Chapter IV. The Hilbert schemes and the Quot schemes

In this Chapter we turn our attention to *global deformations*. The Hilbert schemes and the Quot schemes are important examples of parameter schemes for global families of deformations of algebro-geometric objects. They are used to describe and classify "extrinsic" deformations, i.e. deformations of objects within a given ambient space (e.g. closed subschemes of a given scheme). Their study is preliminary to the construction of "moduli spaces". Moreover they provide some of the most typical examples of constructions in algebraic geometry by the functorial approach. We will study some of their properties and consider a few applications.

IV.1. CASTELNUOVO-MUMFORD REGULARITY

In this Section we introduce the notion of *m-regularity*, also called *Castelnuovo-Mumford regularity*, and we prove its main properties. They will be needed for the construction of the Hilbert schemes and of the Quot schemes.

Let $m \in \mathbb{Z}$. A coherent sheaf $\mathcal{F}$ on $\mathbb{P}^r$ is *m-regular* if

$$H^i(\mathcal{F}(m-i)) = (0)$$

for all $i \geq 1$.

Because of Serre's vanishing theorem, every coherent sheaf $\mathcal{F}$ on $\mathbb{P}^r$ is m-regular for some $m \in \mathbb{Z}$.

The definition of m-regularity makes sense for a coherent sheaf on any projective scheme X endowed with a very ample line bundle $\mathcal{O}(1)$. For semplicity we will consider the case $X = \mathbb{P}^r$ only, leaving to the reader the obvious modifications of the statements and of the proofs in the general case.

(IV.1.1) PROPOSITION *If $\mathcal{F}$ is m-regular then*
(i) the natural map

$$H^0(\mathcal{F}(k)) \otimes_{\mathbf{k}} H^0(\mathcal{O}(1)) \to H^0(\mathcal{F}(k+1))$$

is surjective for all $k \geq m$.
(ii) $H^i(\mathcal{F}(k)) = (0)$ for all $i \geq 1$ and $k \geq m-i$; in particular $\mathcal{F}$ is n-regular for all $n \geq m$.
(iii) $\mathcal{F}(m)$, and therefore also $\mathcal{F}(k)$ for all $k \geq m$, is generated by its global sections.

Proof
We prove (i) and (ii) by induction on r. If $r = 0$ there is nothing to prove. Assume $r \geq 1$ and let H be a hyperplane not containing any point of $\mathrm{Ass}(\mathcal{F})$; it exists because $\mathrm{Ass}(\mathcal{F})$ is a finite set. Tensoring by $\mathcal{F}(k)$ the exact sequence:

$$0 \to \mathcal{O}(-H) \to \mathcal{O} \to \mathcal{O}_H \to 0$$

we get an exact sequence:

$$0 \to \mathcal{F}(k-1) \to \mathcal{F}(k) \to \mathcal{F}_H(k) \to 0$$

where $\mathcal{F}_H = \mathcal{F} \otimes \mathcal{O}_H$. For each $i > 0$ we obtain an exact sequence

$$H^i(\mathcal{F}(m-i)) \to H^i(\mathcal{F}_H(m-i)) \to H^{i+1}(\mathcal{F}(m-i-1))$$

which implies that $\mathcal{F}_H$ is m-regular on H. It follows by induction that (i) and (ii) are true for $\mathcal{F}_H$.

Let's consider the exact sequence

$$H^{i+1}(\mathcal{F}(m-i-1)) \to H^{i+1}(\mathcal{F}(m-i)) \to H^{i+1}(\mathcal{F}_H(m-i))$$

If $i \geq 0$ the two extremes are zero (the right one by (ii) for $\mathcal{F}_H$, the left one by the m regularity of $\mathcal{F}$), therefore $\mathcal{F}$ is $(m+1)$-regular. By iteration this proves (ii).

To prove (i) we consider the commutative diagram:

$$\begin{array}{cccc} & H^0(\mathcal{F}(k)) \otimes_{\mathbf{k}} H^0(\mathcal{O}(1)) & \underset{u}{\longrightarrow} & H^0(\mathcal{F}_H(k)) \otimes_{\mathbf{k}} H^0(\mathcal{O}_H(1)) \\ & \downarrow w & & \downarrow t \\ H^0(\mathcal{F}(k)) \to & H^0(\mathcal{F}(k+1)) & \underset{v}{\longrightarrow} & H^0(\mathcal{F}_H(k+1)) \end{array}$$

The map u is surjective for $k \geq m$ because $H^1(\mathcal{F}(k-1)) = (0)$; moreover t is surjective for $k \geq m$ by (i) for $\mathcal{F}_H$. Therefore $v \circ w$ is surjective. It follows that $H^0(\mathcal{F}(k+1))$ is generated by $\mathrm{Im}(w)$ and by $H^0(\mathcal{F}(k))$ for all $k \geq m$. But $H^0(\mathcal{F}(k)) \subset \mathrm{Im}(w)$ because the inclusion $H^0(\mathcal{F}(k)) \subset H^0(\mathcal{F}(k+1))$ is multiplication by H. Therefore w is surjective.

Let's prove (iii). Let $h \gg 0$ be such that $\mathcal{F}(m+h)$ is generated by its global sections. Then the composition

$$H^0(\mathcal{F}(m)) \otimes_{\mathbf{k}} H^0(\mathcal{O}(h)) \otimes_{\mathbf{k}} \mathcal{O} \to H^0(\mathcal{F}(m+h)) \otimes_{\mathbf{k}} \mathcal{O} \to \mathcal{F}(m+h)$$

is surjective because from (i) it follows that the first map is; we deduce that the composition

$$H^0(\mathcal{F}(m)) \otimes_{\mathbf{k}} H^0(\mathcal{O}(h)) \otimes_{\mathbf{k}} \mathcal{O}(-h) \to H^0(\mathcal{F}(m)) \otimes_{\mathbf{k}} \mathcal{O} \to \mathcal{F}(m)$$

is also surjective, hence the second map is surjective too. *q.e.d.*

Note that if $\mathcal{F}$ is m-regular then the graded $\mathbf{k}[X_1, \ldots, X_r]$-module

$$\Gamma_*(\mathcal{F}) := \bigoplus_{k \in \mathbb{Z}} H^0(\mathcal{F}(k))$$

can be generated by elements of degree $\leq m$. In fact this is equivalent to the surjectivity of the multiplication maps

$$H^0(\mathcal{F}(m)) \otimes_{\mathbf{k}} H^0(\mathcal{O}(h)) \to H^0(\mathcal{F}(m+h))$$

for $h \geq 1$, and follows from part (i) of the proposition. In particular, if an ideal sheaf $\mathcal{I} \subset \mathcal{O}_{\mathbb{P}^r}$ is m-regular then the homogeneous ideal

$$I = \Gamma_*(\mathcal{I}) \subset \mathbf{k}[X_0, \ldots, X_r]$$

is generated by elements of degree $\leq m$.

