu)$) nor surjective (i.e. τ is not transitive on the fibres of $F(\mu)$). This depends on the properties of the map

$$\alpha': F(A'\times_A A') \longrightarrow F(A')\times_{F(A)} F(A')$$

If F is left exact then α' is bijective, hence β is bijective, and the action of t_F on the fibres of $F(\mu)$ is free and transitive, as expected since F is prorepresentable by Prop. (III.2.1).

Conversely what this analysis shows is that for τ to be free and transitive on the fibres of $F(\mu)$ we only need α' bijective, i.e. the condition H, weaker than H_ℓ. $\bar{H}$ only guarantees the transitivity of such action: the failure from prorepresentability is therefore related to the existence of fixed points of this action. In applications this is usually due to the existence of automorphisms of geometric objects associated to an element $\xi\in F(A)$ which don't lift to automorphisms of objects associated to an element $\xi'\in F(\mu)^{-1}(\xi)$ (see §III.4).

As an application of this analysis we prove a Proposition which is often applied in concrete situations.

(III.2.4) PROPOSITION *Let $f: F\to G$ be a morphism of functors of Artin rings having a semiuniversal element. Assume that*
i) df is surjective

ii) F and G have obstruction spaces $v(F)$ and $v(G)$ respectively
iii) there is an injective linear map

$$o(f) : v(F) \to v(G)$$

such that for each A in $\mathrm{ob}(\mathcal{A})$ *and for each $\xi \in F(A)$ the diagram:*

$$\begin{array}{ccccc} & & \mathrm{Ex}_{\mathbf{k}}(A, \mathbf{k}) & & \\ & \swarrow \xi_v & & \searrow \eta_v & \\ v(F) & & \xrightarrow{o(f)} & & v(G) \end{array} \qquad [III.2.2]$$

is commutative, where we have denoted $\eta = f(A)(\xi)$.
Then f is smooth.

Proof
Let $\varphi : \tilde{A} \to A$ be a small extension. Consider the map

$$F(\tilde{A}) \to F(A) \times_{G(A)} G(\tilde{A}) \qquad [III.2.3]$$

and let $(\xi, \tilde{\eta}) \in F(A) \times_{G(A)} G(\tilde{A})$. Since $\tilde{\eta} \mapsto \eta = F(A)(\xi)$ we have $\eta_v(\varphi) = 0$. By the commutativity of [III.1.2] we also have $\xi_v(\varphi) = 0$ and therefore there exists $\tilde{\xi} \in F(\tilde{A})$ such that $\tilde{\xi} \mapsto \xi$. Let $\tilde{\eta}' = F(\tilde{A})(\tilde{\xi}) \in G(\tilde{A})$. We have

$$\begin{array}{ccc} \tilde{\eta} & & \tilde{\eta}' \\ & \searrow \swarrow & \\ & \eta & \end{array}$$

Since by Theorem (III.2.3) the functor G satisfies condition $\bar{H})$, there is $w \in t_G$ such that $\tau(w, \tilde{\eta}') = \tilde{\eta}$. From the surjectivity of *df* it follows that there is $v \in t_F$ such that $v \mapsto w$. Now $\tau(v, \tilde{\xi}) \in F(\tilde{A})$ satisfies

$$\tau(v, \tilde{\xi}) \mapsto (\xi, \tilde{\eta})$$

because the action τ is functorial (as it easily follows from its definition). This shows that [III.2.3] is surjective and f is smooth. *q.e.d.*

Note that when F and G are prorepresentable Proposition (III.2.4) follows directly from Theorem (I.3.5).

Proof of Theorem (III.2.2)
(i) Let's assume that F has a semiuniversal formal element $(R, \hat{u})$. Consider a homomorphism $f : A' \to A$ and a small extension $\pi : A'' \to A$ both in $\mathcal{A}_\Lambda$, and let

$$(\xi', \xi'') \in F(A') \times_{F(A)} F(A'')$$

with

$$F(f)(\xi') = F(\pi)(\xi'') =: \xi \in F(A)$$

By the versality of $(R, \hat{u})$ the following maps are surjective:

$$\mathrm{Hom}_{\Lambda-alg}(R, A') \longrightarrow F(A')$$

$$[III.2.4] \qquad \mathrm{Hom}_{\Lambda-alg}(R, A'') \longrightarrow \mathrm{Hom}_{\Lambda-alg}(R, A) \times_{F(A)} F(A'')$$

therefore there are

$$g' \in \mathrm{Hom}_{\Lambda-alg}(R, A'), \quad g'' \in \mathrm{Hom}_{\Lambda-alg}(R, A'')$$

such that

$$\hat{F}(g')(\hat{u}) = \xi'$$

and $g'' \mapsto (fg', \xi'')$ under the map [III.2.4]. This last condition gives:

$$\pi g'' = fg', \quad \hat{F}(g'')(\hat{u}) = \xi''$$

and consequently $\hat{F}(\pi g'')(\hat{u}) = \xi$. Using the morphism $g' \times g'' : R \to A' \times_A A''$ we obtain an element

$$\zeta := \hat{F}(g' \times g'')(\hat{u}) \in F(A' \times_A A'')$$

which, by construction, is mapped to (ξ', ξ'') by [III.2.1]. This proves that $(R, \hat{u})$ satisfies condition $\bar{H})$.

If $A'' = \mathbf{k}[\epsilon]$ and $A = \mathbf{k}$ the map [III.2.4] reduces to the bijection

$$[III.2.5] \qquad \mathrm{Hom}_{\Lambda-alg}(R, \mathbf{k}[\epsilon]) \longrightarrow t_F$$

In this case if $\zeta_1, \zeta_2 \in F(A' \times_A \mathbf{k}[\epsilon])$ are such that $\alpha(\zeta_1) = \alpha(\zeta_2) = (\xi', \xi'')$ choose $g' \in \mathrm{Hom}_{\Lambda-alg}(R, A')$ as before. By the versality the map

$$\mathrm{Hom}_{\Lambda-alg}(R, A' \times_{\mathbf{k}} \mathbf{k}[\epsilon]) \longrightarrow F(A' \times_{\mathbf{k}} \mathbf{k}[\epsilon]) \times_{F(\mathbf{k}[\epsilon])} \mathrm{Hom}_{\Lambda-alg}(R, \mathbf{k}[\epsilon]) = F(A' \times_{\mathbf{k}} \mathbf{k}[\epsilon])$$

induced by the projection $A' \times_{\mathbf{k}} \mathbf{k}[\epsilon] \to \mathbf{k}[\epsilon]$ is surjective. Hence we obtain two morphisms:

$$g' \times g_i : R \longrightarrow A' \times_{\mathbf{k}} \mathbf{k}[\epsilon]$$

such that

$$\hat{F}(g' \times g_i)(\hat{u}) = \zeta_i$$

$i = 1, 2$. But then $\hat{F}(g_i)(\hat{u}) = \xi''$, $i = 1, 2$, hence, by the bijectivity of [III.2.5], $g_1 = g_2$, i.e. $\zeta_1 = \zeta_2$. This proves that F satisfies condition $H_\epsilon)$. Condition $H_f)$ is satisfied because the differential $t_{R/\Lambda} \to t_F$ is linear and is a bijection by definition of semiuniversal formal couple.

Conversely, let's assume that F satisfies conditions $\bar{H})$, $H_\epsilon)$ and $H_f)$. We will find a semiuniversal formal couple $(R, \hat{u})$ by constructing a projective system

$\{R_n; p_{n+1} : R_{n+1} \to R_n\}_{n\geq 0}$ of Λ-algebras in $\mathcal{A}_\Lambda$ and a sequence $\{u_n \in F(R_n)\}_{n\geq 0}$ such that $F(p_n)(u_n) = u_{n-1}$, $n \geq 1$.

We take $R_0 = \mathbf{k}$ and $u_0 \in F(\mathbf{k})$ its unique element. Let $r = \dim_{\mathbf{k}}(t_F)$, $\{e_1, \dots, e_r\}$ a basis of t_F and denoting $\Lambda[[\underline{X}]] = \Lambda[[X_1, \dots, X_r]]$ we set

$$R_1 = \Lambda[[\underline{X}]]/((\underline{X})^2 + m_\Lambda \Lambda[[\underline{X}]])$$

Since we have

$$R_1 = \mathbf{k}[\epsilon] \times_{\mathbf{k}} \cdots \times_{\mathbf{k}} \mathbf{k}[\epsilon] \qquad (r \quad \text{times})$$

from H_ϵ) we deduce that $F(R_1) = t_F \times \cdots \times t_F$.

Let's take $u_1 = (e_1, \dots, e_r) \in F(R_1)$. Note that the map induced by u_1:

$$\mathbf{k}^r \cong ((\underline{X})/(\underline{X})^2)^\vee = \mathrm{Hom}_{\Lambda-alg}(R_1, \mathbf{k}[\epsilon]) \longrightarrow t_F$$

is the isomorphism

$$(\lambda_1, \dots, \lambda_r) \longmapsto \sum_j \lambda_j e_j$$

Let's proceed by induction on n: assume that couples $(R_0, u_0), (R_1, u_1), \dots, (R_{n-1}, u_{n-1})$ such that

$$R_h = \Lambda[[\underline{X}]]/J_h, \quad u_h \in F(R_h), \quad u_h \mapsto u_{h-1}$$

$h = 1, \dots, n-1$ have been already constructed. In order to construct (R_n, u_n) we consider the family $\mathcal{I}$ of all ideals $J \subset \Lambda[[\underline{X}]]$ having the following properties:

a) $J_{n-1} \supset J \supset (\underline{X})J_{n-1}$:
b) there exists $u \in F(\Lambda[[\underline{X}]]/J)$ such that the map

$$F(\Lambda[[\underline{X}]]/J) \longrightarrow F(R_{n-1})$$

sends $u \mapsto u_{n-1}$.

$\mathcal{I} \neq \emptyset$ because $J_{n-1} \in \mathcal{I}$. Moreover $\mathcal{I}$ has a minimal element J_n. This will be proved if we show that $\mathcal{I}$ is closed with respect to finite intersections. Let $I, J \in \mathcal{I}$ and $K = I \cap J$. It is obvious that K satisfies condition a). We may assume that $I + J = J_{n-1}$ (making J larger without changing K if necessary). This implies that the natural homomorphism

$$\Lambda[[\underline{X}]]/K \longrightarrow \Lambda[[\underline{X}]]/I \times_{R_{n-1}} \Lambda[[\underline{X}]]/J$$

is an isomorphism. By $\bar{H}$) the map

$$\alpha : F(\Lambda[[\underline{X}]]/K) \longrightarrow F(\Lambda[[\underline{X}]]/I) \times_{F(R_{n-1})} F(\Lambda[[\underline{X}]]/J)$$

is surjective, therefore there exists $u \in F(\Lambda[[\underline{X}]]/K)$ whose image in $F(R_{n-1})$ is u_{n-1}; this means that K satisfies condition b) as well, hence it is in $\mathcal{I}$.

We take $R_n = \Lambda[[\underline{X}]]/J_n$ and $u_n \in F(R_n)$ an element which is mapped to u_{n-1}. By induction we have constructed a formal couple $(R, \hat{u})$. We now show that it is a semiuniversal formal couple for F.

As already remarked, $\hat{u}$ induces an isomorphism of tangent spaces $t_{R/\Lambda} \cong t_F$. Therefore we only have to prove versality, namely that the map

$$\hat{u}_\pi : \mathrm{Hom}_{\Lambda-alg}(R, A') \longrightarrow \mathrm{Hom}_{\Lambda-alg}(R, A) \times_{F(A)} F(A')$$

is surjective for every small extension $\pi : A' \to A$.

Let $(f, \xi') \in \mathrm{Hom}_{\Lambda-alg}(R, A) \times_{F(A)} F(A')$, i.e. $\hat{F}(f)(\hat{u}) = F(\pi)(\xi')$. We must find $f' \in \mathrm{Hom}_{\Lambda-alg}(R, A')$ such that $\hat{u}_\pi(f') = (f, \xi')$, i.e. such that

$$1) \quad \pi f' = f; \quad 2) \quad \hat{F}(f')(\hat{u}) = \xi'$$

Let's consider the commutative diagram:
[III.2.6]

$$\begin{array}{ccc} \mathrm{Hom}_{\Lambda-alg}(R, \mathbf{k}[\epsilon]) \times \mathrm{Hom}_{\Lambda-alg}(R, A') & \xrightarrow{\beta_1} & \mathrm{Hom}_{\Lambda-alg}(R, A') \times_{\mathrm{Hom}(R,A)} \mathrm{Hom}_{\Lambda-alg}(R, A') \\ \downarrow & & \downarrow \\ t_F \times F(A') & \xrightarrow{\beta_2} & F(A') \times_{F(A)} F(A') \end{array}$$

where the vertical arrows are induced by $\hat{u}$. The map β_1 is a bijection because the action of $\mathrm{Hom}_{\Lambda-alg}(R, \mathbf{k}[\epsilon])$ on the fibres of $\mathrm{Hom}_{\Lambda-alg}(R, A') \to \mathrm{Hom}_{\Lambda-alg}(R, A)$ is free and transitive (see Remark (III.2.3)). From $\bar{H}$) it follows that β_2 is surjective (Remark (III.2.3) again). Therefore if $f' \in \mathrm{Hom}_{\Lambda-alg}(R, A')$ satisfies 1) then, letting

$$\eta' := F(f')(\hat{u}) \in F(A')$$

there exist $v \in \mathrm{Hom}_{\Lambda-alg}(R, \mathbf{k}[\epsilon]) = t_F, \quad f'' \in \mathrm{Hom}_{\Lambda-alg}(R, A')$ such that in diagram [III.2.6] we have:

$$\begin{array}{ccc} (v, f') & \longmapsto & (f'', f') \\ \downarrow & & \downarrow \\ (v, \eta') & \longmapsto & (\xi', \eta') \end{array}$$

This means that $\hat{u}_\pi(f'') = (f, \xi')$. It follows that it suffices to find f' satisfying condition 1).

Let $n > 0$ be such that f factors as

$$R \to R_{n-1} \xrightarrow{f_{n-1}} A$$

Then f' exists if and only if there exists φ which makes the following diagram commutative:

$$\begin{array}{ccc} R_n & \xrightarrow{\varphi} & A' \\ \downarrow p_n & & \downarrow \pi \\ R_{n-1} & \longrightarrow & A \end{array}$$

equivalently if and only if the extension

$$\begin{array}{lccccccccc} f_n^*(A',\pi): & 0\to & \ker(\pi) & \to & R_n\times_A A' & \xrightarrow{\pi'} & R_n & \to 0 \\ & & \| & & \downarrow f_n' & & \downarrow f_n & \\ & 0\to & \ker(\pi) & \to & A' & \xrightarrow{\pi} & A & \to 0 \end{array}$$

is trivial.

Suppose not. This means that π' induces an isomorphism of tangent spaces (see (I.1.2)(ii)), i.e. that there exists an ideal $I\subset\Lambda[[\underline{X}]]$ such that

$$R_n\times_A A'=\Lambda[[\underline{X}]]/I$$

By construction

$$J_{n-1}\supset I\supset(\underline{X})J_{n-1}$$

Moreover, since by $\bar{H}$) the map

$$F(R_n\times_A A')\longrightarrow F(R_n)\times_{F(A)}F(A')$$

is surjective, there exists $u\in F(R_n\times_A A')$ inducing $u_n\in F(R_n)$, hence inducing $u_{n-1}\in F(R_{n-1})$. It follows that I satisfies condition a) and b) and, by the minimality of J_n in $\mathcal{I}$, it follows that $J_n\subset I$. But this is a contradiction because from the fact that π is a surjection with non trivial kernel it follows that I is properly contained in J_n. This proves that $f_n^*(A',\pi)$ is trivial and concludes the proof of the fact that $(R,\hat{u})$ is semiuniversal and of part (i) of the theorem.

(ii) If F is prorepresentable then it trivially satisfies conditions H), H_ϵ) and H_f), as already remarked.

Conversely, suppose that F satisfies conditions $\bar{H}$), H), H_ϵ) and H_f). We have just proved that F has a semiuniversal formal couple $(R,\hat{u})$. We will prove that this is a universal formal couple by showing that for every A in $\mathcal{A}_\Lambda$ the map

$$\hat{u}(A):\mathrm{Hom}_{\Lambda-alg}(R,A)\longrightarrow F(A)$$

induced by $\hat{u}$ is bijective.

This is clearly true if $A=\mathbf{k}$. We will proceed by induction on $\dim_{\mathbf{k}}(A)$. Let $\pi:A'\to A$ be a small extension in $\mathcal{A}_\Lambda$. By the inductive hypothesis

$$\mathrm{Hom}_{\Lambda-alg}(R,A)\longrightarrow F(A)$$

is bijective and, by the versality, the map

$$\hat{u}_\pi:\mathrm{Hom}_{\Lambda-alg}(R,A')\to\mathrm{Hom}_{\Lambda-alg}(R,A)\times_{F(A)}F(A')\cong F(A')$$

is surjective. The map β_2 in diagram [III.2.6] is bijective by condition H), and this implies that $\hat{u}_\pi$ is bijective. *q.e.d.*

NOTES

1. Theorem (III.2.2) has been published in Schlessinger(1968). It had also appeared in Schlessinger(1964). See also Levelt(1969).

2. From Theorem (III.2.2) it follows that if F has a semiuniversal element then it has a tangent space which is of finite dimension, because F satisfies H_0), H_ϵ) and H_f). This property was not implicit in the definition.

III.3. DEFORMATION FUNCTORS

The local moduli functors

In this Section we associate functors of Artin rings to the deformation problems considered in Chapter II and we verify that, under certain restrictions, they satisfy Schlessinger's conditions. As an application we will obtain "formal (semi)universal deformations" of the objects considered.