Note also that in the way of proving (IV.1.1) we have proved the following:

(IV.1.2) PROPOSITION *If $\mathcal{F}$ is m-regular and*

$$0 \to \mathcal{F}(-1) \to \mathcal{F} \to \mathcal{G} \to 0$$

is an exact sequence, then $\mathcal{G}$ is m-regular.

Conversely, we have the following:

(IV.1.3) PROPOSITION *Let*

$$0 \to \mathcal{F}(-1) \to \mathcal{F} \to \mathcal{G} \to 0$$

be an exact sequence of coherent sheaves on $\mathbb{P}^r$, and assume that $\mathcal{G}$ is m-regular. Then:

$$\begin{array}{lll} (i) & H^i(\mathcal{F}(k)) = 0 & \text{for } i \geq 2 \text{ and } k \geq m-i \\ (ii) & h^1(\mathcal{F}(k-1)) \geq h^1(\mathcal{F}(k)) & \text{for } k \geq m-1 \\ (iii) & H^1(\mathcal{F}(k)) = 0 & \text{for } k \geq (m-1) + h^1(\mathcal{F}(m-1)) \end{array}$$

In particular $\mathcal{F}$ is $m + h^1(\mathcal{F}(m-1))$-regular.

Proof

(i) In the exact sequence

$$H^{i-1}(\mathcal{G}(k)) \to H^i(\mathcal{F}(k-1)) \to H^i(\mathcal{F}(k)) \to H^i(\mathcal{G}(k))$$

the first and the last group are zero for $i \geq 2$ and $k \geq m-(i-1)$. Therefore

$$H^i(\mathcal{F}(m-i)) \cong H^i(\mathcal{F}(m-i+1)) \cong H^i(\mathcal{F}(m-i+2)) \cong \cdots$$

From Serre's vanishing theorem we get $H^i(\mathcal{F}(m-i+h)) = 0$ for all $h \gg 0$ and (i) follows.

(ii) For $k \geq m-1$ we have the exact sequence

$$0 \to H^0(\mathcal{F}(k-1)) \to H^0(\mathcal{F}(k)) \xrightarrow{v_k} H^0(\mathcal{G}(k)) \to H^1(\mathcal{F}(k-1)) \to H^1(\mathcal{F}(k)) \to 0$$

which implies (ii).

(iii) Assume v_k surjective, and consider the commutative diagram:

$$\begin{array}{ccc} H^0(\mathcal{F}(k)) \otimes H^0(\mathcal{O}(1)) & \xrightarrow{v_k \otimes id} & H^0(\mathcal{G}(k)) \otimes H^0(\mathcal{O}(1)) \\ \downarrow & & \downarrow w_k \\ H^0(\mathcal{F}(k+1)) & \xrightarrow{v_{k+1}} & H^0(\mathcal{G}(k+1)) \end{array}$$

Since w_k is surjective for $k \geq m$, we have that v_{k+1} is surjective too. Therefore

$$H^1(\mathcal{F}(k-1)) \cong H^1(\mathcal{F}(k)) \cong H^1(\mathcal{F}(k+1)) \cong \cdots \cong 0$$

If v_k is not surjective then $h^1(\mathcal{F}(k-1)) > h^1(\mathcal{F}(k))$. Therefore the function $k \mapsto h^1(\mathcal{F}(k))$ is strictly decreasing for $k \geq m-1$, and this implies (iii). *q.e.d.*

The following is a useful characterization of m-regularity.

(IV.1.4) THEOREM *A coherent sheaf $\mathcal{F}$ on $\mathbb{P}^r$ is m-regular if and only if it has a resolution of the form:*

$$[IV.1.1] \quad 0 \to \mathcal{O}(-m-r-1)^{b_{r+1}} \to \cdots \to \mathcal{O}(-m-1)^{b_1} \to \mathcal{O}(-m)^{b_0} \to \mathcal{F} \to 0$$

for some $b_0, \ldots, b_{r+1} \geq 1$.

Proof

Assume that $\mathcal{F}$ has a resolution [IV.1.1] and let

$$\mathcal{R}_1 = \ker\left[\mathcal{O}(-m)^{b_0} \to \mathcal{F}\right]$$

$$\mathcal{R}_j = \ker\left[\mathcal{O}(-m-j+1)^{b_{j-1}} \to \mathcal{O}(-m-j+2)^{b_{j-2}}\right] \qquad j = 2, \ldots, r$$

$$\mathcal{R}_{r+1} = \mathcal{O}(-m-r)^{b_{r+1}}$$

Using the short exact sequences:

$$0 \to \mathcal{R}_1(m-i) \to \mathcal{O}(-i)^{b_0} \to \mathcal{F}(m-i) \to 0$$

$$0 \to \mathcal{R}_j(m-i) \to \mathcal{O}(-i-j+1)^{b_{j-1}} \to \mathcal{R}_{j-1}(m-i) \to 0$$

$$0 \to \mathcal{O}(-i-r-1)^{b_{r+1}} \to \mathcal{O}(-i-r)^{b_r} \to \mathcal{R}_r(m-i) \to 0$$

we see that for all $1 \leq i \leq r$ we have:

$$H^i(\mathcal{F}(m-i)) \cong H^{i+1}(\mathcal{R}_1(m-i)) \cong \cdots$$

$$\cdots \cong H^r(\mathcal{R}_{r-i}(m-i)) \cong H^{r+1}(\mathcal{R}_{r-i+1}(m-i)) = (0)$$

and $\mathcal{F}$ is m-regular.

Assume conversely that $\mathcal{F}$ is m-regular. By (IV.1.1)(iii) we have an exact sequence:

$$0 \to \mathcal{R}_1 \to \mathcal{O}(-m)^{b_0} \to \mathcal{F} \to 0$$

with $b_0 = h^0(\mathcal{F}(m))$, which defines $\mathcal{R}_1$.

If $\mathcal{R}_1 = 0$ we are done; if $\mathcal{R}_1 \neq 0$ from the sequences:

$$0 \to \mathcal{R}_1(m-i+1) \to \mathcal{O}(-i+1)^{b_0} \to \mathcal{F}(m-i+1) \to 0$$

we deduce that

$$H^i(\mathcal{R}_1(m-i+1)) \cong H^{i-1}(\mathcal{F}(m-i+1)) \qquad 1 \leq i \leq r$$

hence $\mathcal{R}_1$ is $(m+1)$-regular. Applying the same argument to $\mathcal{R}_1$ we find an exact sequence:

$$0 \to \mathcal{R}_2 \to \mathcal{O}(-m-1)^{b_1} \to \mathcal{O}(-m)^{b_0} \to \mathcal{F} \to 0$$

with $\mathcal{R}_2$ $(m+2)$-regular. This process can be repeated for at most $r+1$ steps, by the Hilbert syzygy theorem, and gives a resolution as required. *q.e.d.*

* * * * * *

We will now turn to the problem of finding numerical criteria of m-regularity for a coherent sheaf $\mathcal{F}$ on $\mathbb{P}^r$.