If X is an algebraic scheme then for every A in $\mathcal{A}$ we let

$$\mathrm{Def}_X(A) = \{\text{deformations of } X \text{ over } A\}/\text{isomorphism}$$

By the functoriality properties already observed in §II.1 this defines a functor of Artin rings

$$\mathrm{Def}_X : \mathcal{A} \to (\text{sets})$$

This is called the *local moduli functor* of X. If $X = \mathrm{Spec}(B_0)$ is affine, we will often write Def_{B_0} instead of Def_X. We can define the subfunctor

$$\mathrm{Def}'_X : \mathcal{A} \to (\text{sets})$$

by

$$\mathrm{Def}'_X(A) = \{\text{locally trivial deformations of } X \text{ over } A\}/\text{isomorphism}$$

called the *locally trivial moduli functor* of X.

(III.3.1) PROPOSITION *(i) For any algebraic scheme X the functors Def_X and Def'_X satisfy Schlessinger's conditions H_0, $\bar{H}$, H_ϵ. Therefore, if $\mathrm{Def}_X(\mathbf{k}[\epsilon])$ (resp. $\mathrm{Def}'_X(\mathbf{k}[\epsilon])$) is finite dimensional, then Def_X (resp. Def'_X) has a semiuniversal element.*
(ii) There is a canonical identification of $\mathbf{k}$-vector spaces

$$\mathrm{Def}'_X(\mathbf{k}[\epsilon]) = H^1(X, T_X) \qquad [III.3.1]$$

In particular if X is nonsingular then

$$\mathrm{Def}_X(\mathbf{k}[\epsilon]) = \mathrm{Def}'_X(\mathbf{k}[\epsilon]) = H^1(X, T_X)$$

(iii) If $X = \mathrm{Spec}(B_0)$ is affine there is a canonical identification of $\mathbf{k}$-vector spaces

$$\mathrm{Def}_{B_0}(\mathbf{k}[\epsilon]) = T^1_{B_0} \qquad [III.3.2]$$

(iv) If X is an arbitrary algebraic scheme then we have a natural identification

$$\mathrm{Def}_X(\mathbf{k}[\epsilon]) = \mathrm{Ex}_{\mathbf{k}}(X, \mathcal{O}_X)$$

and an exact sequence:

$$0 \to H^1(X, T_X) \xrightarrow{\tau} \mathrm{Def}_X(\mathbf{k}[\epsilon]) \xrightarrow{\ell} H^0(X, T^1_X) \xrightarrow{\rho} H^2(X, T_X)$$

(v) If X is a reduced algebraic scheme then there is an isomorphism

$$\mathrm{Def}_X(\mathbf{k}[\epsilon]) \cong \mathrm{Ext}^1_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_X)$$

Proof

Obviously Def_X and Def'_X satisfy condition H_0. To verify the other conditions we assume first that $X = \mathrm{Spec}(B_0)$ is affine.

Let's prove that Def_{B_0} *satisfies* $\bar{H}$. Let

$$\begin{array}{ccc} A' & & A'' \\ & \searrow\swarrow & \\ & A & \end{array}$$

be homomorphisms in $\mathcal{A}$, with $A'' \to A$ a small extension. Letting $\bar{A} = A' \times_A A''$ we have a commutative diagram with exact rows:

[*III*.3.3]
$$\begin{array}{ccccccccc} 0 & \to & (\epsilon) & \to & \bar{A} & \to & A' & \to & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \to & (\epsilon) & \to & A'' & \to & A & \to & 0 \end{array}$$

Consider an element of

$$\mathrm{Def}_{B_0}(A') \times_{\mathrm{Def}_{B_0}(A)} \mathrm{Def}_{B_0}(A'')$$

which is represented by a pair of deformations $f' : A' \to B'$ and $f'' : A'' \to B''$ of B_0 such that $A \to B' \otimes_{A'} A$ and $A \to B'' \otimes_{A''} A$ are isomorphic deformations. Assume that the isomorphism is given by A-isomorphisms $B' \otimes_{A'} A \cong B \cong B'' \otimes_{A''} A$, where $A \to B$ is a deformation. In order to check $\bar{H}$ it suffices to find a deformation $\bar{f} : \bar{A} \to \bar{B}$ inducing (f', f''). Let

$$\bar{B} = B' \times_B B''$$

endowed with the obvious homomorphism $\bar{f} : \bar{A} \to \bar{B}$. It is elementary to check that there are an A'-isomorphism $\bar{B} \otimes_{\bar{A}} A' \cong B'$ and an A''-isomorphism $\bar{B} \otimes_{\bar{A}} A'' \cong B''$.

Therefore we only need to check that $\bar{f}$ is flat. Tensoring diagram [III.3.3] with $\otimes_{\bar{A}} \bar{B}$ we obtain the following diagram with exact rows:

$$\begin{array}{ccccccccc} & & (\epsilon) \otimes_{\bar{A}} \bar{B} & \to & \bar{B} & \to & B' & \to & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \to & B_0 & \to & B'' & \to & B & \to & 0 \end{array}$$

where the second row is given by Lemma (A.2.8). This diagram shows that

$$(\epsilon)\otimes_{\bar{A}}\bar{B}\to\bar{B}$$

is injective: the flatness of $\bar{f}$ follows from Lemma (A.2.8).

Let's prove that Def_{B_0} *satisfies* H_ϵ. Assume in the above situation that $A''=\mathbf{k}[\epsilon]$ and $A=\mathbf{k}$, and let $\tilde{f}:\bar{A}\to\tilde{B}$ be a deformation such that $\alpha(\tilde{f})=(f',f'')$. Then the diagram

$$\begin{array}{ccc}\tilde{B} & \to & \tilde{B}\otimes_{\bar{A}}A''\cong B''\\ \downarrow & & \downarrow\\ \tilde{B}\otimes_{\bar{A}}A'\cong B' & \to & B_0\end{array}$$

commutes: the universal property of the fiber product implies that we have a homomorphism $\gamma:\tilde{B}\to\bar{B}$ of deformations, hence an isomorphism by Lemma (A.2.3). This proves that the fibres of α contain only one element, i.e. α is bijective. Therefore Def_{B_0} satisfies condition H_ϵ.

Let's prove (i) for X arbitrary. Let's consider a diagram in $\mathcal{A}$:

$$\begin{array}{ccc}A' & & A''\\ & \searrow\ \swarrow & \\ & A & \end{array}$$

with $A''\to A$ a small extension and let $\bar{A}=A'\times_A A''$. Consider an element

$$([\mathcal{X}'],[\mathcal{X}''])\in\mathrm{Def}_X(A')\times_{\mathrm{Def}_X(A)}\mathrm{Def}_X(A'')$$

Therefore we have a diagram of deformations:

$$\begin{array}{ccccc}\mathcal{X}' & & & & \mathcal{X}''\\ & \nwarrow f' & & \nearrow f'' & \\ \downarrow & & \mathcal{X} & & \downarrow\\ \mathrm{Spec}(A') & & \downarrow & & \mathrm{Spec}(A'')\\ & \nwarrow & & \nearrow & \\ & & \mathrm{Spec}(A) & & \end{array}$$

where the morphisms f' and f'' induce isomorphisms of deformations

$$\mathcal{X}'\times_{\mathrm{Spec}(A')}\mathrm{Spec}(A)\cong\mathcal{X}\cong\mathcal{X}''\times_{\mathrm{Spec}(A'')}\mathrm{Spec}(A)$$

Consider the sheaf of $\bar{A}$-algebras $\mathcal{O}_{\mathcal{X}'}\times_{\mathcal{O}_{\mathcal{X}}}\mathcal{O}_{\mathcal{X}''}$ on X. Then $\bar{\mathcal{X}}:=(|X|,\mathcal{O}_{\mathcal{X}'}\times_{\mathcal{O}_{\mathcal{X}}}\mathcal{O}_{\mathcal{X}''})$ is a scheme over $\mathrm{Spec}(\bar{A})$ (by the proof of the affine case). Reducing to the affine case one shows that $\bar{\mathcal{X}}$ is flat over $\mathrm{Spec}(\bar{A})$. Therefore $\bar{\mathcal{X}}$ is a deformation of X over $\mathrm{Spec}(\bar{A})$ inducing the pair $([\mathcal{X}'],[\mathcal{X}''])$. This shows that the map

$$\mathrm{Def}_X(\bar{A})\to\mathrm{Def}_X(A')\times_{\mathrm{Def}_X(A)}\mathrm{Def}_X(A'')$$

is surjective, proving $\bar{H}$ for Def_X. Moreover if the deformations $\mathcal{X}'$ and $\mathcal{X}''$ are locally trivial then so is $\bar{\mathcal{X}}$, and therefore $\bar{H}$ holds for the functor Def'_X as well.

Now assume that $A'' = \mathbf{k}[\epsilon]$ and $A = \mathbf{k}$. Then the previous diagram becomes

$$\begin{array}{ccccc} \mathcal{X}' & & & & \mathcal{X}'' \\ & \nwarrow f' & & \nearrow f'' & \\ & & X & & \end{array}$$

In this case any $\tilde{\mathcal{X}} \to \mathrm{Spec}(\bar{A})$ inducing the pair $([\mathcal{X}'], [\mathcal{X}'']) \in \mathrm{Def}_X(A') \times_{\mathrm{Def}_X(A)} \mathrm{Def}_X(A'')$ is such that the isomorphisms

$$\tilde{\mathcal{X}} \times_{\mathrm{Spec}(\bar{A})} \mathrm{Spec}(A') \cong \mathcal{X}', \quad \tilde{\mathcal{X}} \times_{\mathrm{Spec}(\bar{A})} \mathrm{Spec}(A'') \cong \mathcal{X}''$$

induce the identity on $X = \tilde{\mathcal{X}} \times_{\mathrm{Spec}(\bar{A})} \mathrm{Spec}(\mathbf{k})$. Therefore $\tilde{\mathcal{X}}$ fits into a commutative diagram

$$\begin{array}{ccccc} & & \tilde{\mathcal{X}} & & \\ & \nearrow & & \nwarrow & \\ \mathcal{X}' & & & & \mathcal{X}'' \\ & \nwarrow f' & & \nearrow f'' & \\ & & X & & \end{array}$$

By the universal property of the fibered sum of schemes we then get a morphism of deformations $\bar{\mathcal{X}} \to \tilde{\mathcal{X}}$, which is necessarily an isomorphism. This proves property H_ϵ for Def_X. The proof for Def'_X is similar.

(ii) and (iii) The identifications [III.3.1] and [III.3.2] have been proved in (II.1.6) and (II.4.4) respectively. The verification that they are $\mathbf{k}$-linear are elementary and will be left to the reader.

(iv) If

$$\begin{array}{ccc} X & \subset & \mathcal{X} \\ \downarrow & & \downarrow \\ \mathrm{Spec}(\mathbf{k}) & \to & \mathrm{Spec}(\mathbf{k}[\epsilon]) \end{array}$$

is a first order deformation of X then $X \subset \mathcal{X}$ is an extension of X by $\mathcal{O}_X$ because by the flatness of $\mathcal{X}$ over $\mathrm{Spec}(\mathbf{k}[\epsilon])$ we have $\epsilon\mathcal{O}_{\mathcal{X}} \cong \mathcal{O}_X$ (Lemma (A.2.8)). Conversely, given such an extension $X \subset \mathcal{X}$ we have an exact sequence

$$0 \to \mathcal{O}_X \xrightarrow{j} \mathcal{O}_{\mathcal{X}} \to \mathcal{O}_X \to 0$$

$\mathcal{O}_{\mathcal{X}}$ has a natural structure of $\mathbf{k}[\epsilon]$-algebra by sending, for any open $U \subset X$, $\epsilon \mapsto j(1)$. It follows from Lemma (A.2.8) that $\mathcal{X}$ is flat over $\mathrm{Spec}(\mathbf{k}[\epsilon])$.

The map τ corresponds to the inclusion $\mathrm{Def}'_X(\mathbf{k}[\epsilon]) \subset \mathrm{Def}_X(\mathbf{k}[\epsilon])$ in view of [III.3.1]. The map ℓ associates to a first order deformation ξ of X the section of T^1_X defined by the restrictions $\{\xi_{|U_i}\}$ for some affine open cover $\{U_i\}$. It is clear that $\mathrm{Im}(\tau) = \ker(\ell)$.

We now define ρ. Let $h \in H^0(X, T^1_X)$ be represented, in a suitable affine open cover $\mathcal{U} = \{U_i = \mathrm{Spec}(B_i)\}$ of X, by a collection of $\mathbf{k}$-extensions $\mathcal{E}_i$ of B_i by B_i. Since h is a global section there exist isomorphisms $\sigma_{ij} : \mathcal{E}_{j|U_i \cap U_j} \cong \mathcal{E}_{i|U_i \cap U_j}$. These isomorphisms patch together to give an extension $\mathcal{E}$ if and only if $h \in \mathrm{Im}(\ell)$ if and only if we can find new isomorphisms σ'_{ij} such that

$$[III.3.4] \qquad \sigma'_{ij}\sigma'_{jk} = \sigma'_{ik}$$

on $U_i \cap U_j \cap U_k$. Such isomorphisms are of the form

$$\sigma'_{ij} = \sigma_{ij}\theta_{ij}$$

where θ_{ij} is an automorphism of the extension $\mathcal{E}_{j|U_i \cap U_j}$. The collection of automorphisms (θ_{ij}) corresponds, via Lemma (II.1.5), to a 1-cochain $(t_{ij}) \in \mathcal{C}^1(\mathcal{U}, T_X)$; conversely every 1-cochain (t_{ij}) defines a system of isomorphisms (σ'_{ij}); and the condition [III.3.4] is satisfied if and only if (t_{ij}) is a 1-cocycle. Therefore we define $\rho(h)$ to be the class of the 2-cocycle $(t_{ij} + t_{jk} - t_{ik})$. With this definition we clearly have $\ker(\rho) = \mathrm{Im}(\ell)$. We leave to the reader to verify that the definition of ρ does not depend on the choices made.

(v) Since we have a natural identification $\mathrm{Def}_X(\mathbf{k}[\epsilon]) = \mathrm{Ex}_{\mathbf{k}}(X, \mathcal{O}_X)$ we conclude by Theorem (I.4.3). *q.e.d.*

(III.3.2) COROLLARY *Assume that X is one of the following:*
(i) a projective scheme.
(ii) an affine algebraic scheme with isolated singularities.
Then Def_X *has a semiuniversal element.*

Proof
Either condition implies that $H^1(X, T_X)$ and $H^0(X, T^1_X)$ are finite dimensional vector spaces. Therefore the conclusion follows from Theorem (III.3.1). *q.e.d.*

The stronger property of being prorepresentable is not satisfied in general by Def_X. We will discuss this matter in section III.4.

If $(R, \hat{u})$ is a semiuniversal couple for Def_X then the Krull dimension of R (i.e. the maximum of the dimensions of the irreducible components of Spec(R)) is called the *number of moduli* of X and it is denoted by $\mu(X)$.

* * * * * *

The local Hilbert functor

Let $X \subset Y$ be a closed embedding of algebraic schemes. For each A in $\mathcal{A}$ we let

$$H^Y_X(A) = \{\text{deformations of } X \text{ in } Y \text{ over } A\}$$

It is immediate to verify that this defines a functor of Artin rings

$$H^Y_X : \mathcal{A} \to (\text{sets})$$

called the *local Hilbert functor* of X in Y.

(III.3.3) PROPOSITION *Given a closed embedding of schemes $X \subset Y$ then:*
(i) the local Hilbert functor H_X^Y satisfies conditions H_0, H_ϵ, $\bar{H}$, H.
(ii) There is a natural identification

$$H_X^Y(\mathbf{k}[\epsilon]) = H^0(X, N_{X/Y})$$

where $N_{X/Y}$ is the normal sheaf of X in Y.