Consider a sequence $\sigma_1, \ldots, \sigma_N$ of N sections of $\mathcal{O}_{\mathbb{P}^r}(1)$. We will call it *$\mathcal{F}$-regular* if the sequences of sheaf homomorphisms induced by multiplication by $\sigma_1, \ldots, \sigma_N$:

$$0 \to \mathcal{F} \xrightarrow{\sigma_1} \mathcal{F} \to \mathcal{F}_1 \to 0$$

$$0 \to \mathcal{F}_1 \xrightarrow{\sigma_2} \mathcal{F}_1 \to \mathcal{F}_2 \to 0$$

etc., are exact.

By choosing σ_{i+1} not containing any point of $\mathrm{Ass}(\mathcal{F}_i)$ one shows that $\mathcal{F}$-regular sequences of any lenght exist. Therefore any general N-tuple $(\sigma_1, \ldots, \sigma_N) \in H^0(\mathcal{O}_{\mathbb{P}^r}(1))^N$ is an $\mathcal{F}$-sequence.

(IV.1.5) DEFINITION *Let $\mathcal{F}$ be a coherent sheaf on $\mathbb{P}^r$, and $(\mathbf{b}) = (b_0, b_1, \ldots, b_N)$ a sequence of nonnegative integers such that $N \geq \dim[\mathrm{Supp}(\mathcal{F})]$. We will call $\mathcal{F}$ a* $(\mathbf{b})$-sheaf *if there exists an $\mathcal{F}$-regular sequence $\sigma_1, \ldots, \sigma_N$ of sections of $\mathcal{O}_{\mathbb{P}^r}(1)$ such that $h^0(\mathcal{F}_i(-1)) \leq b_i$, $i = 0, \ldots, N$ where $\mathcal{F}_0 = \mathcal{F}$, and $\mathcal{F}_i$, $i \geq 1$, is the restriction of $\mathcal{F}$ to the scheme of zeros of $\sigma_1, \ldots, \sigma_i$ ($\mathcal{F}_i = 0$ if this scheme is empty).*

Note that from the definition it follows immediately that if $\mathcal{F}$ is a $(\mathbf{b})$-sheaf then $\mathcal{F}_1$ is a $(b_1, \ldots, b_N)$-sheaf.

It is likewise clear that for very coherent sheaf $\mathcal{F}$ on $\mathbb{P}^r$ there is a sequence $(\mathbf{b})$ such that $\mathcal{F}$ is a $(\mathbf{b})$-sheaf. Moreover a subsheaf of a $(\mathbf{b})$-sheaf is easily seen to be a $(\mathbf{b})$-sheaf.

For example, every ideal sheaf $\mathcal{I} \subset \mathcal{O}_{\mathbb{P}^r}$ is a $(\mathbf{0})$-sheaf, because $\mathcal{O}_{\mathbb{P}^r}$ is clearly a $(\mathbf{0})$-sheaf.

(IV.1.6) LEMMA *Let*

$$0 \to \mathcal{F}(-1) \to \mathcal{F} \to \mathcal{G} \to 0$$

be an exact sequence of coherent sheaves on $\mathbb{P}^r$. If

$$\chi(\mathcal{F}(k)) = \sum_{i=0}^{r} a_i \binom{k+i}{i}$$

then

$$\chi(\mathcal{G}(k)) = \sum_{i=0}^{r-1} a_{i+1} \binom{k+i}{i}$$

The proof is left to the reader.

(IV.1.7) PROPOSITION *Let $\mathcal{F}$ be a* ($\mathbf{b}$)*-sheaf, let s =dim[Supp($\mathcal{F}$)] and*

$$\chi(\mathcal{F}(k)) = \sum_{i=0}^{s} a_i \binom{k+i}{i}$$

Then
(i) For each $k \geq -1$ we have $h^0(\mathcal{F}(k)) \leq \sum_{i=0}^{s} b_i \binom{k+i}{i}$.
(ii) $a_s \leq b_s$ and $\mathcal{F}$ is also a $(b_0, \ldots, b_{s-1}, a_s)$-sheaf.

Proof
(i) By induction on s. If $s = 0$ then $a_0 = h^0(\mathcal{F}) = h^0(\mathcal{F}(-1)) \leq b_0$ and the conclusion is obvious.

Assume $s \geq 1$. We have an exact sequence

$$0 \to \mathcal{F}(-1) \to \mathcal{F} \to \mathcal{F}_1 \to 0$$

with $\mathcal{F}_1$ a $(b_1, \ldots, b_N)$-sheaf and dim[Supp($\mathcal{F}_1$)] $= s - 1$. Then:

$$h^0(\mathcal{F}(k)) - h^0(\mathcal{F}(k-1)) \leq h^0(\mathcal{F}_1(k))$$

and

$$h^0(\mathcal{F}_1(k)) \leq \sum_{i=0}^{s-1} b_{i+1} \binom{k+i}{i}$$

by the inductive hypothesis. Since $h^0(\mathcal{F}(-1)) \leq b_0$ by induction on $k \geq -1$ we get the conclusion.

(ii) By Lemma (IV.1.6) and by induction on s we get $a_s \leq b_s$ and $\mathcal{F}_1$ is a $(b_1, \ldots, b_{s-1}, a_s)$-sheaf. The conclusion follows. *q.e.d.*

(IV.1.8) DEFINITION *The following polynomials, defined by induction for each integer $r \geq -1$:*

$$\begin{aligned} P_{-1} &= 0 \\ P_r(X_0, \ldots, X_r) &= P_{r-1}(X_1, \ldots, X_r) + \textstyle\sum_{i=0}^{r} X_i \binom{P_{r-1}(X_1,\ldots,X_r)-1+i}{i} \end{aligned}$$

are called ($\mathbf{b}$)-polynomials.

One immediately sees that

$$P_r(X_0, \ldots, X_t, 0, \ldots, 0) = P_t(X_0, \ldots, X_t) \qquad [IV.1.2]$$

for each $t < r$.

The following Theorem gives a numerical criterion of m-regularity.