Proof
(i) Obviously H_X^Y satisfies condition H_0. Let

$$\begin{array}{ccc} A' & & A'' \\ & \searrow\swarrow & \\ & A & \end{array}$$

be homomorphisms in $\mathcal{A}$, with $A'' \to A$ a small extension. Letting $\bar{A} = A' \times_A A''$ we have a commutative diagram with exact rows:

$$[III.3.5] \qquad \begin{array}{ccccccccc} 0 & \to & (\epsilon) & \to & \bar{A} & \to & A' & \to & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \to & (\epsilon) & \to & A'' & \to & A & \to & 0 \end{array}$$

Take an element of

$$H_X^Y(A') \times_{H_X^Y(A)} H_X^Y(A'')$$

which is represented by a pair of deformations $\mathcal{X}' \subset Y \times \mathrm{Spec}(A')$ and $\mathcal{X}'' \subset Y \times \mathrm{Spec}(A'')$ such that

$$\mathcal{X}' \times_{\mathrm{Spec}(A')} \mathrm{Spec}(A) = \mathcal{X}'' \times_{\mathrm{Spec}(A'')} \mathrm{Spec}(A) \subset Y \times \mathrm{Spec}(A)$$

Consider the sheaf of $\bar{A}$-algebras $\mathcal{O}_{\mathcal{X}'} \times_{\mathcal{O}_\mathcal{X}} \mathcal{O}_{\mathcal{X}''}$ on X. Then $\bar{\mathcal{X}} := (|X|, \mathcal{O}_{\mathcal{X}'} \times_{\mathcal{O}_\mathcal{X}} \mathcal{O}_{\mathcal{X}''})$ is a scheme over $\mathrm{Spec}(\bar{A})$, flat over $\mathrm{Spec}(\bar{A})$ (see the proof of (III.3.1)). Therefore $\bar{\mathcal{X}}$ is a deformation of X over $\mathrm{Spec}(\bar{A})$ inducing $\mathcal{X}'$ and $\mathcal{X}''$. We have a commutative diagram:

$$\begin{array}{ccc} \mathcal{X} & \to & \mathcal{X}' \\ \downarrow & & \downarrow \\ \mathcal{X}'' & \to & Y \times \mathrm{Spec}(\bar{A}) \end{array}$$

and the universal property of the fibered sum implies that there is a morphism

$$\Phi : \bar{\mathcal{X}} \to Y \times \mathrm{Spec}(\bar{A})$$

Pulling back Φ over $\mathrm{Spec}(A'')$ (resp. $\mathrm{Spec}(A')$) we obtain the closed embedding $\mathcal{X}'' \subset Y \times \mathrm{Spec}(A'')$ (resp. $\mathcal{X}' \subset Y \times \mathrm{Spec}(A')$). Since $\mathrm{Spec}(A'') \subset \mathrm{Spec}(\bar{A})$ is a closed embedding defined by a square zero ideal, it follows that Φ is a closed embedding as well (details are left to the reader). Therefore $\Phi : \bar{\mathcal{X}} \subset Y \times \mathrm{Spec}(\bar{A})$ defines an element of $H_X^Y(\bar{A})$ which is mapped to $(\mathcal{X}', \mathcal{X}'')$ by the map:

$$\alpha : H_X^Y(A' \times_A A'') \to h_X^Y(A') \times_{H_X^Y(A)} H_X^Y(A'')$$

It follows that α is surjective. Now let $\tilde{\mathcal{X}} \subset Y \times \mathrm{Spec}(\bar{A})$ be another element of $H_X^Y(\bar{A})$ which is mapped to $(\mathcal{X}', \mathcal{X}'')$. Then by the universal property of the fibered sum there is a morphism $\tau : \bar{\mathcal{X}} \to \tilde{\mathcal{X}}$ which, as in the proof of (III.3.1), is easily seen to be an isomorphism. Moreover the following diagram commutes:

$$\begin{array}{ccccc} & & Y \times \mathrm{Spec}(\bar{A}) & & \\ & \nearrow & & \nwarrow & \\ \bar{\mathcal{X}} & & \xrightarrow{\tau} & & \tilde{\mathcal{X}} \end{array}$$

Since the diagonal arrows are closed embeddings it follows that $\bar{\mathcal{X}} = \tilde{\mathcal{X}}$ as closed subschemes of $Y \times \mathrm{Spec}(\bar{A})$. This proves that α is actually a bijection and (i) follows.

(ii) has already been proved in (II.3.3). *q.e.d.*

(III.3.4) COROLLARY *Let $X \subset Y$ be a closed embedding. If $h^0(X, N_{X/Y}) < \infty$, for example if X is projective, then H_X^Y is prorepresentable.*

Proof

Follows from Proposition (III.3.3) and from Schlessinger's Theorem. *q.e.d.*

If $X \subset Y$ is a closed embedding of projective schemes then the prorepresentability of H_X^Y follows directly from the existence of the Hilbert scheme Hilb^Y because H_X^Y is prorepresented by the complete local ring $\hat{\mathcal{O}}_{\mathrm{Hilb}^Y,[X]}$ (see §IV.4).

* * * * * *

Formal deformations

The Theorems proved in this Section can be interpreted in terms of "formal deformations" which we now introduce.

Let $\bar{A}$ be in $\hat{\mathcal{A}}$. A *formal deformation* of X over $\bar{A}$ is a sequence of infinitesimal deformations of X

$$\eta_n : \quad \begin{array}{ccc} X & \xrightarrow{f_n} & \mathcal{X}_n \\ \downarrow & & \downarrow \pi_n \\ \mathrm{Spec}(\mathbf{k}) & \to & \mathrm{Spec}(\bar{A}_n) \end{array}$$

where $\bar{A}_n = \bar{A}/m_{\bar{A}}^{n+1}$, such that for all $n \geq 1$ η_n induces η_{n-1} by pullback under the natural inclusion $\mathrm{Spec}(\bar{A}_{n-1}) \to \mathrm{Spec}(\bar{A}_n)$, i.e. we have:

$$\eta_{n-1} : \quad \begin{array}{ccc} X & \xrightarrow{f_{n-1}} & \mathcal{X}_n \otimes_{\bar{A}_n} \bar{A}_{n-1} = \mathcal{X}_{n-1} \\ \downarrow & & \downarrow \pi_{n-1} \\ \mathrm{Spec}(\mathbf{k}) & \to & \mathrm{Spec}(\bar{A}_{n-1}) \end{array}$$

We will denote such a formal deformation by $(\bar{A}, \{\eta_n\})$. It can be also viewed as the morphism of formal schemes $\bar{\pi} : \mathfrak{X} \to \mathrm{Specf}(\bar{A})$ where

$$\mathfrak{X} = (X, \lim_{\leftarrow} \mathcal{O}_{\mathcal{X}_n}), \quad \bar{\pi} = \lim_{\leftarrow} \pi_n$$

Warning: a formal deformation $(\bar{A}, \{\eta_n\})$ is not to be confused with a deformation of X over $\mathrm{Spec}(\bar{A})$ (see §III.5 for a discussion of this point).

It follows from the definitions that a formal deformation $(\bar{A}, \{\eta_n\})$ defines an element $\hat{\eta} \in \widehat{\mathrm{Def}}_X(\bar{A})$ and conversely every such element is defined by some formal deformation. Equivalently $(\bar{A}, \hat{\eta})$ is a formal couple for Def_X.

The formal deformation $(\bar{A}, \{\eta_n\})$ will be called *trivial* (resp. *locally trivial*) if each η_n is trivial (resp. locally trivial). For every algebraic scheme X and for each $\bar{A}$ in $\hat{\mathcal{A}}$ the trivial formal deformation of X over $\bar{A}$ can be identified with the formal completion of $X \times \mathrm{Spec}(\bar{A})$ along $X = X \times \mathrm{Spec}(\mathbf{k})$, and this is in turn identified with the product $X \times \mathrm{Specf}(\bar{A})$.

Let $\bar{A}$ be in $\hat{\mathcal{A}}$ and let $X \subset Y$ be a closed embedding of algebraic schemes. *A formal deformation of X in Y* is a sequence

$$\xi_n : \begin{array}{ccc} \mathcal{X}_n & \subset & Y \times \mathrm{Spec}(\bar{A}_n) \\ \downarrow & & \\ \mathrm{Spec}(\bar{A}_n) & & \end{array}$$

of infinitesimal deformations of X in Y over $\bar{A}_n = \bar{A}/m_{\bar{A}}^{n+1}$ such that for all $n \geq 1$ ξ_n induces ξ_{n-1} by pullback under the natural inclusion $\mathrm{Spec}(\bar{A}_{n-1}) \to \mathrm{Spec}(\bar{A}_n)$. We can describe the formal deformation $(\bar{A}, \{\xi_n\})$ of X in Y as a diagram of formal schemes

$$\begin{array}{cc} \mathcal{X} & \subset Y \times \mathrm{Specf}(\bar{A}) \\ \downarrow & \\ \mathrm{Specf}(\bar{A}) & \end{array}$$

As in the case of the functor Def_X, a formal deformation $(\bar{A}, \{\xi_n\})$ of X in Y defines an element $\hat{\xi} \in \widehat{H_X^Y}(\bar{A})$, i.e. a formal couple $(\bar{A}, \hat{\xi})$ for H_X^Y, and conversely every such element is defined by a formal deformation of X in Y.

(III.3.5) DEFINITION *Let X be an algebraic scheme. A formal deformation $(R, \{\eta_n\})$ of X is called* universal *(resp.* semiuniversal, versal*) if it defines a universal (resp. semiuniversal, versal) formal element of* Def_X*, i.e. if $(R, \hat{\eta})$ is a universal (resp. semiuniversal, versal) formal couple for* Def_X.
Let $X \subset Y$ be a closed embedding of algebraic schemes. A formal deformation $(R, \{\xi_n\})$ of X in Y is called universal *(resp.* semiuniversal, versal*) if $(R, \hat{\eta})$ is a universal (resp. semiuniversal, versal) formal couple of H_X^Y.*

Of course Corollaries (III.3.2) and (III.3.4) can be rephrased as existence Theorems for (semi)universal formal deformations.

* * * * * *

Obstruction spaces

The elementary analysis of obstructions to lift infinitesimal deformations carried out in Chapter II can be interpreted as the description of obstruction spaces for the corresponding deformation functors. More precisely we have the following

(III.3.6) PROPOSITION *(i) Let X be a nonsingular algebraic variety. Then $H^2(X, T_X)$ is an obstruction space for the functor* Def_X*. If X is an arbitrary algebraic scheme then $H^2(X, T_X)$ is an obstruction space for the functor* Def'_X.

(ii) Let $X = \mathrm{Spec}(B)$ *be an affine algebraic scheme. Then* T^2_B *is an obstruction space for the functor* $\mathrm{Def}_X = \mathrm{Def}_B$.
(iii) Let $X \subset Y$ *be a closed regular embedding of algebraic schemes. Then* $H^1(X, N_{X/Y})$ *is an obstruction space for the local Hilbert functor* H^Y_X.

Proof
The Proposition is just a rephrasing of (II.1.8), (II.5.3) and (II.3.3). *q.e.d.*

In particular we see that X as in (i) or (ii) is unobstructed if and only if the functor Def_X is smooth. Similarly if $X \subset Y$ are as in (iii), then X is unobstructed in Y if and only if H^Y_X is smooth.

(III.3.7) COROLLARY *(i) Let* X *be a nonsingular projective algebraic variety and let* $(R, \{\eta_n\})$ *be a formal semiuniversal deformation of* X. *Then*

$$h^1(X, T_X) \geq \dim(R) \geq h^1(X, T_X) - h^2(X, T_X)$$

The first equality holds if and only if X *is unobstructed*
(ii) Let $X = \mathrm{Spec}(B)$ *be an affine algebraic scheme with isolated singularities and let* $(R, \{\eta_n\})$ *be a formal semiuniversal deformation of* X. *Then*

$$\dim_{\mathbf{k}}(T^1_B) \geq \dim(R) \geq \dim_{\mathbf{k}}(T^1_B) - \dim_{\mathbf{k}}(T^2_B)$$

The first equality holds if and only if X *is unobstructed.*
(iii) Let $X \subset Y$ *be a closed regular embedding of algebraic schemes with* X *projective and let* $(R, \{\xi_n\})$ *be a formal universal deformation of* X *in* Y. *Then*

$$h^0(X, N_{X/Y}) \geq \dim(R) \geq h^0(X, N_{X/Y}) - h^1(X, N_{X/Y})$$

The first equality holds if and only if X *is unobstructed in* Y.

Proof
It is an immediate consequence of (III.3.6) and of Corollary (III.1.11). *q.e.d.*

In the case of a closed embedding $X \subset Y$ which is not regular Proposition (III.3.6) and Corollary (III.3.7) say nothing about the obstructions of H^Y_X. We refer the reader to §IV.5 and §IV.6 for some information about the general case.

If X is an algebraic scheme which is neither affine nor nonsingular the previous results give no information about obstructions of the functor Def_X. The following Proposition addresses the case of a reduced l.c.i. scheme.

(III.3.8) PROPOSITION *Let* X *be a reduced l.c.i. algebraic scheme* X, *and assume char*$(\mathbf{k}) = 0$. *Then* $\mathrm{Ext}^2_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_X)$ *is an obstruction space for the functor* Def_X.

Proof
Let A be in $\mathcal{A}$ and let

$$\xi: \begin{array}{ccc} X & \to & \mathcal{X} \\ \downarrow & & \downarrow f \\ \mathrm{Spec}(\mathbf{k}) & \to & \mathrm{Spec}(A) \end{array}$$

be a family of deformations of X over A. We need to define a $\mathbf{k}$-linear map

$$o_\xi : \mathrm{Ex}_{\mathbf{k}}(A, \mathbf{k}) \to \mathrm{Ext}^2_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_X)$$

having the properties of an obstruction map according to Definition (III.1.9). Consider an element of $\mathrm{Ex}_{\mathbf{k}}(A, \mathbf{k})$ represented by an extension

$$0 \to (t) \to \tilde{A} \xrightarrow{\eta} A \to 0$$

Consider the conormal sequence of $\mathbf{k} \to \tilde{A} \to A$

[*III*.3.6] $$0 \to (t) \to \Omega_{\tilde{A}/\mathbf{k}} \otimes_{\tilde{A}} A \to \Omega_{A/\mathbf{k}} \to 0$$

which is exact by Example (A.1.9)(v). By pulling it back to $\mathcal{X}$ we obtain the exact sequence

$$0 \to \mathcal{O}_X \to f^*(\Omega^1_{\mathrm{Spec}(\tilde{A})|\mathrm{Spec}(A)}) \to f^*(\Omega^1_{\mathrm{Spec}(A)}) \to 0$$

If we combine the above sequence with the relative cotangent sequence of f we obtain the following diagram:

[*III*.3.7] $$\begin{array}{ccccccc} 0 \to & \mathcal{O}_X & \to & f^*(\Omega^1_{\mathrm{Spec}(\tilde{A})|\mathrm{Spec}(A)}) & \to & f^*(\Omega^1_{\mathrm{Spec}(A)}) & \to 0 \\ & & & & & \downarrow & \\ & & & & & \Omega^1_{\mathcal{X}} & \\ & & & & & \downarrow & \\ & & & & & \Omega^1_{\mathcal{X}/\mathrm{Spec}(A)} & \\ & & & & & \downarrow & \\ & & & & & 0 & \end{array}$$

and from this diagram we obtain the 2-term extension

$$0 \to \quad \mathcal{O}_X \quad \to \quad f^*(\Omega^1_{\mathrm{Spec}(\tilde{A})|\mathrm{Spec}(A)}) \quad \to \quad \Omega^1_{\mathcal{X}} \quad \to \quad \Omega^1_{\mathcal{X}/\mathrm{Spec}(A)} \quad \to 0$$

which defines an element

$$o_\xi(\eta) \in \mathrm{Ext}^2_{\mathcal{O}_{\mathcal{X}}}(\Omega^1_{\mathcal{X}/\mathrm{Spec}(A)}, \mathcal{O}_X)$$

This defines the map o_ξ. The linearity of o_ξ is a consequence of the linearity of the map $\mathrm{Ex}_{\mathbf{k}}(A, \mathbf{k}) \to \mathrm{Ext}^1_A(\Omega_{A/\mathbf{k}}, \mathbf{k})$ associating to an extension η the conormal sequence [III.3.6].

Assume that there is a deformation of X over $\tilde{A}$ extending ξ, i.e. that we have a diagram:

$$\begin{array}{ccc} \mathcal{X} & \subset & \tilde{\mathcal{X}} \\ \downarrow f & & \downarrow \tilde{f} \\ \mathrm{Spec}(A) & \subset & \mathrm{Spec}(\tilde{A}) \end{array}$$

Then diagram [III.3.7] can be completed to a diagram as follows:

$$\begin{array}{ccccccccc}
0\to & \mathcal{O}_X & \to & \tilde{f}^*(\Omega^1_{\mathrm{Spec}(\tilde{A})})_{|\mathrm{Spec}(A)} & \to & f^*(\Omega^1_{\mathrm{Spec}(A)}) & \to 0 \\
 & \| & & \downarrow & & \downarrow & \\
0\to & \mathcal{O}_X & \to & \Omega^1_{\tilde{\mathcal{X}}|\mathcal{X}} & \to & \Omega^1_{\mathcal{X}} & \to 0 \\
 & & & \downarrow & & \downarrow & \\
 & & & (\Omega^1_{\tilde{\mathcal{X}}/\mathrm{Spec}(\tilde{A})})_{|\mathcal{X}} & = & \Omega^1_{\mathcal{X}/\mathrm{Spec}(A)} & \\
 & & & \downarrow & & \downarrow & \\
 & & & 0 & & 0 &
\end{array}$$

In this diagram the first row is the pullback of the second row and this implies that $o_\xi(\eta) = 0$.

Conversely, assume that $o_\xi(\eta) = 0$. Then diagram [III.3.7] can be completed as follows:

$$\begin{array}{ccccccccc}
0\to & \mathcal{O}_X & \to & f^*(\Omega^1_{\mathrm{Spec}(\tilde{A})|\mathrm{Spec}(A)}) & \to & f^*(\Omega^1_{\mathrm{Spec}(A)}) & \to 0 \\
 & \| & & \downarrow & & \downarrow & \\
0\to & \mathcal{O}_X & \to & \mathcal{E} & \to & \Omega^1_{\mathcal{X}} & \to 0 \\
 & & & & & \downarrow & \\
 & & & & & \Omega^1_{\mathcal{X}/\mathrm{Spec}(A)} & \\
 & & & & & \downarrow & \\
 & & & & & 0 &
\end{array}$$

for some coherent sheaf $\mathcal{E}$ on $\mathcal{X}$. By the construction of Theorem (I.4.3) one finds a sheaf of A-algebras $\mathcal{O}_{\tilde{\mathcal{X}}}$ and an extension of sheaves of A-algebras

$$0 \to \mathcal{O}_X \to \mathcal{O}_{\tilde{\mathcal{X}}} \to \mathcal{O}_{\mathcal{X}} \to 0$$

such that $\mathcal{E} = \Omega^1_{\tilde{\mathcal{X}}|\mathcal{X}}$. It remains to be shown that $\mathcal{O}_{\tilde{\mathcal{X}}}$ can be given a structure of sheaf of flat $\tilde{A}$-algebras. This can be done by means of the homomorphism

$$f^*(\Omega^1_{\mathrm{Spec}(\tilde{A})|\mathrm{Spec}(A)}) \to \mathcal{E} = \Omega^1_{\tilde{\mathcal{X}}|\mathcal{X}}$$

(details are left to the reader). *q.e.d.*

* * * * * *

The forgetful morphism

Let $X \subset Y$ be a closed embedding of algebraic schemes. The *forgetful morphism*

$$\Phi : H^Y_X \to \mathrm{Def}_X$$

is the morphism which associates to an infinitesimal deformation of X in Y:

$$\xi : \begin{array}{ccc} \mathcal{X} & \subset & Y \times \mathrm{Spec}(A) \\ \downarrow & & \\ \mathrm{Spec}(A) & & \end{array}$$

the deformation of X:

$$\begin{array}{ccc} X & \to & \mathcal{X} \\ \downarrow & & \downarrow \\ \mathrm{Spec}(\mathbf{k}) & \to & \mathrm{Spec}(A) \end{array}$$

(III.3.9) PROPOSITION *Assume that X and Y are nonsingular and that X is projective. Consider the exact sequence*

$$[III.3.8] \qquad 0 \to T_X \to T_{Y|X} \to N_{X/Y} \to 0$$

Then
(i) $d\Phi = \delta : H^0(X, N_{X/Y}) \to H^1(X, T_X)$ the coboundary map coming from [III.3.8].
(ii) If X is unobstructed in Y and δ is surjective then Φ is smooth, X is unobstructed as an abstract variety and has $rk(\delta)$ number of moduli.
(iii) If $H^1(X, T_{Y|X}) = 0$ then Φ is smooth.