(IV.1.9) THEOREM *Let $\mathcal{F}$ be a* ($\mathbf{b}$)*-sheaf on $\mathbb{P}^r$, with* $(\mathbf{b}) = (b_0, b_1, \ldots, b_N)$, *and let*

$$\chi(\mathcal{F}(k)) = \sum_{i=0}^{r} a_i \binom{k+i}{i}$$

be its Hilbert polynomial. Let $(c_0, \ldots, c_r)$ *be a sequence of integers such that* $c_i \geq b_i - a_i$, *for* $i = 0, \ldots, r$, *and* $m = P_r(c_0, \ldots, c_r)$. *Then* $m \geq 0$ *and* $\mathcal{F}$ *is* m*-regular. In particular* $\mathcal{F}$ *is* $P_{s-1}(c_0, \ldots, c_{s-1})$*-regular, if* s *=dim[Supp(*$\mathcal{F}$*)].*

Proof

By induction on r. If $r = 0$ then $m = 0$ and $\mathcal{F}$ is n-regular for every $n \in \mathbb{Z}$, so the theorem is true in this case. Assume $r \geq 1$. We have an exact sequence:

$$0 \to \mathcal{F}(-1) \to \mathcal{F} \to \mathcal{F}_1 \to 0$$

with $\mathcal{F}_1$ a $(b_1, \ldots, b_N)$-sheaf supported on $\mathbb{P}^{r-1}$. From Lemma (IV.1.6) and from the inductive hypothesis we deduce that $n \geq 0$ and $\mathcal{F}_1$ is n-regular, where $n = P_{r-1}(c_1, \ldots, c_r)$. From (IV.1.3) we deduce that $\mathcal{F}$ is $[n + h^1(\mathcal{F}(n-1)]$-regular and $h^i(\mathcal{F}(n-1)) = 0$ for $i \geq 2$. Therefore:

$$h^1(\mathcal{F}(n-1)) = h^0(\mathcal{F}(n-1)) - \chi(\mathcal{F}(n-1)) \leq \sum_{i=0}^{r}(b_i - a_i)\binom{n-1+i}{i}$$

by (IV.1.7)(i). It follows that $\mathcal{F}$ is $n + \sum_{i=0}^{r} c_i \binom{n-1+i}{i}$-regular, by (IV.1.1)(ii). This proves the first assertion.

The last assertion follows from (IV.1.7)(ii) and from [IV.1.2]. *q.e.d.*

Note that the integer m in the statement of the Theorem depends on the coefficients of the Hilbert polynomial of $\mathcal{F}$ as well as on the integers b_i. In the special case when $\mathcal{F}$ is a sheaf of ideals we can determine an m for which $\mathcal{F}$ is m-regular which depends only on the Hilbert polynomial of $\mathcal{F}$, as stated in the next Corollary.

(IV.1.10) COROLLARY *For each* $r \geq 0$ *there exists a polynomial* $F_r(X_0, \ldots, X_r)$ *such that every sheaf of ideals* $\mathcal{I} \subset \mathcal{O}_{\mathbb{P}^r}$ *having Hilbert polynomial*

$$\chi(\mathcal{I}(k)) = \sum_{i=0}^{r} a_i \binom{k+r}{i}$$

is m*-regular, where* $m = F_r(a_0, \ldots, a_r)$, *and* $m \geq 0$.

Proof

It suffices to observe that $\mathcal{I}$ is a $(\mathbf{0})$-sheaf. Therefore the Corollary follows from Theorem (IV.1.9) taking $F_r(X_0, \ldots, X_r) = P_r(-X_0, \ldots, -X_r)$. *q.e.d.*

NOTES

1. Corollary (IV.1.10) is in general false for coherent sheaves which are not sheaves of ideals. An example (Mumford (1966)) is

$$\mathcal{F} = \mathcal{O}_{\mathbb{P}^1}(k) \oplus \mathcal{O}_{\mathbb{P}^1}(-k)$$

Infact $\chi(\mathcal{F}) = 2$ is independent of k but the least m such $\mathcal{F}$ is m-regular is $|k|$.

2. If $\mathcal{I}$ is the sheaf of ideals of the closed subscheme $X \subset \mathbb{P}^r$ and $\mathcal{I}$ is m-regular with $m \geq 0$, then $\mathcal{O}_X$ is $(m-1)$-regular. Conversely, if $\mathcal{O}_X$ is $(m-1)$-regular and the restriction map

$$H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(m-1)) \to H^0(X, \mathcal{O}_X(m-1))$$

is surjective, then $\mathcal{I}$ is m-regular. This follows from the exact sequences

$$0 \to \mathcal{I}(k) \to \mathcal{O}_{\mathbb{P}^r}(k) \to \mathcal{O}_X(k) \to 0$$

$k \geq m-1$.

3. The notion of m-regularity is related with that of *bounded collection of sheaves*, important in moduli theory.

A collection of coherent sheaves $\{F_j\}_{j \in J}$ on a projective scheme X is said to be bounded if there is an algebraic scheme S and a coherent sheaf $\mathcal{F}$ on $X \times S$ such that for each $j \in J$ there is a closed point $s \in S$ such that F_j is isomorphic to the sheaf $\mathcal{F}(s) = \mathcal{F}_{|X \times \mathrm{Spec}(s)}$. One also says that the collection $\{F_j\}_{j \in J}$ is bounded by the sheaf $\mathcal{F}$ on $X \times S$. For details we refer to Kleiman(1971)

4. The original source for the notion of Castelnuovo-Mumford regularity is Mumford(1966). The treatment of (**b**)-sheaves has been taken from Kleiman(1971).

IV.2. FLATTENING STRATIFICATIONS

This Section is devoted to the proof of a powerful technical result due to Grothendieck and Mumford, the existence of flattening stratifications, which is a key ingredient in the construction of the Hilbert schemes, of the Quot schemes, and of related schemes like the Severi varieties. We will start by brefly recalling some properties of flat families of projective schemes which will be needed in this Chapter.

* * * * * *

Flatness and Hilbert polynomials

The following result gives the name to the "Hilbert scheme".

(IV.2.1) PROPOSITION *(i) Let S be a scheme, $\mathcal{F}$ a coherent sheaf on $I\!P^r \times S$ and $p : I\!P^r \times S \to S$ the projection. Then $\mathcal{F}$ is flat over S if and only if $p_*\mathcal{F}(h)$ is locally free on S for all $h \gg 0$.*
(ii) Assume that S is connected. For each $s \in S$ let

$$P_s(t) = \chi(\mathcal{F}(s)(t)) = \sum_i (-1)^i h^i(I\!P^r(s), \mathcal{F}(s)(t))$$

be the Hilbert polynomial of $\mathcal{F}(s)$. If $\mathcal{F}$ is flat over S then $P_s(t)$ is independent of $s \in S$. Conversely, if S is integral and $P_s(t)$ is independent of s for all $s \in S$, then $\mathcal{F}$ is flat over S. If S is integral and algebraic and $P_s(t)$ is independent of s for all closed $s \in S$, then $\mathcal{F}$ is flat over S.

For the proof of this Proposition we refer the reader to Hartshorne(1977), Theorem III.9.9.

(IV.2.2) COROLLARY *If*

$$\begin{array}{ccc} \mathcal{X} & \subset & I\!P^r \times S \\ \downarrow & & \\ S & & \end{array}$$

is a flat family of closed subschemes of $I\!P^r$ with S connected, then all fibres $\mathcal{X}(s)$ have the same Hilbert polynomial; in particular they have the same degree.

Proof
It follows from (IV.2.1) applied to $\mathcal{F} = \mathcal{O}_{\mathcal{X}}$. *q.e.d.*

(IV.2.3) EXAMPLES (i) Let $U_i = \{(z_0, z_1) \in \mathbb{P}^1 : z_i \neq 0\}$, $U = U_0 \coprod U_1$ and $f : U \to \mathbb{P}^1$ the natural morphism. Then f is flat surjective and quasi-finite. The fibres of f are 0-dimensional, hence projective, but their degree is not constant. This is not a contradiction with Corollary (IV.2.2) because the morphism f is not projective, since U is an affine variety.