Proof
(i) follows from (II.3.6).
(ii) follows from (i) and from (III.2.4)).
(iii) follows from (II.3.7) and (III.2.4)). *q.e.d.*

Part (iii) of the Proposition often gives a very effective way of proving that a given $X \subset Y$ is unobstructed as an abstract variety. If X is a curve in $\mathbb{P}^r$ the vanishing of $H^1(X, T_{\mathbb{P}^r|X})$ is related with the Petri map (see example (II.3.10)(ii)). The following examples are further applications of this principle.

(III.3.10) EXAMPLES (i) (Kodaira-Spencer(1958)) Let's give a non trivial application of (III.3.9). Let $X \subset \mathbb{P}^r$, $r \geq 3$, be a nonsingular hypersurface of degree $d \geq 2$. Then $h^1(N_{X/\mathbb{P}^r}) = h^1(\mathcal{O}_X(d)) = 0$ and therefore X is unobstructed in $\mathbb{P}^r$. On the other hand from the exact sequence:

$$0 \to T_{\mathbb{P}^r}(-d) \to T_{\mathbb{P}^r} \to T_{\mathbb{P}^r|X} \to 0$$

and the Euler sequence:

$$0 \to \mathcal{O}_{\mathbb{P}^r} \to \mathcal{O}_{\mathbb{P}^r}(1)^{r+1} \to T_{\mathbb{P}^r} \to 0$$

we deduce that:

$$h^1(T_{\mathbb{P}^r|X}) = h^2(T_{\mathbb{P}^r}(-d)) = 0 \quad \text{if } r \geq 4$$

while for $r = 3$ we have the exact sequence:

$$\begin{array}{ccccccc} 0 \leftarrow & H^2(T_{\mathbb{P}^3}(-d))^\vee & \leftarrow & H^0(\mathcal{O}_{\mathbb{P}^3}(d-4)) & \leftarrow & H^0(\mathcal{O}_{\mathbb{P}^3}(d-5))^4 \\ & \| & & & & \\ & H^1(T_{\mathbb{P}^3|X})^\vee & & & & \end{array}$$

Therefore we see that

$$h^1(T_{\mathbb{P}^r|X}) = \begin{cases} 1 & \text{if } r = 3 \text{ and } d = 4; \\ 0 & \text{otherwise} \end{cases}$$

From (III.3.9) we therefore deduce that $H_X^{\mathbb{P}^r} \to \mathrm{Def}_X$ is smooth and X is unobstructed as an abstract variety unless $r = 3$ and $d = 4$ (this is precisely the case when X is a K3 surface). An analogous result holds more generally for complete intersections (Sernesi(1975)).

Using [III.3.8] one computes easily that $H^2(T_X) \neq 0$ if X is a nonsingular surface of degree $d \geq 5$ in $\mathbb{P}^3$: therefore the unobstructedness of X could not have been deduced from (III.3.7) in this case.

(ii) The previous example can be easily generalized to nonsingular hypersurfaces of $\mathbb{P}^n \times \mathbb{P}^m$, $1 \leq n \leq m$, $n + m \geq 3$. Let

$$\begin{array}{ccc} \mathbb{P}^n \times \mathbb{P}^m & \xrightarrow{q} & \mathbb{P}^m \\ \downarrow p & & \\ \mathbb{P}^n & & \end{array}$$

be the projections. Consider a nonsingular hypersurface $X \subset \mathbb{P}^n \times \mathbb{P}^m$ of bidegree (a, b), i.e. defined by an equation $\sigma = 0$ for some $\sigma \in H^0(\mathcal{O}(a, b))$, where

$$\mathcal{O}(a, b) := p^*\mathcal{O}(a) \otimes q^*\mathcal{O}(b)$$

From the exact sequence

$$0 \to \mathcal{O} \to \mathcal{O}(a, b) \to N_{X/\mathbb{P}^n \times \mathbb{P}^m} \to 0$$

one deduces that

$$H^1(N_{X/\mathbb{P}^n \times \mathbb{P}^m}) = (0)$$

and therefore X is unobstructed in $\mathbb{P}^n \times \mathbb{P}^m$. For any coherent sheaf $\mathcal{F}$ on $\mathbb{P}^n \times \mathbb{P}^m$ we use the notation

$$\mathcal{F}(\alpha, \beta) = \mathcal{F} \otimes \mathcal{O}(\alpha, \beta)$$

Using the fact that

$$T_{\mathbb{P}^n \times \mathbb{P}^m} = p^*T_{\mathbb{P}^n} \oplus q^*T_{\mathbb{P}^m}$$

and the Leray spectral sequence with respect to any one of the projections, one easily computes that

$$h^i(T_{\mathbb{P}^n \times \mathbb{P}^m}(\alpha, \beta)) = 0$$

when $n + m \geq 4$, $i = 1, 2$ and (α, β) arbitrary. Moreover, when $(n, m) = (1, 2)$ one finds:

$$\begin{array}{lll} h^i(T_{\mathbb{P}^1 \times \mathbb{P}^2}) & = 0 & \text{all } i \geq 1 \\ h^2(T_{\mathbb{P}^1 \times \mathbb{P}^2}(-a, -b)) & = 0 & \text{unless } (a, b) = (2, 3) \end{array}$$

Putting all these informations together and using the exact sequence

$$0 \to T_{\mathbb{P}^n\times\mathbb{P}^m}(-a,-b) \to T_{\mathbb{P}^n\times\mathbb{P}^m} \to T_{\mathbb{P}^n\times\mathbb{P}^m|X} \to 0$$

one deduces that

$$h^1(T_{\mathbb{P}^n\times\mathbb{P}^m|X}) = 0$$

unless $(n,m) = (1,2)$ and $(a,b) = (2,3)$ (these are precisely tha cases when X is a K3 surface). Now as before we conclude that the forgetful morphism $H_X^{\mathbb{P}^n\times\mathbb{P}^m} \to \mathrm{Def}_X$ is smooth and X is unobstructed as an abstract variety.

* * * * * *

The local relative Hilbert functor - Stability

Given a projective morphism $p : \mathcal{X} \to S$ of schemes and a $\mathbf{k}$-rational point $s \in S$, consider the fibre $\mathcal{X}(s)$ and a closed subscheme $Z \subset \mathcal{X}(s)$. For each A in $\mathrm{ob}(\mathcal{A})$ an *infinitesimal deformation of Z in $\mathcal{X}$ relative to p parametrized by A* is a commutative diagram:

$$\begin{array}{ccccc} \mathcal{Z} & \subset & \mathcal{X}_A & \to & \mathcal{X} \\ & \searrow & \downarrow & & \downarrow p \\ & & \mathrm{Spec}(A) & \xrightarrow{s} & S \end{array}$$

where the right square is cartesian, the left diagonal morphism is flat and its closed fibre is Z; this means in particular that the morphism s has image $\{s\}$ and therefore that A is an $\mathcal{O}_{S,s}$-algebra. Then, letting $\Lambda = \mathcal{O}_{S,s}$, we can define the *local relative Hilbert functor*

$$H_Z^{\mathcal{X}/S} : \mathcal{A}_\Lambda \to (\mathrm{sets})$$

by

$$H_Z^{\mathcal{X}/S}(A) = \left\{ \begin{array}{c} \text{infinitesimal deformations of } Z \text{ in } \mathcal{X} \\ \text{relative to } p \text{ parametrized by } A \end{array} \right\}$$

By definition the functor $H_Z^{\mathcal{X}/S}$ comes equipped with a structural morphism

$$H_Z^{\mathcal{X}/S} \to h_\Lambda$$

We have the following generalization of Corollaries (III.3.4) and (III.3.8):

(III.3.11) THEOREM *Let $p : \mathcal{X} \to S$ be a projective morphism of schemes, $s \in S$ a $\mathbf{k}$-rational point, and $Z \subset \mathcal{X}(s)$ a closed subscheme of the fibre $\mathcal{X}(s)$. Denote by $\Lambda = \mathcal{O}_{S,s}$. Then*
(i) the local relative Hilbert functor $H_Z^{\mathcal{X}/S} : \mathcal{A}_\Lambda \to$ (sets) is prorepresentable and has tangent space $H^0(Z, N_{Z/\mathcal{X}(s)})$.
(ii) If Z is regularly embedded in $\mathcal{X}(s)$ and p is flat then $H^1(Z, N_{Z/\mathcal{X}(s)})$ is an obstruction space for $H_Z^{\mathcal{X}/S}$, and we have an exact sequence:

$$[III.3.9] \qquad 0 \to H^0(Z, N_{Z/\mathcal{X}(s)}) \to t_R \to T_sS \to H^1(Z, N_{Z/\mathcal{X}(s)})$$

where R is the local Λ-algebra prorepresenting $H_Z^{\mathcal{X}/S}$.

Proof

(i) The proof of (III.3.3) can be followed almost verbatim showing that $H_Z^{\mathcal{X}/S}$ satisfies conditions H_0, H_ϵ, $\bar{H}$, H and that $H^0(Z, N_{Z/\mathcal{X}(s)})$ is its tangent space.

(ii) The proof of (III.3.6) can be easily adapted to this case. The exact sequence [III.3.9] follows from the above and from [I.3.2]. *q.e.d.*

(III.3.12) DEFINITION *In the above situation $Z \subset X$ is called* stable with respect to the family p *if the morphism $H_Z^{\mathcal{X}/S} \to h_\Lambda$ is smooth. Z is* stable in X *if it is stable with respect to every flat projective family $p : \mathcal{X} \to S$ of deformations of X, with $\mathcal{X}$ and S algebraic.*

This Definition generalizes a notion introduced and studied in Kodaira(1963) for a compact complex submanifold of a complex manifold. Stability implies that Z extends to every local deformation of X or, as stated in Kodaira(1963), that "no local deformation of X makes Z disappear". With this terminology Theorem (III.3.11) implies the following:

(III.3.13) COROLLARY *Let $Z \subset X$ be a regular embedding of projective schemes. If $H^1(Z, N_{Z/X}) = (0)$ then Z is stable in X.*

(III.3.14) EXAMPLES (i) (Kodaira(1963), Th. 5) Let Y be a projective nonsingular variety, $\gamma \subset Y$ a nonsingular closed subvariety and $\pi : X \to Y$ the blow-up of Y with center γ. Let $E = \pi^{-1}(\gamma) \subset X$ be the exceptional divisor; then $E \cong \mathbb{P}(N_{\gamma/Y})$ is a projective bundle over γ: let $q : E \to \gamma$ be the structure morphism. Then $N_{E/X} = \mathcal{O}_E(E)$ and it is well known that the restriction of $N_{E/X}$ to each fibre $\mathbb{P}$ of q is $\mathcal{O}_{\mathbb{P}}(-1)$. Therefore by the Leray spectral sequence of q we immediately deduce that

$$h^i(E, N_{E/X}) = 0$$

for all i. From Corollary (III.3.13) we obtain that E is a stable subvariety of X.

(ii) Let X be a projective nonsingular algebraic surface and $Z \subset X$ an irreducible nonsingular rational curve with self intersection $\nu = Z^2$. Then Z is stable in X if $\nu \geq -1$ because $H^1(Z, N_{Z/X}) = 0$ in this case. On the other hand if $\nu \leq -2$ then in general Z is not stable in X. An example is provided by the negative section E in the rational ruled surface F_m, for $m \geq 2$. In fact $E^2 = -m$ and we have seen in Example (II.1.1)(ii) that there is a family $f : \mathcal{W} \to \mathbf{A}^1$ of deformations of F_m for which $[E]$ is an isolated point of $\mathrm{Hilb}^{\mathcal{W}/\mathbf{A}^1}$ because E does not extend to the other fibres $\mathcal{W}(t)$, $t \neq 0$, since they are isomorphic to F_n for some $0 \leq n < m$. This shows that $\pi : \mathrm{Hilb}^{\mathcal{W}/\mathbf{A}^1} \to \mathbf{A}^1$ is not smooth at $[E]$, i.e. E is not stable w.r. to f.

Another generalization can be obtained with no extra effort. Consider a projective scheme X and a formal deformation of X

$$\bar{\pi} : \mathfrak{X} \to \mathrm{Specf}(R)$$

where R is in $\hat{\mathcal{A}}$ and $\bar{\pi}$ is a flat projective morphism of formal schemes; let $Z \subset X$ be a closed subscheme. For each A in ob$(\mathcal{A})$ define an *infinitesimal deformation of Z in $\mathcal{X}$ relative to $\bar{\pi}$ parametrized by A* as a commutative diagram:

$$\begin{array}{ccccc} \mathcal{Z} & \subset & \mathcal{X}_A & \to & \mathcal{X} \\ & \searrow & \downarrow & & \downarrow \bar{\pi} \\ & & \mathrm{Spec}(A) & \xrightarrow{s} & S \end{array}$$

where the right square is cartesian, the left diagonal morphism is flat and its closed fibre is Z. Note that Spec(A) = Specf(A) and the morphism s is defined by a surjective homomorphism $R \to A$, so that $\mathcal{X}_A$ is just an ordinary scheme projective and flat over Spec(A). We can define the *local relative Hilbert functor*

$$H_Z^{\mathcal{X}/\mathrm{Specf}(R)} : \mathcal{A} \to (\mathrm{sets})$$

as above. A result analogous to (III.3.11) can be proved in this case as well with a similar proof. Details of this straightforward generalization are left to the reader.

Note that the functor $H_Z^{\mathcal{X}/\mathrm{Specf}(R)}$ comes equipped with a structural morphism

$$H_Z^{\mathcal{X}/\mathrm{Specf}(R)} \to h_R$$

* * * * * *

Algebraic surfaces

In this subsection we will assume char$(\mathbf{k}) = 0$. We will denote by S a projective nonsingular connected algebraic surface. Let $(R, \hat{u})$ be a semiuniversal deformation of S, and denote by

$$\mu(S) := \dim(R)$$

the number of moduli of S.

(III.3.15) PROPOSITION

[*III*.3.10] $$10(p_a + 1) - 2(K^2) + h^0(S, T_S) \leq \mu(S) \leq h^1(S, T_S)$$

where $p_a = p_a(S) := \chi(\mathcal{O}_S) - 1$ is the arithmetic genus of S. If $h^2(S, T_S) = 0$ then both inequalities are equalities.

Proof
A direct application of the Riemann-Roch formula gives

$$h^1(S, T_S) - h^2(S, T_S) = 10(p_a + 1) - 2(K^2) + h^0(S, T_S)$$

By applying Corollary (III.3.7) we obtain the conclusion. *q.e.d.*

The first inequality was proved by Enriques (see Enriques(1949)).

(III.3.16) EXAMPLES (i) If S is a minimal ruled surface and $S \to C$ is the ruling over a projective nonsingular curve C, then letting $S(x) \cong \mathbb{P}^1$ be the fibre of any $x \in C$ we have

$$h^1(S(x), T_{S|S(x)}) = 0$$

as an immediate consequence of the exact sequence

$$0 \to T_{S(x)} \to T_{S|S(x)} \to N_{S(x)/S} \to 0, \qquad N_{S(x)/S} = \mathcal{O}_{S(x)}$$

Therefore $R^1 p_* T_S = 0$ by Corollary (IV.2.6) and the Leray spectral sequence implies that $H^2(S, T_S) = 0$. Therefore S is unobstructed and $\mu(S) = h^1(S, T_S)$. This computation for the rational ruled surfaces F_m is also done in example (A.1.10)(iii).

(ii) Assume that S is a K3-surface. Then

$$h^2(S, T_S) = h^0(S, \Omega^1_S) = h^1(S, \mathcal{O}_S) = 0$$

Therefore S is unobstructed. Moreover

$$h^0(S, T_S) = h^2(S, \Omega^1_S) = h^1(S, \omega_S) = 0$$

and Def_S is prorepresentable (Corollary (III.4.3)). Formula [III.3.10] gives in this case

$$\mu(S) = h^1(S, T_S) = 20$$

(iii) Let $\pi : X \to S$ be the blow-up of S at a point s and $E = \pi^{-1}(s)$ the exceptional curve. Then we have an exact sequence

$$0 \to T_X \to \pi^* T_S \to N_\pi \to 0$$

Since $N_\pi = \mathcal{O}_E(1)$ (use the exact sequence [III.6.13]) we see that

$$h^2(X, T_X) = h^2(S, T_S)$$

This implies for example that non-minimal rational or ruled surfaces and blow-ups of K3-surfaces are unobstructed.

(iv) When κ-dim$(S) \geq 1$ then in general $h^2(S, T_S) = h^0(S, \Omega^1 \otimes \omega_S) \neq 0$ and infact such surfaces can be obstructed. We will give some examples below. If we assume that $h^0(S, T_S) = 0$ the estimate for $\mu(S)$ given by Proposition (III.3.15) becomes

$$10(p_a + 1) - 2(K^2) \leq \mu(S) \leq h^1(S, T_S) = 10(p_a + 1) - 2(K^2) + h^0(S, \Omega^1 \otimes \omega_S)$$

and to give an upper bound for $\mu(S)$ amounts to giving one for $h^0(S, \Omega^1 \otimes \omega_S)$ in terms of p_a, K^2, q. We refer the reader to Catanese(1988) for a more detailed discussion of this point.

(v) If $h^0(S,\omega_S) > 0$ and $q > 0$ then certainly

$$h^0(S,\Omega^1 \otimes \omega_S) > 0$$

This is because there is a bilinear pairing

$$H^0(S,\Omega^1_S) \times H^0(S,\omega_S) \to H^0(S,\Omega^1 \otimes \omega_S)$$

which is non-degenerate on each factor. For example, if S is an abelian surface then

$$h^0(S,\Omega^1 \otimes \omega_S) = 2 = h^0(S,T_S)$$

Formula [III.3.10] gives

$$2 \leq \mu(S) \leq h^1(S,T_S) = 4$$

In fact the second equality holds because abelian surfaces are unobstructed. This is a property common to all abelian varieties and can be proved along the lines of Proposition (III.4.4) (we refer the reader to Oort(1972)).

(vi) One should keep in mind that $\mu(S)$ is defined as the number of moduli of S in a formal sense. This is because the semiuniversal deformation $(R,\hat{u})$ can be non-algebraizable. For example $\mu(S) = 20$ for a K3-surface, but every algebraic family of K3-surfaces has dimension ≤ 19. Similarly an abelian surface has $\mu(S) = 4$ but every algebraic family of abelian surfaces has dimension ≤ 3. In order to give an algebraic meaning to the number of moduli one should count the maximum dimension of a semiuniversal deformation of a pair (S,L) where L is an ample invertible sheaf on S. See the appendix by Mumford to Chapter V of Zariski(1971).

NOTES

1. Let X be a reduced scheme and let $\xi : \mathcal{X} \to \mathrm{Spec}(\mathbf{k}[\epsilon])$ be a first order deformation of X. Then the conormal sequence of $X \subset \mathcal{X}$

$$[III.3.11] \qquad 0 \to \mathcal{O}_X \to \Omega^1_{\mathcal{X}|X} \to \Omega^1_X \to 0$$

is exact and defines the element of $\mathrm{Ext}^1_{\mathcal{O}_X}(\Omega^1_X,\mathcal{O}_X)$ which corresponds to ξ in the identification $\mathrm{Def}_X(\mathbf{k}[\epsilon]) = \mathrm{Ext}^1_{\mathcal{O}_X}(\Omega^1_X,\mathcal{O}_X)$ of Proposition (III.3.1)(v) (see also Theorem (I.4.3)).

Given an infinitesimal deformation $\xi : \mathcal{X} \to \mathrm{Spec}(A)$ of X we have a *Kodaira-Spencer map* $\kappa_\xi : t_A \to \mathrm{Ext}^1_{\mathcal{O}_X}(\Omega^1_X,\mathcal{O}_X)$ which associates to a tangent vector $\theta \in t_A$ the conormal sequence of the pullback of ξ to $\mathrm{Spec}(\mathbf{k}[\epsilon])$ defined by θ.

2. Let $X \subset Y$ be a regular embedding of algebraic schemes with X reduced and Y nonsingular, $\mathcal{I} \subset \mathcal{O}_Y$ the ideal sheaf of X, and let $\Phi : H^Y_X \to \mathrm{Def}_X$ be the forgetful morphism. The differential

$$d\Phi : H^0(X,N_{X/Y}) = \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{I}/\mathcal{I}^2,\mathcal{O}_X) \to \mathrm{Ext}^1_{\mathcal{O}_X}(\Omega^1_X,\mathcal{O}_X)$$

is the $\mathbf{k}$-linear map which associates to $\sigma : \mathcal{I}/\mathcal{I}^2 \to \mathcal{O}_X$ the pushout $\sigma_*(S)$ where

$$S : 0 \to \mathcal{I}/\mathcal{I}^2 \to \Omega^1_{Y|X} \to \Omega^1_X \to 0$$

is the conormal sequence of $X \subset Y$. This generalizes Proposition (III.3.9). The proof consists in considering, for a first order deformation of X in Y

$$X \subset \mathcal{X} \subset Y \times \mathrm{Spec}(\mathbf{k}[\epsilon])$$

the induced diagram of conormal sequences

$$\begin{array}{ccccccc} 0 \to & \mathcal{I}/\mathcal{I}^2 \oplus \mathcal{O}_X & \to & \Omega^1_{Y|X} \oplus \mathcal{O}_X & \to & \Omega^1_X & \to 0 \\ & \downarrow & & \downarrow & & \| & \\ 0 \to & \mathcal{O}_X & \to & \Omega^1_{\mathcal{X}|X} & \to & \Omega^1_X & \to 0 \end{array}$$

and in recognizing the second row as the pushout of the first.

3. Consider a reduced hypersurface $X \subset \mathbb{P}^r$, $r \geq 3$, of degree $d \geq 2$. Then the conormal sequence is

$$0 \to \mathcal{O}_X(-d) \to \Omega^1_{\mathbb{P}^r|X} \to \Omega^1_X \to 0$$

so that $H^1(X, N_{X/\mathbb{P}^r}) = 0$ and we have the exact sequence

$$\begin{array}{ccc} H^0(X, N_{X/\mathbb{P}^r}) & \xrightarrow{d\Phi} \mathrm{Ext}^1_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_X) \to & \mathrm{Ext}^1(\Omega^1_{\mathbb{P}^r|X}, \mathcal{O}_X) \\ \| & & \| \\ H^0(X, \mathcal{O}_X(d)) & & H^1(X, T_{\mathbb{P}^r|X}) \end{array}$$

where the equality on the right is because $\Omega^1_{\mathbb{P}^r|X}$ is locally free. Therefore as in example (III.3.10)(i) we see that $\Phi : H^Y_X \to \mathrm{Def}_X$ is smooth and X is unobstructed if $(r, d) \neq (3, 4)$.

III.4. AUTOMORPHISMS AND PROREPRESENTABILITY

The following Theorem gives a criterion on an algebraic scheme X to decide whether Def_X, resp. Def'_X, has a universal element and not merely a semiuniversal one.

(III.4.1) THEOREM *Assume that X is an algebraic scheme such that* Def_X *has a semiuniversal element (e.g. X affine with isolated singularities or X projective). Then the following conditions are equivalent:*
(i) Def_X *is prorepresentable*
(ii) for each small extension $A' \to A$ in $\mathcal{A}$, and for each deformation $\mathcal{X}'$ of X over $\mathrm{Spec}(A')$*, every automorphism of the deformation* $\mathcal{X}' \times_{\mathrm{Spec}(A')} \mathrm{Spec}(A)$ *is induced by an automorphism of $\mathcal{X}'$.*

A similar statement holds for the functor Def'_X.

Proof
(i) $\Rightarrow$ (ii) Let $\mathcal{X} = \mathcal{X}' \times_{\mathrm{Spec}(A')} \mathrm{Spec}(A)$ and let $f : \mathcal{X} \to \mathcal{X}'$ be the induced morphism; assume that θ is an automorphism of $\mathcal{X}$. Letting $\bar{A} = A' \times_A A'$, one can construct two deformations $\mathcal{Z}$ and $\mathcal{W}$ of X over $\bar{A}$ as we did in the proof of Proposition (III.3.1) as fibered sums fitting into the two diagrams:

$$\begin{array}{ccccc} & & \mathcal{Z} & & \\ & \nearrow & & \nwarrow & \\ \mathcal{X}' & & & & \mathcal{X}' \\ & \nwarrow f\theta & & \nearrow f & \\ & & \mathcal{X} & & \end{array} \qquad \begin{array}{ccccc} & & \mathcal{W} & & \\ & \nearrow & & \nwarrow & \\ \mathcal{X}' & & & & \mathcal{X}' \\ & \nwarrow f & & \nearrow f & \\ & & \mathcal{X} & & \end{array}$$

Since $[\mathcal{Z}], [\mathcal{W}] \in \mathrm{Def}_X(\bar{A})$ have the same image $([\mathcal{X}'], [\mathcal{X}'])$ under the map

$$\mathrm{Def}_X(\bar{A}) \to \mathrm{Def}_X(A') \times_{\mathrm{Def}_X(A)} \mathrm{Def}_X(A')$$

and since this map is bijective by (i), we have an isomorphism of deformations $\rho : \mathcal{Z} \cong \mathcal{W}$. The isomorphism ρ induces automorphisms θ_1 and θ_2 of $\mathcal{X}'$ and an automorphism ψ of $\mathcal{X}$ such that:

$$\theta_1 f \theta = f \psi, \quad \theta_2 f = f \psi$$

This equality implies $f\theta = \theta_1^{-1}\theta_2 f$:

$$\begin{array}{ccc} \mathcal{X}' & \xrightarrow{\theta_1^{-1}\theta_2} & \mathcal{X}' \\ \uparrow f & & \uparrow f \\ \mathcal{X} & \xrightarrow{\theta} & \mathcal{X} \end{array}$$

and therefore $\theta_1^{-1}\theta_2$ induces θ.

(ii) $\Rightarrow$ (i) Since Def_X has a semiuniversal element, it suffices to show that it satisfies condition H of Theorem (III.2.2). Let $A' \to A$ be a small extension in $\mathcal{A}$: letting $\bar{A} = A' \times_A A'$ we must show that the map

$$\alpha : \mathrm{Def}_X(\bar{A}) \to \mathrm{Def}_X(A') \times_{\mathrm{Def}_X(A)} \mathrm{Def}_X(A')$$

is bijective. Given deformations $\mathcal{X}'$ and $\tilde{\mathcal{X}}'$ of X over A' inducing the deformation $\mathcal{X}$ over A, we have the "fibered sum" deformation $\bar{\mathcal{X}}$ over $\bar{A}$, which fits into the diagram:

$$\begin{array}{ccccc} & & \bar{\mathcal{X}} & & \\ & \nearrow & & \nwarrow & \\ \mathcal{X}' & & & & \tilde{\mathcal{X}}' \\ & \nwarrow f & & \nearrow \tilde{f} & \\ & & \mathcal{X} & & \end{array}$$

and satisfies $\alpha([\bar{\mathcal{X}}]) = ([\mathcal{X}'], [\mathcal{X}'])$. Suppose that $\mathcal{Z}$ is another deformation of X over $\bar{A}$ such that $\alpha([\mathcal{Z}]) = ([\mathcal{X}'], [\tilde{\mathcal{X}}'])$. We have isomorphisms of deformations induced by the two projections:

$$\mathcal{X}' \cong \mathcal{Z} \times_{\mathrm{Spec}(\bar{A})} \mathrm{Spec}(A') \cong \tilde{\mathcal{X}}'$$

There remains induced an automorphism θ of $\mathcal{X}$ as the composition:

$$\mathcal{X} \cong \mathcal{X}' \times_{\mathrm{Spec}(A')} \mathrm{Spec}(A) \cong \mathcal{Z} \times_{\mathrm{Spec}(\bar{A})} \mathrm{Spec}(A) \cong \tilde{\mathcal{X}}' \times_{\mathrm{Spec}(A')} \mathrm{Spec}(A) \cong \mathcal{X}$$

and θ fits into the commutative diagram:

$$\begin{array}{ccccccc} & & & \mathcal{Z} & & & \\ & \nearrow & & & & \nwarrow & \\ \mathcal{X}' & & & & & & \tilde{\mathcal{X}}' \\ & \nwarrow f & & & & \nearrow \tilde{f} & \\ & & \mathcal{X} & \xrightarrow{\theta} & \mathcal{X} & & \end{array}$$

By (ii) we can lift θ to an automorphism $\sigma : \mathcal{X}' \cong \mathcal{X}'$. Replacing the lower left map f by σf we obtain the commutative diagram

$$\begin{array}{ccccc} & & \mathcal{Z} & & \\ & \nearrow & & \nwarrow & \\ \mathcal{X}' & & & & \tilde{\mathcal{X}}' \\ & \nwarrow \sigma f & & \nearrow \tilde{f} & \\ & & \mathcal{X} & & \end{array}$$

By the universal property of the fibered sum we obtain an isomorphism $\bar{\mathcal{X}} \cong \mathcal{Z}$ which is an isomorphism of deformations. Therefore $[\mathcal{Z}] = [\bar{\mathcal{X}}]$ and α is bijective.

In the case of Def'_X the proof is similar. *q.e.d.*

When X is a projective scheme condition (ii) of Theorem (III.4.1) can be stated in a different way by means of the *automorphism functor*, which we now introduce.

Assume that X is an algebraic scheme such that Def_X has a semiuniversal couple $(R, \hat{u})$. Consider the functor of Artin rings

$$\begin{array}{rcl} \mathrm{Aut}_{\hat{u}} : \mathcal{A}_R & \to & (\text{ sets}) \\ \mathrm{Aut}_{\hat{u}}(A) & = & \text{the group of automorphisms of the deformation } \mathcal{X}_A \end{array}$$

where $\mathcal{X}_A$ is the deformation induced by $\hat{u}$ under the morphism $R \to A$. Then we have the following

(III.4.2) PROPOSITION *If X is projective then* $\mathrm{Aut}_{\hat{u}}$ *is prorepresented by a complete local R-algebra S. Moreover the deformation functor* Def_X *is prorepresentable if and only if S is a formally smooth R-algebra, i.e. if it is a power series ring over R.*

Proof

Obviously $\mathrm{Aut}_{\hat{u}}$ satisfies condition H_0 because by definition the only automorphism of the deformation $\mathcal{X}_{\mathbf{k}} = X$ is the identity. Now consider a diagram in $\mathcal{A}_R$:

$$\begin{array}{ccc} A' & & A'' \\ & \searrow\swarrow & \\ & A & \end{array}$$

with $A'' \to A$ a small extension and let $\bar{A} = A' \times_A A''$. There is induced a diagram of deformations:

$$\begin{array}{ccc} & \mathcal{X}_{\bar{A}} & \\ & \nearrow\nwarrow & \\ \mathcal{X}_{A'} & & \mathcal{X}_{A''} \\ & \nwarrow\nearrow & \\ & \mathcal{X}_A & \end{array}$$

and therefore a natural homomorphism:

[III.4.1] $$\mathcal{O}_{X_{\bar{A}}} \to \mathcal{O}_{\mathcal{X}_{A'}} \times_{\mathcal{O}_{\mathcal{X}_A}} \mathcal{O}_{\mathcal{X}_{A''}}$$

which is compatible with the structure of deformations. Since $\mathcal{O}_{\mathcal{X}_{A'}} = \mathcal{O}_{\mathcal{X}_{\bar{A}}} \otimes_{\bar{A}} A'$ it follows from Lemma (A.2.3) that [III.4.1] is an isomorphism; in particular we obtain an induced isomorphism

$$\mathcal{O}^*_{\mathcal{X}_{\bar{A}}} \cong \mathcal{O}^*_{\mathcal{X}_{A'}} \times_{\mathcal{O}^*_{\mathcal{X}_A}} \mathcal{O}^*_{\mathcal{X}_{A''}}$$

and therefore

[III.4.2] $$H^0(\mathcal{O}^*_{\mathcal{X}_{\bar{A}}}) \cong H^0(\mathcal{O}^*_{\mathcal{X}_{A'}}) \times_{H^0(\mathcal{O}^*_{\mathcal{X}_A})} H^0(\mathcal{O}^*_{\mathcal{X}_{A''}})$$

Now note that for every A in $\mathcal{A}_R$ the elements of $\mathrm{Aut}_{\hat{u}}(A)$ are identified with the subgroup of $H^0(\mathcal{O}^*_{\mathcal{X}_A})$ consisting of those elements which restrict to $1 \in \mathcal{O}^*_X$. Hence, considering that [III.4.1] is compatible with the structure of deformations we see that [III.4.2] immediately implies the bijection

$$\mathrm{Aut}_{\hat{u}}(\bar{A}) \cong \mathrm{Aut}_{\hat{u}}(A') \times_{\mathrm{Aut}_{\hat{u}}(A)} \mathrm{Aut}_{\hat{u}}(A'')$$

Therefore the functor $\mathrm{Aut}_{\hat{u}}$ also satisfies conditions $H)$ and $H_\epsilon)$.

From Lemma (II.1.5) it follows that

$$[III.4.3] \qquad \mathrm{Aut}_{\hat{u}}(\mathbf{k}[\epsilon]) \cong H^0(X, T_X)$$

which has finite dimension since X is projective, and also $H_f)$ is satisfied. This concludes the proof of the first assertion.

Condition (ii) of Theorem (III.4.1) can be rephrased by saying that the functor $\mathrm{Aut}_{\hat{u}}$ is smooth over Def_X. The conclusion follows. *q.e.d.*

An important application of the above Proposition is the following result, which is the scheme-theoretic version of a classical Theorem due to Kodaira-Nirenberg-Spencer:

(III.4.3) COROLLARY *If X is a projective scheme such that $h^0(X, T_X) = 0$ then Def_X is prorepresentable. If moreover X is nonsingular and $h^2(X, T_X) = 0$ then Def_X is prorepresented by a formal power series ring.*

Proof

From [III.4.3] it follows that $S = R$ if $H^0(X, T_X) = (0)$ and in particular S is a formally smooth R-algebra. Then the first part follows from Proposition (III.4.2). The last assertion is a consequence of Corollary (III.3.7). *q.e.d.*

The condition $H^0(X, T_X) = 0$ means that $\mathrm{Aut}(X)$ is finite and reduced. In this case, as we saw in the proof of (III.4.3), it follows from Proposition (III.4.2) that $\mathrm{Aut}_{\hat{u}} \cong \mathrm{Def}_X$; in other words every infinitesimal deformation of X has no non-trivial automorphisms, and in particular Def_X is prorepresentable by Theorem (III.4.1). We thus see that the existence of a finite and reduced automorphism group does not prevent Def_X from being prorepresentable. On the other hand the existence of automorphisms whatsoever is a source of difficulties when one considers *local* deformations (see §III.5).