(ii) In $\mathbb{P}^3$ with homogeneous coordinates $\underline{X} = (X_0, X_1, X_2, X_3)$ consider the curve

$$C_u = \mathrm{Proj}(\mathbf{k}[\underline{X}]/(X_2, X_3)) \cup \mathrm{Proj}(\mathbf{k}[\underline{X}]/(X_1, X_3 - uX_0))$$

for every $u \in \mathbf{A}^1$. If $u \neq 0$ then C_u consists of two disjoint lines, while

$$C_0 = \mathrm{Proj}(\mathbf{k}[\underline{X}]/(X_1X_2, X_3))$$

is a reducible conic in the plane $X_3 = 0$. The Hilbert polynomials are

$$\begin{aligned} P_u(t) &= 2t + 2 \qquad u \neq 0 \\ P_0(t) &= 2t + 1 \end{aligned}$$

From Corollary (IV.2.2) it follows that $\{C_u\}$ cannot be the set of fibres of a flat family of closed subschemes of $\mathbb{P}^3$.

We may try to construct a morphism whose fibres are the C_u's considering the closed subscheme $\mathcal{X} \subset \mathbb{P}^3 \times \mathbf{A}^1$ defined by the ideal

$$J = (X_2, X_3) \cap (X_1, X_3 - uX_0) = (X_1X_2, X_1X_3, X_2(X_3 - uX_0), X_3(X_3 - uX_0))$$

of $\mathbf{k}[u, X_0, \ldots, X_3]$. From Hartshorne(1977), Prop. III.9.7, it follows that $\mathcal{X}$ is flat over $\mathbf{A}^1$. We have:

$$\begin{aligned} \mathcal{X}(u) &= C_u \qquad u \neq 0 \\ \mathcal{X}(0) &= \mathrm{Proj}(\mathbf{k}[\underline{X}]/(X_1X_2, X_1X_3, X_2X_3, X_3^2)) \end{aligned}$$

and $\mathcal{X}(0) \neq C_0$: indeed

$$\mathcal{X}(0) = C_0 \cup \mathrm{Proj}(\mathbf{k}[\underline{X}]/(X_1, X_2, X_3^2))$$

is a non-reduced scheme obtained from C_0 by adjoining an embedded point in $(1, 0, 0, 0)$. In particular we see that $\mathcal{X}(0)$ and C_0 have the same support. Prop. III.9.8 of Hartshorne(1977) implies that $\mathcal{X}(0)$ is uniquely determined by the other fibres, i.e. by $\mathcal{X} \cap [\mathbb{P}^3 \times (\mathbf{A}^1 \backslash \{0\})]$.

Fix a scheme S and a coherent sheaf $\mathcal{F}$ on $\mathbb{P}^r \times S$. Consider a morphism $g : T \to S$ and the diagram

$$\begin{array}{ccc} \mathbb{P}^r \times T & \xrightarrow{h} & \mathbb{P}^r \times S \\ \downarrow q & & \downarrow p \\ T & \xrightarrow{g} & S \end{array}$$

where $h = id \times g$. For every open set $U \subset S$ we have homomorphisms

$$H^j(\mathbb{P}^r \times U, \mathcal{F}) \to H^j(\mathbb{P}^r \times g^{-1}(U), h^*\mathcal{F}) \to H^0(g^{-1}(U), R^j q_*(h^*\mathcal{F}))$$

and therefore a homomorphism

$$R^j p_* \mathcal{F} \to g_*[R^j q_*(h^*\mathcal{F})]$$

which corresponds to a homomorphism

$$g^*(R^j p_* \mathcal{F}) \to R^j q_*(h^*\mathcal{F})$$

In case $j = 0$ we have the following asymptotic result which will applied later in this Section:

(IV.2.4) PROPOSITION *For all $m \gg 0$ the homomorphism*

$$g^*(p_*\mathcal{F}(m)) \to q_*(h^*\mathcal{F}(m))$$

is an isomorphism and, if T is noetherian, $R^j q_*(h^*\mathcal{F}(m)) = 0$ all $j \geq 1$.

Proof
We have

$$h^*\mathcal{F} = \Gamma_*(h^*\mathcal{F})\tilde{} := \left[\oplus_m q_*(h^*\mathcal{F}(m))\right]\tilde{}$$

Since $\mathcal{F} = \Gamma_*(\mathcal{F})\tilde{}$ we also have

$$h^*\mathcal{F} = h^*\left[\Gamma_*(\mathcal{F})\tilde{}\right] = \left[\oplus_m g^*(p_*\mathcal{F})(m)\right]\tilde{}$$

and therefore for all $m \gg 0$

$$g^*(p_*\mathcal{F}(m)) \cong q_*(h^*\mathcal{F}(m))$$

For the last assertion cover T by finitely many affine open sets and apply Theorem III.5.2 of Hartshorne(1977). *q.e.d.*

The homomorphism of Proposition (IV.2.4) is particularly important when $g : \mathrm{Spec}(\mathbf{k}(s)) \to S$ is the inclusion in S of a point $s \in S$; it is denoted

$$t^j(s) : R^j p_*(\mathcal{F})_s \otimes \mathbf{k}(s) \to H^j(\mathbb{P}^r(s), \mathcal{F}(s))$$

The study of these homomorphisms is carried out in [EGA], Ch. III_2 (see also Chapter III, section 12, of Hartshorne(1977)). Their main properties are summarized in the following Theorem and in its Corollary.

(IV.2.5) THEOREM *Let S be a scheme, $\mathcal{F}$ a coherent sheaf on $\mathbb{P}^r \times S$, flat over S, $s \in S$ and $j \geq 0$ an integer. Then:*
(i) If $t^j(s)$ is surjective then it is an isomorphism.
(ii) If $t^{j+1}(s)$ is an isomorphism then $R^{j+1}p_*(\mathcal{F})$ is free at s if and only if $t^j(s)$

is an isomorphism.
(iii) If $R^jp_*(\mathcal{F})$ is free at s for all $j \geq j_0+1$ then $t^j(s)$ is an isomorphism for all $j \geq j_0$.