Note also that the condition $H^0(X, T_X) = 0$ is not necessary for the prorepresentability of Def_X. An example is given by $X = \mathbb{P}^r$, $r \geq 1$: in this case Def_X is trivially prorepresentable because X is rigid, but $h^0(X, T_X) = (r+1)^2 - 1 > 0$. For another example see the following (III.4.4).

Corollary (III.4.3) can be generalized in a straightforward way to conclude that any functor of Artin rings F classifying isomorphism classes of deformations of a scheme with some additional structure or of any other algebro-geometric object Ξ (a morphism, etc.) is prorepresentable provided F has a semiuniversal element and Ξ has a finite and reduced automorphism group. As an application of this remark we have the following

(III.4.4) PROPOSITION *Let X be a projective irreducible and nonsingular curve of genus 1. Then* Def_X *is prorepresentable.*

Proof

Fix a closed point $p \in X$. For each A in $\mathcal{A}$ we define a deformation of the pointed curve (X,p) to be a pair (ξ, σ) where

$$\xi : \begin{array}{ccc} X & \to & \mathcal{X} \\ \downarrow & & \downarrow \pi \\ \mathrm{Spec}(\mathbf{k}) & \to & \mathrm{Spec}(A) \end{array}$$

is an infinitesimal deformation of X over A and $\sigma : \mathrm{Spec}(A) \to \mathcal{X}$ is a section of π such that $\mathrm{Im}(\sigma) = \{p\}$. We have an obvious definition of isomorphism of two deformations of (X,p) over A, and we define a functor of Artin rings

$$\mathrm{Def}_{(X,p)} \to (\mathrm{sets})$$

by

$$\mathrm{Def}_{(X,p)}(A) = \{\text{deformations of } (X,p) \text{ over } A\}/\text{isomorphism}$$

We have a morphism of functors:

$$\phi : \mathrm{Def}_{(X,p)} \to \mathrm{Def}_X$$

induced by the correspondence

$$(\xi, \sigma) \mapsto \xi$$

which forgets the section σ. The Proposition is a consequence of the following two facts:

a) ϕ is an isomorphism of functors.

b) $\mathrm{Def}_{(X,p)}$ is prorepresentable.

To prove a) let $A \in \mathrm{ob}(\mathcal{A})$ and consider an infinitesimal deformation ξ of X over A. The point p defines a morphism $\mathrm{Spec}(\mathbf{k}) \to \mathcal{X}$ making the following diagram commutative:

$$\begin{array}{ccc} & & \mathcal{X} \\ & p \nearrow & \downarrow \pi \\ \mathrm{Spec}(\mathbf{k}) & \to & \mathrm{Spec}(A) \end{array}$$

By the smoothness of $\mathcal{X}$ over $\mathrm{Spec}(A)$ there is an extension of p to a section $\sigma : \mathrm{Spec}(A) \to \mathcal{X}$ of π: this proves that $\mathrm{Def}_{(X,p)}(A) \to \mathrm{Def}_X(A)$ is surjective. Now let (ξ, σ) and (η, τ) be two deformations with section of X over A, where

$$\eta : \begin{array}{ccc} X & \to & \mathcal{Y} \\ \downarrow & & \downarrow q \\ \mathrm{Spec}(\mathbf{k}) & \to & \mathrm{Spec}(A) \end{array}$$

and suppose that there is an isomorphism of deformations

$$\begin{array}{ccccc} & & X & & \\ & \swarrow & & \searrow & \\ \mathcal{X} & & \stackrel{\psi}{\longrightarrow} & & \mathcal{Y} \\ & \searrow & & \swarrow & \\ & & \mathrm{Spec}(A) & & \end{array}$$

Then $\psi\sigma, \tau : \mathrm{Spec}(A) \to \mathcal{Y}$ are two sections of q. We now use that fact that $\mathcal{Y}$ has a structure of group scheme over Spec(A) with identity τ (in outline this can be seen as follows: X is a group scheme with identity p and the group structure is given by a multiplication morphism $\mu : X \times X \to X$; the group operation on $\mathcal{Y}$ is defined by a morphism $\mu_A : \mathcal{Y} \times_A \mathcal{Y} \to \mathcal{Y}$ which extends μ and which exists because we have a commutative diagram:

$$\begin{array}{ccccc} X \times X & \stackrel{\mu}{\longrightarrow} & X & \subset & \mathcal{Y} \\ \cap & & & & \downarrow \\ \mathcal{Y} \times_A \mathcal{Y} & & \to & & \mathrm{Spec}(A) \end{array}$$

and $\mathcal{Y}$ is smooth over Spec(A)). Replacing ψ by by $\psi' = \psi(\psi\sigma)^{-1}$ we obtain an isomorphism of deformations ψ' such that $\psi'\sigma = \tau$ and therefore ψ' defines an isomorphism of (ξ, σ) with (η, τ): this proves that $\mathrm{Def}_{(X,p)}(A) \to \mathrm{Def}_X(A)$ is injective as well, and a) is proved.

In particular it follows that $\mathrm{Def}_{(X,p)}$ has a semiuniversal element because Def_X does. Now observe that the vector space of automorphisms of the trivial deformation of (X,p) can be identified with the vector subspace of $H^0(X, T_X) = H^0(X, \mathcal{O}_X)$ consisting of the derivations $D : \mathcal{O}_X \to \mathcal{O}_X$ vanishing at p, and this is equal to $H^0(X, \mathcal{O}_X(-p)) = (0)$. Now the remark following Corollary (III.4.3) applies to conclude that $\mathrm{Def}_{(X,p)}$ is prorepresentable, i.e. b) holds. *q.e.d.*

The following Corollary computes in particular the number of moduli of projective nonsingular curves.

(III.4.5) COROLLARY *If X is a projective nonsingular connected curve of genus g then* Def_X *is prorepresentable. More precisely,* $\mathrm{Def}_X = h_R$ *where*

$$R = \begin{cases} \mathbf{k} & \textit{if } g = 0 \\ \mathbf{k}[[X]] & \textit{if } g = 1 \\ \mathbf{k}[[X_1, \ldots, X_{3g-3}]] & \textit{if } g \geq 2 \end{cases}$$

.

Proof

X is unobstructed by Proposition (III.3.6) and

$$h^1(X, T_X) = \begin{cases} 0 & \text{if } g = 0 \\ 1 & \text{if } g = 1 \\ 3g - 3 & \text{if } g \geq 2 \end{cases}$$

So it remains to show that Def_X is prorepresentable. In case $g = 0$ this is because $I\!P^1$ is rigid; in case $g \geq 2$ since $\deg(T_X) = 2 - 2g < 0$ we have $H^0(X, T_X) = 0$ and therefore (III.4.3) applies. If $g = 1$ we use (III.4.4). *q.e.d.*

(III.4.6) EXAMPLES

(i) (Schlessinger(1964)) Let $C = \mathrm{Spec}(B_0)$, where $B = \mathbf{k}[x,y]/(xy)$, be a reducible affine plane conic. Then Def_C is not prorepresentable although C has a semiuniversal deformation by Corollary (III.3.2). Infact consider the deformation of C over $\mathbf{k}[\epsilon]$ given by $xy + \epsilon = 0$ and its automorphism:

$$\begin{array}{ccc} x & \mapsto & x + x\epsilon \\ y & \mapsto & y \end{array}$$

This automorphism does not extend to an automorphism of $xy+\bar{t} = 0$ over $\mathbf{k}[t]/(t^3)$: if it did it would be of the form

$$\begin{array}{ccc} x & \mapsto & x + x\bar{t} + a\bar{t}^2 \\ y & \mapsto & y + b\bar{t}^2 \end{array}$$

for some $a, b \in \mathbf{k}[x,y]$. But this implies that $bx + ay = -1$ in $\mathbf{k}[x,y]$, which is impossible. From Theorem (III.4.1) we deduce that Def_C is not prorepresentable.

(ii) The condition of Corollary (III.4.3) is not satisfied by the surfaces F_m, $m \geq 0$ (see Example (A.1.10)(iii)). Since $h^1(T_{F_0}) = 0 = h^1(T_{F_1})$ we find that F_0 and F_1 are rigid; in particular Def_{F_0} and Def_{F_1} are prorepresentable. On the other hand when $m \geq 2$ Def_{F_m} is unobstructed (since $h^2(T_{F_m}) = 0$) and has a semiuniversal element but it is not prorepresentable. To see it we can argue as follows. For simplicity let's consider the case $m = 2$. By Note 1 of §A.1we can identify F_2 with the hypersurface Σ_2 of $I\!P^1 \times I\!P^2$ of equation

$$x^2v - y^2u = 0$$

where $(x, y; u, v, w)$ are bihomogeneous coordinates in $I\!P^1 \times I\!P^2$. The linear pencil

$$\mathcal{V} \subset I\!P^1 \times I\!P^2 \times \mathbf{A}^1$$

of equation:

$$x^2v - y^2u - txyw = 0$$

defines a flat family $\mathcal{V} \to \mathbf{A}^1$ such that $\mathcal{V}(0) = \Sigma_2$ and $\mathcal{V}(t) \cong \Sigma_0$ for all $t \neq 0$. In fact we have an isomorphism $\mathcal{V}\backslash\mathcal{V}(0) \to \mathcal{V}(1) \times \mathbf{A}^1\backslash\{0\}$ over $\mathbf{A}^1\backslash\{0\}$ sending

$$(x, y; u, v, w; t) \mapsto (x, y; u, v, tw; t);$$

on the other hand $\Sigma_0 \cong \mathcal{V}(1)$ by the isomorphism

$$(x, y; u, v, w) \mapsto (x, y; -x^2uw - xyw^2, xyu^2 + y^2vw, x^2u^2 + 2xyuw + y^2w^2)$$

($\mathcal{V}$ is essentially the family considered in Example (A.2.1)(iii) for $m = 2$). The pullback

$$\mathcal{V}_\epsilon = V(x^2v - y^2u - \epsilon xyw) \subset \mathbb{P}^1 \times \mathbb{P}^2 \times \mathrm{Spec}(\mathbf{k}[\epsilon])$$

has the automorphism defined by sending $w \mapsto w + \epsilon u$ and leaving all the other coordinates unchanged. We leave to the reader to check that this automorphism does not extend to the pullback of $\mathcal{V}$ over $\mathrm{Spec}(\mathbf{k}[t]/(t^3))$. From Theorem (III.4.1) we deduce that Def_{F_2} is not prorepresentable.

NOTES

1. The family $F : \mathcal{Y} \to \mathbf{A}^{m-1}$ of deformation of F_m considered in Example (II.1.1)(iii) can be identified with the family of deformations of Σ_m defined by the linear system:

$$\mathcal{V}^{(m)} \subset \mathbb{P}^1 \times \mathbb{P}^2 \times \mathbf{A}^{m-1}$$

of equation

$$x^m v - y^m u = \sum_{\nu=1}^{m-1} t_\nu x^\nu y^{m-\nu} w$$

The projection $\Pi : \mathcal{V}^{(m)} \to \mathbf{A}^{m-1}$ defines a proper smooth family such that $\mathcal{V}^{(m)}(0) = \Sigma_m$ and

$$\mathcal{V}^{(m)}((0, \ldots, t_k, \ldots, 0)) \cong \mathcal{V}^{(m)}((0, \ldots, 1, \ldots, 0)) \cong \Sigma_{m-2k}$$

for all $t_k \neq 0$ and $k = 1, \ldots, m-1$ (where $\Sigma_{-h} := \Sigma_h$, $h > 0$). The first isomorphism is given by

$$(x, y; u, v, w) \mapsto (x, y; u, v, t_k w)$$

and the inverse of the second isomorphism is

$$(x, y; u, v, w) \mapsto$$

$$\mapsto (x, y; -(x^m vw + x^k y^{m-k} w^2), x^k y^{m-k} uv + y^m vw, x^m uv + x^k y^{m-k} uw + x^{m-k} y^k vw + y^m w^2)$$

Moreover the Kodaira-Spencer map

$$\kappa_{\Pi,\underline{0}} : T_{\underline{0}} \mathbf{A}^{m-1} \to H^1(\Sigma_m, T_{\Sigma_m})$$

is an isomorphism.

2. Corollary (III.4.3) can be also proved directly without using Proposition (III.4.2). Just observe that if $H^0(X, T_X) = 0$ then using Lemma (II.1.5) one shows by induction that every infinitesimal deformation of X has no automorphisms.

III.5. FORMAL VERSUS ALGEBRAIC DEFORMATIONS

We have already mentioned (see Examples (II.1.4) and (II.1.12)(ii)) that infinitesimal deformations do not explain faithfully some of the phenomena which can occur when one considers deformations parametrized by algebraic schemes or by the spectrum of an arbitrary noetherian, or even e.f.t., local ring. In this Section we will make such statements precise and we will explain the main reasons why the functor $\mathrm{Def}_X : \mathcal{A}^* \to$ (sets) considered in the Introduction is not representable in general even when X is a projective scheme. We will start from a few definitions and some terminology.

Let X be a projective scheme and consider a flat family of deformations of X parametrized by an affine scheme $S = \mathrm{Spec}(B)$, with B in ($\mathbf{k}$-algebras)

$$\eta : \begin{array}{ccc} X & \xrightarrow{f} & \mathcal{X} \\ \downarrow & & \downarrow \pi \\ \mathrm{Spec}(\mathbf{k}) & \xrightarrow{s} & S \end{array}$$

namely a cartesian diagram with π projective and flat. The deformation η is called *algebraic* if B is a $\mathbf{k}$-algebra of finite type. If B is in $\mathcal{A}^*$ then η is called a *local deformation* of X. If B is in $\mathcal{A}$ we obtain an infinitesimal deformation of X which is simultaneously local and algebraic. We will identify the deformation η with the couple (S, η) or (B, η) and we will also denote it by (S, s, η) or (B, s, η).

Let (S, s, η) be a deformation of X. Let η_n be the infinitesimal deformation induced by pulling back η under the natural closed embedding

$$\mathrm{Spec}(\mathcal{O}_{S,s}/m^{n+1}) \to S$$

We have $\mathcal{O}_{S,s}/m^{n+1} = \hat{\mathcal{O}}_{S,s}/\hat{m}^{n+1}$ and therefore it follows that $(\hat{\mathcal{O}}_{S,s}, \{\eta_n\}_{n\geq 0})$ is a formal deformation of X. It will be called *the formal deformation defined by* (or *associated to*) η.

(S, s, η) is called *formally trivial* (resp. *formally locally trivial*) if the formal deformation defined by η is trivial (resp. locally trivial).

(III.5.1) DEFINITION *A deformation (S, s, η) of X is called* formally universal *(resp.* formally semiuniversal, formally versal*) if the formal deformation $(\hat{\mathcal{O}}_{S,s}, \{\eta_n\}_{n\geq 0})$ associated to η defines a universal (resp. semiuniversal, versal) formal element of* Def_X. *A formally versal algebraic deformation of X is also called* with general moduli.

A flat family $\pi : \mathcal{X} \to S$ *is called* formally universal *(resp.* formally semiuniversal, formally versal, with general moduli) at a **k**-rational point $s \in S$ *if*

$$\begin{array}{cccc} & \mathcal{X}(s) & \subset & \mathcal{X} \\ \eta : & \downarrow & & \downarrow \pi \\ & \mathrm{Spec}(\mathbf{k}) & \xrightarrow{s} & S \end{array}$$

is a formally universal, (resp. formally semiuniversal, formally versal, with general moduli) deformation of $\mathcal{X}(s)$.

The expression "general moduli" goes back to the classical geometers. Informally, it means that the family parametrizes all possible "sufficiently small" deformations of $\mathcal{X}(s)$; when the family π parametrizes varieties for which there is a moduli space, or just a moduli stack (whatever this means for the reader since we have not introduced these notions), π with general moduli means that the functorial morphism from S to the moduli space or stack is open at s. An expression like "consider a variety X with general moduli" is used to mean: "choose X as a fibre in a family with general moduli".

The early literature on deformation theory of complex analytic manifolds (in the approach of Kodaira and Spencer) considered only families parametrized by complex analytic manifolds. In that context the expression "effectively parametrized" was used to mean what we call here semiuniversal, and the word "complete" was used to mean versal, in the category of germs of complex analytic manifolds.

The following result is very useful in practise:

(III.5.2) PROPOSITION *Let* (S, s, η) *be a deformation of* X. *Then:*
(i) If η *is formally versal (resp. formally semiuniversal or formally universal) then the Kodaira-Spencer map*

$$\kappa_{\pi,s} : T_s S \to \mathrm{Def}_X(\mathbf{k}[\epsilon])$$

is surjective (resp. an isomorphism).
(ii) If S *is nonsingular at* s *and the Kodaira-Spencer map* $\kappa_{\pi,s}$ *is surjective (resp. an isomorphism) then* η *is formally versal (resp. formally semiuniversal) and* X *is unobstructed, i.e. the functor* Def_X *is smooth.*

Proof
(i) is obvious in view of the definitions of versality and semiuniversality of a formal couple applied to $h_{\hat{\mathcal{O}}_{S,s}} \to \mathrm{Def}_X$.
(ii) follows from Proposition (III.1.6)(iii) applied to $f : h_{\hat{\mathcal{O}}_{S,s}} \to \mathrm{Def}_X$. *q.e.d.*

The Proposition applies in particular to an algebraic deformation, giving a criterion for it to have general moduli. A classical result (see Kodaira-Spencer(1958b)) states the completeness of a complex analytic family of compact complex manifolds if the map $\kappa_{\pi,s}$ is surjective. Part (ii) of Proposition (III.5.2) is the algebraic version of this result. It turns out to be very useful because it reduces the verification of formal versality to the computation of the Kodaira-Spencer map.