Proof
(i) and (ii) are Theorem III.12.11 of Hartshorne(1977). (iii) follows from (i) and (ii) by descending induction on j_0. *q.e.d.*

(IV.2.6) COROLLARY *Let $\mathcal{X} \to S$ be a projective morphism, and let $\mathcal{F}$ be a coherent sheaf on $\mathcal{X}$, flat over S. Then:*
(i) If $H^{j+1}(\mathcal{X}(s), \mathcal{F}(s)) = 0$ for some $s \in S$ and $j \geq 0$ then $R^{j+1}p_*(\mathcal{F})_s = (0)$, and

$$t^j(s) : R^jp_*(\mathcal{F})_s \otimes \mathbf{k}(s) \to H^j(\mathcal{X}(s), \mathcal{F}(s))$$

is an isomorphism.
(ii) Let j_0 be an integer such that

$$H^j(\mathcal{X}(s), \mathcal{F}(s)) = 0$$

for all $j \geq j_0+1$ and $s \in S$ (e.g. $j_0 = \max_{s\in S}\{\dim[\mathrm{Supp}(\mathcal{F}(s))]\}$). Then $t^{j_0}(s)$ is an isomorphism for all $s \in S$.
(iii) Let $j_0 \geq 0$ be an integer. Then there is a non empty open set $U \subset S$ such that $t^{j_0}(s)$ is an isomorphism for all $s \in U$.

Proof
(i) follows immediately from (IV.2.5). (ii) is a special case of (i).
(iii) It is the open set $U = \cap_{j\geq j_0} U_j$, where $U_j = \{s \in S : R^jp_*(\mathcal{F})_s \text{ is free}\}$ (apply (IV.2.5)(iii)). *q.e.d.*

* * * * * *

Stratifications

Let S be a scheme. A *stratification* of S consists of a set of finitely many locally closed subschemes $\{S_1, \ldots, S_n\}$ of S, called *strata*, pairwise disjoint and such that $S = S_1 \cup \ldots \cup S_n$.

Let $\mathcal{F}$ be a coherent sheaf on S and for each $s \in S$ let

$$e(s) := \dim_{\mathbf{k}(s)}[\mathcal{F}_s \otimes \mathbf{k}(s)]$$

Fix a point $s \in S$, let $e = e(s)$ and let $a_1, \ldots, a_e \in \mathcal{F}_s$ be such that their images in $\mathcal{F}_s \otimes \mathbf{k}(s)$ form a basis. From Nakayama's Lemma it follows that the homomorphism $f_s : \mathcal{O}^e_{S,s} \to \mathcal{F}_s$ defined by $a_1, \ldots, a_e$ is surjective; therefore there is an open neighborhood U of s to which f extends defining a surjective homomorphism $f : \mathcal{O}^e_U \to \mathcal{F}_{|U}$. With a similar argument applied to $\ker(f_s)$ we may find an affine open neighborhood $U(s)$ of s contained in U and an exact sequence

$$[IV.2.1] \qquad \mathcal{O}^d_{U(s)} \xrightarrow{g} \mathcal{O}^e_{U(s)} \xrightarrow{f} \mathcal{F}_{|U(s)} \to 0$$

It follows that:

(i) $e(s') \leq e(s)$ for all $s' \in U(s)$: therefore $s \mapsto e(s)$ is an upper semicontinuous function from S to $\mathbb{Z}$.

(ii) Let (g_{ij}) be the $e \times d$ matrix with entries in $H^0(U(s), \mathcal{O}_S)$ which defines g. The ideal generated by the g_{ij}'s in $H^0(U(s), \mathcal{O}_S)$ defines a closed subscheme Z_s of $U(s)$ with support equal to $Y_e \cap U(s)$, where for each $e \geq 0$ we have set $Y_e = \{s \in S : e(s) = e\}$. In particular Y_e is a locally closed subset of S.

Moreover

(iii) If $q : T \to U(s)$ is a morphism; $q^*(\mathcal{F})$ is locally free of rank e if and only if q factors through the subscheme Z_s.

Proof

q factors through Z_s if and only if all the functions $q^*(g_{ij})$ are zero on T. Since the sequence

$$\mathcal{O}_T^d \xrightarrow{q^*(g)} \mathcal{O}_T^e \xrightarrow{q^*(f)} q^*(F) \to 0$$

is exact on T, this is equivalent to $q^*(f)$ being an isomorphism and this condition implies that $q^*(f)$ is locally free of rank e. Conversely if $q^*(f)$ is locally free of rank e, let $\mathcal{G} = \ker[q^*(f)]$. At every point $t \in T$ we have an exact sequence:

$$0 \to \mathcal{G} \otimes \mathbf{k}(t) \to \mathbf{k}(t)^e \to q^*(\mathcal{F}) \otimes \mathbf{k}(t) \to 0$$

Since $q^*(\mathcal{F}) \otimes \mathbf{k}(t)$ is a vector space of dimension e we have $\mathcal{G} \otimes \mathbf{k}(t) = (0)$. By Nakayama's lemma $\mathcal{G} = (0)$ in a neighborhood of t and therefore $\mathcal{G} = (0)$ everywhere.

(iv) Since property (iii) characterizes the scheme Z_s and does not depend on the presentation [IV.2.1], for any $s, s' \in S$ the schemes Z_s and $Z_{s'}$ coincide on $U(s) \cap U(s')$; therefore the collection of schemes $\{Z_s : s \in S\}$ defines a locally closed subscheme Z_e of S supported on Y_e. Evidently $\{Z_e : e \geq 0\}$ is a stratification of S.

(v) Because of (i), for each e the closure of Z_e is contained in $\cup_{e' \geq e} Z_{e'}$. In particular, if E is the highest integer such that $Z_E \neq \emptyset$, then Z_E is closed.

We have proved the following

(IV.2.7) THEOREM *Let S be a scheme and $\mathcal{F}$ a coherent sheaf on S. There is a unique stratification $\{Z_e\}_{e \geq 0}$ of S such that if $q : T \to S$ is a morphism the sheaf $q^*(\mathcal{F})$ is locally free if and only if q factors through the disjoint union of the Z_e's: $T \to \coprod_e Z_e \to S$.*
Moreover the strata $Z_0, Z_1, \ldots$ are indexed so that for each $e = 0, 1, \ldots$ the restriction of $\mathcal{F}$ to Z_e is locally free of rank e.
For a given e, $\bar{Z}_e \subset \cup_{e' \geq e} Z_{e'}$. In particular, if E is the highest integer such that $Z_E \neq \emptyset$, then Z_E is closed.

Theorem (IV.2.7) describes a natural way to construct stratifications on a scheme. $\{Z_e\}_{e \geq 0}$ is called *the stratification defined by the sheaf $\mathcal{F}$*.

(IV.2.8) EXAMPLE. Let $\varphi : \mathbf{A} \to \mathbf{B}$ be a homomorphism of locally free sheaves on the scheme S, of ranks a and b respectively. Applying Theorem (IV.2.7) to coker(φ) we obtain a stratification of S with the property that Z_{b-e} is supported on the locus

$$\{s \in S : \mathrm{rk}[\varphi(s) : \mathbf{A}(s) \to \mathbf{B}(s)] = e\}$$

The scheme Z_{b-e} of this stratification will be denoted $\mathrm{D}_e(\varphi)$. Note that in particular the subscheme $\mathrm{D}_0(\varphi)$, called the *vanishing scheme* of φ, is closed in S because of (v) above. It has the property that a morphism $f : T \to S$ satisfies $f^*(\varphi) = 0$ is and only if f factors through $D_0(\varphi)$.

The ideal sheaf of $D_0(\varphi)$ is locally generated by the entries of a matrix representing φ. More intrinsecally it can be obtained as follows. Since $\varphi \in \mathrm{Hom}(\mathbf{A}, \mathbf{B})$, it induces by adjunction a homomorphism:

$$Hom(\mathbf{B}, \mathbf{A}) \xrightarrow{\varphi^\vee} \mathcal{O}_S$$

whose image is just the ideal sheaf of $D_0(\varphi)$.

* * * * * *

Flattening stratifications

(IV.2.9) DEFINITION *Let S be a scheme and $\mathcal{F}$ a coherent sheaf on $\mathbb{P}^r \times S$. A* flattening stratification *for $\mathcal{F}$ is a stratification $\{S_1, \ldots, S_n\}$ of S such that for every morphism $g : T \to S$ the sheaf*

$$\mathcal{F}_g := (1 \times g)^*(\mathcal{F})$$

on $\mathbb{P}^r \times T$ is flat over T if and only if g factors through $\coprod S_i$.

Note that if such a stratification exists it is clearly unique. In the special case $r = 0$ we obtain again the notion of stratification defined by the sheaf $\mathcal{F}$.

The following is a basic technical result.

(IV.2.10) THEOREM *For every coherent sheaf $\mathcal{F}$ on $\mathbb{P}^r \times S$ the flattening stratification exists.*

Proof

The Theorem has already been proved in the case $r = 0$ (Theorem (IV.2.7)). Therefore we may assume $r \geq 1$. We will proceed in several steps.

Step 1): There are finitely many locally closed subsets $Y^1, \ldots, Y^k$ of S such that for each $i = 1, \ldots, k$ if we consider on Y^i the reduced scheme structure then $\mathcal{F} \otimes \mathcal{O}_{Y^i \times \mathbb{P}^r}$ is flat over Y^i.

It follows immediately from a repeated use of the fact that there is a nonempty open subset $U \subset S$ such that $\mathcal{F}_{|\mathbb{P}^r \times U_{red}}$ is flat over U_{red} (see Note 7).

Step 2): Only finitely many polynomials $P^1, \ldots, P^h$ occur as Hilbert polynomials of the sheaves $\mathcal{F}(s)$, $s \in S$.

In fact from Corollary (IV.2.2) it follows that at most as many Hilbert polynomials occur as the number of connected components of the sets $Y^1, \dots, Y^k$.

Step 3): There is an integer N such that for every $m \geq 0$ and for every $s \in S$ we have:

$$H^j(\mathbb{P}^r(s), \mathcal{F}(s)(N+m)) = (0)$$

for $j \geq 1$ and the natural map:

$$[p_*\mathcal{F}(N+m)]_s \otimes \mathbf{k}(s) \to H^0(\mathbb{P}^r(s), \mathcal{F}(s)(N+m))$$

is an isomorphism, where $p : \mathbb{P}^r \times S \to S$ is the projection.

For each $i = 1, \dots, k$ consider the diagram

$$[IV.2.2]_i \qquad \begin{array}{cccc} h^i: & \mathbb{P}^r \times Y^i & \to & \mathbb{P}^r \times S \\ & \downarrow p_i & & \downarrow p \\ & Y^i & \to & S \end{array}$$

and let $n_i \gg 0$ be so that $R^j p_{i*}[h^{i*}\mathcal{F}(n_i + m)] = (0)$ for all $m \geq 0$ and all $j \geq 1$ (apply Proposition (IV.2.4)). Letting

$$N \gg \max\{n_1, \dots, n_k\}$$

we may apply Proposition (IV.2.4) to the diagrams $[IV.2.2]_i$ and to the sheaf $\mathcal{F}$ and we obtain isomorphisms

$$[p_*\mathcal{F}(N+m)] \otimes \mathcal{O}_{Y^i} \cong p_{i*}[h^{i*}\mathcal{F}(N+m)]$$

for all $s \in Y^i$ and for all $i = 1, \dots, k$. In particular we have isomorphisms

$$[IV.2.3] \qquad [p_*\mathcal{F}(N+m)] \otimes \mathbf{k}(s) \cong p_{i*}[h^{i*}\mathcal{F}(N+m)]_s \otimes \mathbf{k}(s)$$

for all $s \in Y^i$ and for all $i = 1, \dots, k$. We may also apply Corollary (IV.2.6) to the sheaves $h^{i*}\mathcal{F}$ and to the projections p_i for $j_0 = 0$ to deduce that

$$H^j(\mathbb{P}^r(s), \mathcal{F}(s)(N+m)) = (0)$$

for all $s \in S$, $j \geq 1$ and $m \geq 0$, and that

$$[IV.2.4] \qquad p_{i*}[h^{i*}\mathcal{F}(N+m)]_s \otimes \mathbf{k}(s) \cong H^0(\mathbb{P}^r(s), \mathcal{F}(s)(N+m))$$

for all $s \in Y^i$ and for all $i = 1, \dots, k$ and all $m \geq 0$.

Comparing [IV.2.3] and [IV.2.4] we obtain the conclusion.

Step 4): Let N be as in Step 3, and let $g : T \to S$ be a morphism. Then $\mathcal{F}_g$ is flat over T if and only if $g^*[p_*\mathcal{F}(N+m)]$ is locally free for all $m \geq 0$.

Suppose that $\mathcal{F}_g$ is flat over T and let $q : I\!P^r \times T \to T$ be the projection. Since

$$H^j(I\!P^r(t), \mathcal{F}_g(t)(N+m)) = H^j(I\!P^r(g(t)), \mathcal{F}(g(t))(N+m)) = (0)$$

for all $t \in T$, $m \geq 0$ and $j \geq 1$, from Corollary (IV.2.6)(ii) we deduce that

$$[IV.2.5] \qquad q_*\mathcal{F}_g(N+m)_t \otimes \mathbf{k}(t) \to H^0(I\!P^r(g(t)), \mathcal{F}(g(t))(N+m))$$

is an isomorphism for all $t \in T$. Theorem (IV.2.5)(ii) applied for $j = -1$ implies that $q_*\mathcal{F}_g(N+m)$ is locally free for all $m \geq 0$. For all $t \in T$ the natural homomorphism

$$\varphi : g^*[p_*\mathcal{F}(N+m)] \to q_*\mathcal{F}_g(N+m)$$

induces an isomorphism:

$$g^*[p_*\mathcal{F}(N+m)]_t \otimes \mathbf{k}(t) \cong q_*\mathcal{F}_g(N+m)_t \otimes \mathbf{k}(t)$$

because both sides are isomorphic to $H^0(I\!P^r(g(t)), \mathcal{F}(g(t))(N+m))$ (the first because of Step 3, the second because of [IV.2.5]). From the fact that $q_*\mathcal{F}_g(N+m)$ is locally free and from Nakayama's Lemma it follows that φ is an isomorphism. Therefore $g^*[p_*\mathcal{F}(N+m)]$ is locally free for every $m \geq 0$.