(III.5.3) DEFINITION *A formal deformation* $(\bar{A}, \{\eta_n\})$ *of* X *is called* algebraizable *if there exists an algebraic deformation* (S, s, ξ) *of* X *such that there exists an isomorphism* $\hat{\mathcal{O}}_{S,s} \cong \bar{A}$ *sending* η_n *to* ξ_n *for all* n *(i.e.* $(\bar{A}, \{\eta_n\})$ *is isomorphic to the formal deformation defined by* ξ*). The deformation* (S, s, ξ) *is called an* algebraization *of* $(\bar{A}, \{\eta_n\})$.

An algebraization, if it exists, is obviously not unique, because replacing S by any etale neighborhood of s and pulling back ξ we obtain another algebraic deformation of X having the same associated formal deformation. It goes without saying that an algebraization of a formal versal (resp. semiuniversal, universal) deformation is formally versal (resp. formally semiuniversal, formally universal).

The existence of algebraizations is a highly non trivial problem. It can be considered as the counterpart of the convergence step in the construction of local families of deformations in the Kodaira-Spencer theory of deformations of compact complex manifolds. But the algebraic case presents some characteristic features which make the two theories radically different in methods and in results. The main results on this matter are due to M. Artin who gave criteria of algebraization which apply to very general functors (see Artin(1969a), Artin(1969b), Artin(1976)).

We will not discuss Artin's Algebraization Theorem here. Our concern will only be to derive the existence of formally universal algebraic deformations in specific geometrical situations. The following is a Theorem of such type which applies only in certain special cases, but it is sufficient for several applications.

(III.5.4) THEOREM *Let* X *be a projective scheme such that*

$$H^0(X, T_X) = (0)$$

and let $(\bar{A}, \{\eta_n\})$ *be a formal universal deformation of* X *(it exists because* Def_X *is prorepresentable by Corollary (III.4.3)). Assume that there is an algebraic deformation* (S, s, ξ) *of* X *having general moduli. Then* $(\bar{A}, \{\eta_n\})$ *is algebraizable.*

Proof

Since $\mathrm{Def}_X \cong h_{\bar{A}}$ and (S, s, ξ) has general moduli we have an isomorphism $\hat{\mathcal{O}}_{S,s} \cong \bar{A}[[T_1, \ldots, T_k]]$ for some $T_1, \ldots, T_k \in m_{S,s}$. Let

$$B = \mathcal{O}_{S,s}/(T_1, \ldots, T_k)$$

Then $\hat{B} = \bar{A}$. Now let $U \subset S$ be an open set containing s where $T_1, \ldots, T_k$ are regular and let $W = V(T_1, \ldots, T_k) \subset U$. Then the restriction of ξ to W has the required properties. *q.e.d.*

Note that we used $H^0(X, T_X) = (0)$ only to ensure the existence of $(\bar{A}, \{\eta_n\})$. The existence of the algebraic deformation (S, s, ξ) is a strong condition. In fact to give conditions which imply its existence is the difficult and delicate part of the algebraization theorem of M. Artin. On the other hand in several concrete cases the existence of (S, s, ξ) is easy to prove and Theorem (III.5.4) addresses those cases.

* * * * * *

The notions of triviality and of formal triviality of an algebraic deformation are related in a quite subtle way, as shown by example (II.1.12)(ii). Such example is a special case of an important phenomenon, called *isotriviality*.

(III.5.5) DEFINITION *Let* $\mathcal{X} \to S$ *be a flat family of schemes, and let* $s \in S$ *be a* **k***-rational point.* π *is called* isotrivial at s *if there is an etale neighborhood* $f : (S', s') \to (S, s)$ *such that the pullback* $\pi_{S'} : S' \times_S \mathcal{X} \to S'$ *of* π *is trivial.* π *is called* isotrivial *if it is isotrivial at every* **k***-rational point of* S.

Every trivial family is isotrivial; the family of example (II.1.12)(ii) is isotrivial (why?) but is not trivial.

(III.5.6) PROPOSITION *Let* $\pi : \mathcal{X} \to S$ *be a flat family of algebraic schemes, and let* $s \in S$ *be a closed point. If* π *is isotrivial at* s *then the formal deformation of* $\mathcal{X}(s)$ *associated to* π *is trivial.*

Proof
It follows immediately from the already remarked fact that an etale base change has no effect on associated formal deformations. *q.e.d.*

If a scheme X has an isotrivial local deformation η which is non-trivial then the local moduli functor

$$\mathrm{Def}_X : \mathcal{A}^* \to (\mathrm{sets})$$

considered in the introduction cannot be representable, i.e. the local deformation υ considered there cannot exist. Infact, since η is non-trivial it must be pulled back from υ by a non-constant morphism $g : \mathrm{Spec}(A) \to \mathrm{Spec}(\mathcal{O})$. On the other hand, since η is isotrivial its pullback to $\mathrm{Spec}(\tilde{A})$ is trivial and is therefore obtained by pulling back υ in two different ways: by the constant morphism and by the composition

$$\mathrm{Spec}(\tilde{A}) \xrightarrow{\varphi} \mathrm{Spec}(A) \xrightarrow{g} \mathcal{O}$$

which is non-constant because φ is faithfully flat hence surjective, and this contradicts the universality of υ.

These remarks explain why we cannot expect to be able to construct families representing functors defined on $\mathcal{A}^*$ or on (**k**-algebras) or on (schemes), which classify isomorphism classes of schemes having non-trivial isotrivial deformations; and the existence of such deformations is closely related to the existence of non-trivial automorphisms of such schemes. A similar remark applies to deformations of other objects, for example vector bundles, which have not been considered here. Here it is interesting to quote a letter of Grothendieck to Serre of november 1959 (see Colmez-Serre(2001), p. 94):

chaque fois que ... une variété de modules ... ne peut exister, malgré de bonnes hypothéses de platitude, propreté, et non singularité eventuellement, la raison en est seulement l'existence d'automorphismes de la structure.

This discussion suggests that while the consideration of isomorphism classes of deformations is not a drawback when one is studying infinitesimal deformations, it becomes inadequate for the classification of algebraic deformations and for global moduli problems. In other words, because of the presence of non trivial automorphisms we cannot in general expect to find a scheme structure on the set M of isomorphism classes of objects we want to classify in such a way that it reflects faithfully the functorial properties of families. For example, in the case of projective nonsingular curves of genus 0 one should have $M = \mathrm{Spec}(\mathbf{k})$ and the universal family should be $\mathbb{P}^1 \to \mathrm{Spec}(\mathbf{k})$ because there is only one isomorphism class of such curves; but the families $F_m \to \mathbb{P}^1$ (Example (A.1.10)(iii)) cannot be pulled back from it.

That's why it would be more natural, instead of taking isomorphism classes of deformations, to consider all families together and analize them and their isomorphisms. This will result in a more general structure, called a *stack*, which contains all the informations about families and deformations of the objects of the set M we want to classify.

NOTES

1. We will not say anything about stacks here. We refer to Deligne-Mumford(1969), Vistoli(1989) and Laumon-Moret Bailly(2000) for details.

2. The notion of isotriviality has been introduced for the first time in Serre(1958). The inverse implication of Proposition (III.5.6) is false in general: there are families which are formally trivial but not isotrivial, as shown by example (II.1.4). But if π is projective then the converse is also true. This follows from Grothendieck's fundamental Theorems and from Artin's algebraization Theorem.

III.6. MORPHISMS

In this Section we study deformations of a morphism between algebraic schemes. In the analytic case the corresponding theory has been developed in Horikawa(1973), Horikawa(1974), and Horikawa(1976). More recent related work is in Ran(1989).

We will consider deformations of a morphism which keep the target fixed. This case is already sufficient to cover many important cases. For the general case we refer the reader to the above quoted papers of Horikawa and Ran.

(III.6.1) DEFINITION *Let $g : X \to Y$ be a morphism of algebraic schemes. A commutative diagram*

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{G} & Y \times S \\ & \pi \searrow \quad \swarrow & \\ & S & \end{array} \qquad [III.6.1]$$

where $S = \mathrm{Spec}(A)$ with A in $\mathrm{ob}(\mathcal{A})$ (resp. in $\mathrm{ob}(\mathcal{A}^*)$), π is flat and the pullback of [III.6.1] over the closed point $o \in S$ is

$$\begin{array}{ccc} X & \xrightarrow{g} & Y \\ & \searrow \quad \swarrow & \\ & \mathrm{Spec}(\mathbf{k}) & \end{array}$$

is called an infinitesimal *(resp.* local) family of deformations of g with target Y parametrized by A *(shortly* a deformation of g over A with target Y*). If we replace S by a pointed scheme (S, o) we will call [III.6.1]* a family of deformations of g with target Y.

We have a well defined functor of Artin rings

$$\mathrm{Def}_{g/Y} : \mathcal{A} \to (\mathrm{sets})$$

$$\mathrm{Def}_{g/Y}(A) = \{\text{deformations of } g \text{ over } A \text{ with target } Y\}$$

which will be called *the functor of infinitesimal deformations of g with target Y.*

Note that this definition differs from Horikawa's and Ran's because they consider deformations modulo isomorphism.

If g is a closed embedding then $\mathrm{Def}_{g/Y} = H_X^Y$. There is an obvious morphism of functors of Artin rings (the "forgetful morphism")

$$\Phi : \mathrm{Def}_{g/Y} \to \mathrm{Def}_X$$

which, for each A in ob($\mathcal{A}$), associates to a family [III.6.1] the isomorphism class of the family

$$[III.6.2] \qquad \begin{array}{ccc} X & \to & \mathcal{X} \\ \downarrow & & \downarrow \\ \mathrm{Spec}(\mathbf{k}) & \to & \mathrm{Spec}(A) \end{array}$$

The main result about $\mathrm{Def}_{g/Y}$ is the following:

(III.6.2) THEOREM *Let $g : X \to Y$ be a morphism between projective algebraic schemes. Let $\bar{\pi} : \mathfrak{X} \to \mathrm{Specf}(R)$ be a semiuniversal formal deformation of X and $\Gamma_o \subset X \times Y$ the graph of g. Then there is a natural identification of functors:*

$$\mathrm{Def}_{g/Y} = H_{\Gamma_o}^{\mathfrak{X} \times Y/\mathrm{Specf}(R)}$$

In particular $\mathrm{Def}_{g/Y}$ is prorepresentable.

Proof

Given a deformation [III.6.1] of g with A in ob($\mathcal{A}$) consider the associated deformation [III.6.2] of X. By the formal semiuniversality of $\bar{\pi}$ there is an induced morphism $\mathrm{Spec}(A) \to \mathrm{Specf}(R)$ such that

$$\begin{array}{ccc} \mathcal{X} & = & \mathfrak{X}_A = \mathfrak{X} \times_{\mathrm{Specf}(R)} \mathrm{Spec}(A) \\ \downarrow & & \\ \mathrm{Spec}(A) & & \end{array}$$

and the graph

$$\Gamma \subset \mathcal{X} \times Y \times \mathrm{Spec}(A) = \mathfrak{X} \times Y \times_{\mathrm{Specf}(R)} \mathrm{Spec}(A)$$

of G is an element of $H_{\Gamma_o}^{\mathfrak{X} \times Y/\mathrm{Specf}(R)}(A)$. Conversely, given a morphism $\mathrm{Spec}(A) \to \mathrm{Specf}(R)$ for some A in ob($\mathcal{A}$), and an element

$$\Gamma \subset \mathfrak{X} \times Y \times_{\mathrm{Specf}(R)} \mathrm{Spec}(A) = \mathfrak{X}_A \times Y \times \mathrm{Spec}(A)$$

of $H_{\Gamma_o}^{\mathfrak{X} \times Y/\mathrm{Specf}(R)}(A)$, we obtain a deformation of g over A by the projection $G_A : \Gamma \to Y \times \mathrm{Spec}(A)$, because $\Gamma \cong \mathfrak{X}_A$ is a deformation of X, and Γ is identified with the graph of G_A. *q.e.d.*

When the functor $\mathrm{Def}_{g/Y}$ is smooth we call g *unobstructed*. The local properties of $\mathrm{Def}_{g/Y}$ can now be easily deduced.

(III.6.3) COROLLARY *Let $g : X \to Y$ be a morphism between projective algebraic schemes, with X reduced and Y nonsingular. Then there is an exact sequence:*

[III.6.3]

$$0 \to H^0(X, g^*T_Y) \to \mathrm{Def}_{g/Y}(\mathbf{k}[\epsilon]) \xrightarrow{d\Phi} \mathrm{Ext}^1_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_X) \xrightarrow{h} H^1(X, g^*T_Y)$$

and if $H^1(X, g^*T_Y) = (0)$ *the forgetful morphism* $\Phi : \mathrm{Def}_{g/Y} \to \mathrm{Def}_X$ *is smooth.*

Proof

As in the proof of Proposition (II.3.8) one shows that $\Gamma_o \subset X \times Y$ is a regular embedding and that $H^0(\Gamma_o, N_{\Gamma_o/X\times Y}) = H^0(X, g^*T_Y)$. Therefore, after identifying $\mathrm{Def}_{g/Y}$ with the relative local Hilbert functor $H_{\Gamma_o}^{\mathcal{X}\times Y/\mathrm{Specf}(R)}$ as we did in the proof of the Theorem, the exact sequence [III.6.3] is just the reformulation in this case of [III.3.5]; moreover if $H^1(X, g^*T_Y) = (0)$ then the structural morphism

$$H_{\Gamma_o}^{\mathcal{X}\times Y/\mathrm{Specf}(R)} \to h_R$$

is smooth and therefore the forgetful morphism, which equals the composition

$$\mathrm{Def}_{g/Y} = H_{\Gamma_o}^{\mathcal{X}\times Y/\mathrm{Specf}(R)} \to h_R \to \mathrm{Def}_X$$

is also smooth. *q.e.d.*

To a morphism $g : X \to Y$ as above there is associated an exact sequence of coherent sheaves on X

$$[III.6.4] \qquad 0 \to T_{X/Y} \to T_X \xrightarrow{dg} g^*T_Y \to N_g \to 0$$

which defines the sheaf N_g called the *normal sheaf of* g. The morphism g is called *nondegenerate* when $T_{X/Y} = 0$; if g is smooth then $N_g = 0$. In these two special cases the information we get is more precise. We start with the nondegenerate case.

(III.6.4) PROPOSITION *Let* $g : X \to Y$ *be a morphism between projective nonsingular algebraic schemes. Assume that* g *is nondegenerate. Then we have an exact sequence*

$$(0) \to H^0(X, T_X) \to \mathrm{Def}_{g/Y}(\mathbf{k}[\epsilon]) \xrightarrow{p} H^0(X, N_g) \to (0)$$

and $H^1(X, N_g)$ *is an obstruction space for* $\mathrm{Def}_{g/Y}$. *In particular if* $H^1(X, N_g) = (0)$ *then* g *is unobstructed.*
Moreover we have a commutative diagram
$[III.6.5]$

$$\begin{array}{ccccc}
(0) \to H^0((X, g^*T_Y) \to & \mathrm{Def}_{g/Y}(\mathbf{k}[\epsilon]) & \xrightarrow{d\Phi} & \mathrm{Def}_X(\mathbf{k}[\epsilon]) & \\
\downarrow & \downarrow p & & \| & \\
(0) \to \frac{H^0(X, g^*T_Y)}{H^0(X, T_X)} \to & H^0(X, N_g) & \xrightarrow{\kappa} & H^1(X, T_X) \to & \\
 & & & \to H^1(X, g^*T_Y) & \to H^1(X, N_g) \xrightarrow{\delta} H^2(X, T_X)
\end{array}$$

where the second row is the exact sequence induced by [III.6.4] and δ *induces the obstruction map of* $\Phi : \mathrm{Def}_{g/Y} \to \mathrm{Def}_X$.
If in particular $H^0(X, T_X) = (0)$ *then*

$$\mathrm{Def}_{g/Y}(\mathbf{k}[\epsilon]) = H^0(X, N_g)$$

Proof

Everything follows easily from Corollary (III.6.3) except for the assertion about obstructions. Let S in $\mathrm{ob}(\hat{\mathcal{A}})$ be the local ring which prorepresents $\mathrm{Def}_{g/Y}$ and R as in the statement of (III.6.2). We have a commutative diagram with exact rows:

$$\begin{array}{ccccccc} t_R & \to & o(S/R) & \to & o(S) & \to & o(R) \\ \| & & \cap & & \downarrow p_1 & & \cap \\ H^1(T_X) & \to & H^1(g^*T_Y) & \to & H^1(N_g) & \to & H^2(T_X) \end{array}$$

The first row is [I.3.2], the second row is [III.6.5] and p_1 is induced by the diagram. From the commutativity it follows that p_1 is injective. *q.e.d.*

In the smooth case we have:

(III.6.5) PROPOSITION *Let $g : X \to Y$ be a morphism between projective nonsingular algebraic schemes. Assume that g is smooth. Then we have an exact sequence*

$$(0) \to H^0(X, T_X) \to \mathrm{Def}_{g/Y}(\mathbf{k}[\epsilon]) \xrightarrow{p} H^1(X, T_{X/Y}) \to (0)$$

and $H^2(X, T_{X/Y})$ is an obstruction space for $\mathrm{Def}_{g/Y}$.
Moreover we have a commutative diagram
$[III.6.6]$

$$\begin{array}{ccccccc} (0) \to H^0((X, g^*T_Y) \to & \mathrm{Def}_{g/Y}(\mathbf{k}[\epsilon]) & \xrightarrow{d\Phi} & \mathrm{Def}_X(\mathbf{k}[\epsilon]) & & & \\ \downarrow & \downarrow p & & \| & & & \\ (0) \to \frac{H^0(X,g^*T_Y)}{H^0(X,T_X)} \to & H^1(X, T_{X/Y}) & \xrightarrow{\kappa} & H^1(X, T_X) \to & & & \\ & & & \to H^1(X, g^*T_Y) & \to H^2(X, T_{X/Y}) & \xrightarrow{\delta} & H^2(X, T_X) \end{array}$$

where the second row is the exact sequence induced by [III.6.4] and δ induces the obstruction map of $\Phi : \mathrm{Def}_{g/Y} \to \mathrm{Def}_X$.
If in particular $H^0(X, T_X) = (0)$ then

$$\mathrm{Def}_{g/Y}(\mathbf{k}[\epsilon]) = H^1(X, T_{X/Y})$$

Proof

Left to the reader (it is similar to the proof of (III.6.4)). *q.e.d.*

* * * * * *

Morphisms from a nonsingular curve

The previous results apply in particular to a morphism

$$\varphi : C \to Y$$

where C and Y are projective and nonsingular, C is a curve, and φ is not constant on each component of C. Consider the exact sequence

$$0 \to T_C \xrightarrow{d\varphi} \varphi^*T_Y \to N_\varphi \to 0$$

The vanishing divisor

$$Z := D_0(d\varphi)$$

of $d\varphi$ is called the *ramification divisor* of φ; the *index of ramification* of φ at $p \in C$ is the coefficient of p in Z. φ is unramified if and only if $Z = 0$. The homomorphism $d\varphi$ extends to a homomorphism

$$T_C(Z) \to \varphi^* T_Y$$

whose cokernel we denote by $\bar{N}_\varphi$; it is locally free. We have

$$\bar{N}_\varphi = N_\varphi / \mathcal{H}_\varphi$$

where $\mathcal{H}_\varphi \subset N_\varphi$ is the torsion subsheaf; it is supported on Z. The following commutative and exact diagram summarizes the situation:

[III.6.7]
$$\begin{array}{ccccccccc} & & & & & & 0 & & \\ & & & & & & \downarrow & & \\ & & & & & & \mathcal{H}_\varphi & & \\ & & & & & & \downarrow & & \\ 0 & \to & T_C & \xrightarrow{d\varphi} & \varphi^* T_Y & \to & N_\varphi & \to & 0 \\ & & \downarrow & & \| & & \downarrow & & \\ 0 & \to & T_C(Z) & \to & \varphi^* T_Y & \to & \bar{N}_\varphi & \to & 0 \\ & & & & & & \downarrow & & \\ & & & & & & 0 & & \end{array}$$

We obtain:

[III.6.8]
$$\chi(N_\varphi) = \chi(\varphi^* T_Y) + 3g - 3$$

Assume that C is connected of genus g, that Y is a projective connected nonsingular curve of genus γ, and that φ has degree d; then

$$\bar{N}_\varphi = 0, \quad N_\varphi = \mathcal{H}_\varphi = \mathcal{O}_Z$$

where $\mathcal{O}(Z) = \varphi^*(T_Y) \otimes K_C$, so that

$$\chi(N_\varphi) = h^0(N_\varphi) = \deg(Z) = 2[g - 1 + (1 - \gamma)d]$$

and φ is unobstructed because $h^1(N_\varphi) = 0$.

Note that φ is rigid if $g \geq 2$ and $Z = 0$, i.e. if it is unramified.

* * * * * *

Morphisms from a curve to a surface

Assume now that $\varphi : C \to S$ is a non-constant morphism from an irreducible projective nonsingular curve C of genus g to a projective nonsingular surface S and that φ is birational onto its image; let $\Gamma = \varphi(C) \subset S$. Then we have a commutative and exact diagram

$$\begin{array}{ccccccccc} & & 0 & & & & & & \\ & & \downarrow & & & & & & \\ 0 \to & & T_C & \xrightarrow{d\varphi} & \varphi^* T_S & \to & N_\varphi & & \to 0 \\ & & \downarrow & & \| & & \downarrow j & & \\ 0 \to & & \varphi^* T_\Gamma & \to & \varphi^* T_S & \to & \varphi^* N_{\Gamma/S} & & \end{array}$$

Since $\varphi^* N_{\Gamma/S}$ is invertible the homomorphism j factors through $\bar{N}_\varphi$ and the above diagram gives rise to the following:

$$\begin{array}{ccccccc} & & 0 & & & & \\ & & \downarrow & & & & \\ 0 \to & & T_C(Z) & \longrightarrow & \varphi^* T_S & \to & \bar{N}_\varphi & \to 0 \\ & & \downarrow & & \| & & \downarrow & \\ 0 \to & & \varphi^* T_\Gamma & \to & \varphi^* T_S & \to & \varphi^* N'_{\Gamma/S} & \to 0 \end{array}$$

where Z is the ramification divisor of φ and $N'_{\Gamma/S} = \ker[N_{\Gamma/S} \to T^1_\Gamma]$ is the *equisingular normal sheaf* (see also §IV.7). This diagram implies the following facts:

$$T_C(Z) \cong \varphi^* T_\Gamma, \quad \bar{N}_\varphi \cong \varphi^* N'_{\Gamma/S}, \quad \mathcal{H}_\varphi \cong \operatorname{coker}[T_C \to \varphi^*(T_\Gamma)] =: N_{\bar{\varphi}}$$

where we have denoted by $\bar{\varphi} : C \to \Gamma$ the morphism induced by φ. In particular $\varphi^* T_\Gamma$ and $\varphi^* N'_{\Gamma/S}$ are invertible and

$$\varphi_*[T_C(Z)] \cong T_\Gamma \otimes \varphi_* \mathcal{O}_C, \quad \varphi_* \bar{N}_\varphi \cong N'_{\Gamma/S} \otimes \varphi_* \mathcal{O}_C$$

On Γ we have a natural exact sequence:

$$0 \to \mathcal{O}_\Gamma \to \varphi_* \mathcal{O}_C \to \mathbf{t} \to 0$$

where $\mathbf{t}$ is a torsion sheaf supported on the singular locus of Γ. Since $N_{\Gamma/S}$ is invertible the homomorphism

$$N_{\Gamma/S} \to N_{\Gamma/S} \otimes \varphi_* \mathcal{O}_C$$

is injective and it follows that we have an exact sequence

$$[III.6.9] \qquad \begin{array}{ccccccc} 0 \to & N'_{\Gamma/S} & \to & \varphi_* \bar{N}_\varphi & \to & N'_{\Gamma/S} \otimes \mathbf{t} & \to 0 \\ & & & \| & & & \\ & & & N'_{\Gamma/S} \otimes \varphi_* \mathcal{O}_C & & & \end{array}$$

This sequence implies in particular:

$$\begin{aligned} h^0(N'_{\Gamma/S}) &\le h^0(\bar{N}_\varphi) &\le h^0(N_\varphi) \\ h^1(N'_{\Gamma/S}) &\ge h^1(\bar{N}_\varphi) &= h^1(N_\varphi) \end{aligned}$$

(III.6.6) LEMMA *If the singularities of* Γ *are nodes and ordinary cusps then* $N'_{\Gamma/S} \otimes \mathbf{t} = 0$, *equivalently*

$$N'_{\Gamma/S} \cong \varphi_* \bar{N}_\varphi$$

Proof

The exact sequence [III.6.9] can be embedded in the following exact and commutative diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 \to & & N'_{\Gamma/S} & \to & \varphi_*\bar{N}_\varphi & \to & N'_{\Gamma/S}\otimes \mathbf{t} & \to 0 \\
 & & \downarrow & & \downarrow a & & \downarrow d & & \\
0 \to & & N_{\Gamma/S} & \to & N_{\Gamma/S}\otimes\varphi_*\mathcal{O}_C & \to & N_{\Gamma/S}\otimes \mathbf{t} & \to 0 \\
 & & \downarrow & & \downarrow & & \downarrow c & & \\
0 \to & & T^1_\Gamma & \xrightarrow{b} & T^1_\Gamma\otimes\varphi_*\mathcal{O}_C & \to & T^1_\Gamma\otimes \mathbf{t} & \to 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

The arrow a is injective because it is a nonzero homomorphism of torsion free rank one sheaves. Because of the assumptions made on the singularities, at each singular point $p \in \Gamma$ we have $\mathbf{t}_p = \varphi_*\mathcal{O}_{C,p}/\mathcal{O}_{\Gamma,p} \cong \mathbf{k}$. Therefore the arrow c is an isomorphism because $N_{\Gamma/S}\otimes\mathbf{t} \cong \mathbf{t} \cong T^1_\Gamma \otimes \mathbf{t}$. Thus $d = 0$. The arrow b is injective because at each singular point $p \in \Gamma$ we have

$$(T^1_\Gamma \otimes \varphi_*\mathcal{O}_C)_p = \varphi_*\varphi^*(T^1_{\Gamma,p}) \cong \begin{cases} \mathbf{k}^2 & \text{if } p \text{ is a node} \\ \mathbf{k}^3 & \text{if } p \text{ is a cusp} \end{cases}$$

(proved by easy local computation) while

$$T^1_{\Gamma,p} \cong \begin{cases} \mathbf{k} & \text{if } p \text{ is a node} \\ \mathbf{k}^2 & \text{if } p \text{ is a cusp} \end{cases}$$

The conclusion now follows from the "Snake Lemma". *q.e.d.*

It is possible to show that conversely if $N'_{\Gamma/S} \otimes \mathbf{t} = 0$ then the singularities of Γ are nodes and ordinary cusps (see Greuel(1984)).

If $Y = \mathbb{P}^2$ then, letting $L = \varphi^*\mathcal{O}(1)$, $d = \deg(L)$, from the Euler sequence restricted to C:

$$0 \to \mathcal{O}_C \to L^{\oplus 3} \to \varphi^* T_{\mathbb{P}^2} \to 0$$

we deduce $\chi(\varphi^* T_{\mathbb{P}^2}) = 3d + 2 - 2g$ and from [III.6.8]

$$[III.6.10] \qquad \chi(N_\varphi) = 3d + g - 1$$

By Proposition (III.6.4) the unobstructedness of φ is related to the vanishing of $H^1(C, N_\varphi)$. From [III.6.7] we see that

$$c_1(N_\varphi) = c_1(\varphi^* T_{\mathbb{P}^2}) - \deg(T_C) = 3d + 2g - 2$$

and that

$$h^1(N_\varphi) = h^1(\bar{N}_\varphi)$$

But

$$\deg(\bar{N}_\varphi) = c_1(N_\varphi) - \deg(Z)$$

and therefore $\bar{N}_\varphi$ is a non-special line bundle whenever $\deg(Z) < 3d$. We can therefore state the following result:

(III.6.7) THEOREM *Let $\varphi : C \to \mathbb{P}^2$ be a non constant morphism from an irreducible projective nonsingular curve C of genus g. Let $d = \deg(\varphi^*\mathcal{O}(1))$, and let Z be the ramification divisor of φ. Then the formal semiuniversal deformation of φ has dimension $\geq 3d+g-1+h^0(C,T_C)$. If $\deg(Z) < 3d$ then φ is unobstructed and the above inequality is an equality.*

* * * * * *

Deformations of a blow-up

Let Y be a projective nonsingular variety, $\gamma \subset Y$ a nonsingular closed subvariety of pure codimension $r \geq 2$ and $\pi : X \to Y$ the blow-up of Y with center γ. Let $E = \pi^{-1}(\gamma) \subset X$ be the exceptional divisor. We have $T_{X/Y} = 0$ because $T_{X/Y}$ is a subsheaf of the locally free T_X and is supported on E; therefore π is nondegenerate. Since $\pi_*\mathcal{O}_X = \mathcal{O}_Y$ and $R^i\pi_*\mathcal{O}_X = 0$ for $i \geq 1$, from the Leray spectral sequence we deduce that

$$[III.6.11] \qquad H^i(X, \pi^*T_Y) = H^i(Y, (\pi_*\pi^*\mathcal{O}_X) \otimes T_Y) = H^i(Y, T_Y), \quad i \geq 0$$

Also recall that

$$H^i(E, N_{E/X}) = (0)$$

all i (see Example (III.3.14)(i)).

Let $q := \pi_{|E} : E \to \gamma$; then we have an exact and commutative diagram of locally free sheaves on E:

$$[III.6.12] \qquad \begin{array}{ccccccccc} & & & 0 & & 0 & & & \\ & & & \downarrow & & \downarrow & & & \\ 0 \to & T_{E/\gamma} & \to & T_E & \to & q^*T_\gamma & \to & 0 & \\ & \| & & \downarrow & & \downarrow & & & \\ 0 \to & T_{E/\gamma} & \to & T_{X|E} & \to & q^*T_Y & \to & N_\pi & \to 0 \\ & & & \downarrow & & \downarrow & & \| & \\ & 0 & \to & N_{E/X} & \to & q^*N_{\gamma/Y} & \to & N_\pi & \to 0 \\ & & & \downarrow & & \downarrow & & & \\ & & & 0 & & 0 & & & \end{array}$$

In particular we see that we have an exact sequence of locally free sheaves on E:

$$[III.6.13] \qquad 0 \to N_{E/X} \to q^*N_{\gamma/Y} \to N_\pi \to 0$$

The verification of these facts is straightforward and it is left to the reader.

(III.6.8) PROPOSITION *In the above situation, assume that* $H^1(Y,T_Y) = (0)$, *i.e. that* Y *is rigid. Then*
(i) The forgetful morphism $\Phi : \mathrm{Def}_{\pi/Y} \to \mathrm{Def}_X$ *is smooth.*
(ii) There is a natural morphism of functors

$$\Psi : H^Y_\gamma \to \mathrm{Def}_{\pi/Y}$$

which is an isomorphism if $H^0(X,T_X) = (0)$.
In particular, if γ *is obstructed in* Y *then* X *is obstructed as an abstract variety.*

Proof
(i) follows from Corollary (III.6.3) and from [III.6.13].
(ii) We have a well defined morphism of functors

$$\Psi : H^Y_\gamma \to \mathrm{Def}_{\pi/Y}$$

which associates to a family of deformations

$$\begin{array}{ccc} \gamma_A & \subset & Y \times \mathrm{Spec}(A) \\ \downarrow & & \\ \mathrm{Spec}(A) & & \end{array}$$

of γ in Y over A the blow-up

$$\pi_A : X_A := Bl_{\gamma_A}(Y \times \mathrm{Spec}(A)) \to Y \times \mathrm{Spec}(A)$$

of $Y \times \mathrm{Spec}(A)$ along γ_A. Assume that $H^0(X,T_X) = (0)$: then the differential of Ψ is the composition

$$d\Psi : H^0(\gamma, N_{\gamma/Y}) \cong H^0(E, q^*N_{\gamma/Y}) \to H^0(X, N_\pi)$$

where the first map is the obvious isomorphism and the second one comes from the exact sequence [III.6.13]; in a similar way one describes the obstruction map of Ψ as the one induced by the composition

$$H^1(\gamma, N_{\gamma/Y}) \cong H^1(E, q^*N_{\gamma/Y}) \to H^1(X, N_\pi)$$

deduced from the exact sequence [III.6.13]. These facts can be easily verified by chasing diagram [III.6.12]. From our assumptions we see that these maps are both bijective, and the conclusion follows.

The last assertion is an obvious consequence of the fact that the composition $\Phi\Psi : H^Y_\gamma \to \mathrm{Def}_X$ is smooth. *q.e.d.*

Observe that it follows from (ii) that the complete local ring prorepresenting H^Y_γ is a power series ring over the ring R prorepresenting Def_X.

As a consequence we get:

(III.6.9) COROLLARY *Let $\pi : X \to \mathbb{P}^3$ be the blow-up with center a nonsingular irreducible and nondegenerate curve $\gamma \subset \mathbb{P}^3$ which is obstructed. Then X is obstructed.*

Note that $H^0(X, T_X) = (0)$ because γ is nondegenerate.

(III.6.10) EXAMPLE Let $\pi : X = Bl_{[1,0,0]}\mathbb{P}^2 \to \mathbb{P}^2$ be the blow-up of $\mathbb{P}^2$ with center the point $[1,0,0]$. From the exact sequence [III.6.13] we deduce that $N_\pi = \mathcal{O}_E(1)$. Therefore

$$h^0(X, N_\pi) = 2, \quad h^i(X, N_\pi) = 0, \ i \geq 1$$

Moreover $h^0(X, T_X) = h^0(\mathbb{P}^2, T_{\mathbb{P}^2} \otimes \mathcal{I}_{[1,0,0]}) = 6$ as can be easily checked using the Euler sequence. Therefore from the exact sequence [III.6.5] we see that $h^1(X, T_X) = 0$, i.e. X is rigid. It also follows from Proposition (III.6.4) that $\mathrm{Def}_{\pi/\mathbb{P}^2}$ is smooth of dimension 8.

NOTES

1. Let Y be a scheme, $\Phi : \mathcal{X} \to Y \times S$ a family of morphisms into Y parametrized by an algebraic scheme S, with $\mathcal{X} \to S$ projective. Assume that for a **k**-rational point $o \in S$ the fibre $\Phi_o : \mathcal{X}(o) \to Y$ is a closed embedding. Then there is an open neighborhood $U \subset S$ of o such that the restriction $\Phi(U) : \mathcal{X}(U) \to Y \times U$ is a family of closed subschemes of Y. Suppose moreover that Φ_o an isomorphism. Then there is an open neighborhood $U \subset S$ of o such that the restriction $\Phi(U) : \mathcal{X}(U) \to Y \times U$ is an isomorphism. (use Note 2 of §IV.2).

2. By applying Corollary (III.6.9) to the curve $\gamma \subset \mathbb{P}^2$ of degree 14 and genus 24 described in §IV.6 we obtain an example of obstructed projective variety of dimension 3. This has been the first published example of an obstructed projective nonsingular variety (see Mumford(1962)). The analysis of deformations of blow-ups given here is due to Kodaira (see Kodaira(1963)).

3. The analysis of morphisms from a nonsingular curve is taken from Arbarello-Cornalba(1981). For Lemma (III.6.6) see also Tannenbaum(1984).