Conversely suppose that $g^*[p_*\mathcal{F}(N+m)]$ is locally free for all $m \geq 0$. Since for all $m \gg 0$ the natural map φ is an isomorphism (Prop. (IV.2.4)) it follows that $q_*\mathcal{F}_g(N+m)$ is locally free for all $m \gg 0$: Proposition (IV.2.1) implies that $\mathcal{F}_g$ is flat.

Step 5): For every $m \geq 0$ apply Theorem (IV.2.7) to the sheaf $p_*\mathcal{F}(N+m)$ and let $Y_{m,j}$ be the component of the corresponding stratification of S where $p_*\mathcal{F}(N+m)$ becomes locally free of rank j. Then for each $j = 1, \ldots, h$ we have the following equality of subsets of S:

$$\cap_{m\geq 0}\text{Supp}(Y_{m,P^i(N+m)}) = \cap_{m=0,\ldots,r}\text{Supp}(Y_{m,P^i(N+m)})$$

The inclusion $\subset$ is obvious. For $s \in S$ let $P_s(t)$ be the Hilbert polynomial of $\mathcal{F}(s)$. Then $s \in \cap_{m\geq 0}\text{Supp}(Y_{m,P^i(N+m)})$ if and only if

$$P_s(N+m) = h^0(I\!P^r(s), \mathcal{F}(s)(N+m)) = \dim[p_*\mathcal{F}(N+m)_s \otimes \mathbf{k}(s)] = P^i(N+m)$$

for all $m \geq 0$, and this happens if and only if $P_s(t) = P^i(t)$ as polynomials. On the other hand $s \in \cap_{m=0,\ldots,r}\text{Supp}(Y_{m,P^i(N+m)})$ if and only if $P_s(N+m) = P^i(N+m)$ for $m = 0, \ldots, r$. Since both $P_s(t)$ and $P^i(t)$ have degree $\leq r$, it follows that $P_s(t) = P^i(t)$ and therefore $s \in \cap_{m\geq 0}\text{Supp}(Y_{m,P^i(N+m)})$.

Step 6): Fix i between 1 and h. For each integer $c \geq 0$ the finite intersection

$$\cap_{m=0,\ldots,c}Y_{m,P^i(N+m)}$$

is a well defined locally closed subscheme of S. Because of Step 5 the subschemes

$$\cap_{m=0,\ldots,c} Y_{m,P^i(N+m)} \qquad c = r, r+1, \ldots$$

for a descending chain with fixed support; in particular they form a descending chain of closed subschemes of a *fixed* open set $V \subset S$, and therefore they stabilize. In other words the intersection

$$Z^i = \cap_{m\geq 0} \mathrm{Supp}(Y_{m,P^i(N+m)})$$

is a well defined locally closed subscheme of S. By Step 5 we have:

$$\mathrm{Supp}(Z^i) = \{s \in S : P_s(t) = P_i(t)\}$$

Step 7): The subschemes $Z^1, \ldots, Z^h$ form a stratification of S. It follows immediately from Step 4 that this is the flattening stratification for $\mathcal{F}$. This concludes the proof of Theorem (IV.2.10). *q.e.d.*

NOTES

1. From the proof of Theorem (IV.2.10) it follows that the strata $Z^1, \ldots, Z^h$ of the flattening stratification for $\mathcal{F}$ are indexed by the Hilbert polynomials of the sheaves $\mathcal{F}(s)$, $s \in S$.

2. Let

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{\Phi} & \mathcal{Y} \\ & \searrow\swarrow & \\ & S & \end{array}$$

be a commutative diagram of morphisms of algebraic schemes, with $\mathcal{X}$ and $\mathcal{Y}$ S-flat and $\mathcal{X}$ projective over S. Assume that $\Phi_o : \mathcal{X}(o) \to \mathcal{Y}(o)$ is a closed embedding, for some $\mathbf{k}$-rational point $o \in S$. Then there is an open neighborhood $U \subset S$ of o such that the restriction $\Phi(U) : \mathcal{X}(U) \to \mathcal{Y}(U)$ is a closed embedding.

Proof

Let $\mathcal{K} = \mathrm{coker}[\mathcal{O}_{\mathcal{Y}} \to \Phi_*(\mathcal{O}_{\mathcal{X}})]$. Since Φ is projective $\Phi_*(\mathcal{O}_{\mathcal{X}})$ is a coherent sheaf and so is $\mathcal{K}$. Moreover $\mathcal{K}(o) = (0)$ because Φ_o is a closed embedding. It follows that there is an open subset $U \subset S$ containing o such that $\mathcal{K}_{|\mathcal{Y}(U)} = (0)$. Let $\mathcal{Z} = \mathrm{Spec}(\Phi_*(\mathcal{O}_{\mathcal{X}}))$, $h : \mathcal{Z} \to \mathcal{Y}$ the induced S-morphism and

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{\Phi} & \mathcal{Y} \\ & g \searrow\nearrow h & \\ & \mathcal{Z} & \end{array}$$

the Stein factorization of Φ. Then it follows that $h(U) : \mathcal{Z}(U) \to \mathcal{Y}(U)$ is a closed embedding. Moreover, since g has connected fibres and is bijective over $\mathcal{Z}(o)$, it follows that, modulo shrinking U if necessary, $g(U) : \mathcal{X}(U) \to \mathcal{Z}(U)$ is an isomorphism. The conclusion follows. *q.e.d.*

3. Let $\mathcal{X} \to S$ be a flat projective morphism of algebraic schemes, and $\mathcal{L}$ an invertible sheaf on $\mathcal{X}$. Assume that for some $\mathbf{k}$-rational point $o \in S$ the sheaf $\mathcal{L}(o)$ is very ample on $\mathcal{X}(o)$ and satisfies $H^1(\mathcal{X}(o), \mathcal{L}(o)) = 0$. Then there is an open neighborhood $V \subset S$ of o such that $\mathcal{L}_V := \mathcal{L}_{|\mathcal{X}(V)}$ is very ample relative to V. In particular $\mathcal{L}(s)$ is very ample on $\mathcal{X}(s)$ for every $s \in V$.

Proof

By Corollary (IV.2.6) there is an open neighborhood $U \subset S$ of o such that $(R^1 f_*\mathcal{L})_{|U} = (0)$ and

$$t^0(u) : (f_*\mathcal{L})_u \otimes \mathbf{k}(u) \to H^0(\mathcal{X}(u), \mathcal{L}(u))$$

is an isomorphism for all $u \in U$. We may even assume that $f_*\mathcal{L}$ is locally free of rank $h^0(\mathcal{X}(o), \mathcal{L}(o))$ on U. From the surjectivity of the map $t^0(o)$ and from the fact that $\mathcal{L}(o)$ is globally generated we deduce that the canonical homomorphism :

$$f^*(f_*\mathcal{L}) \to \mathcal{L}$$

is surjective on $\mathcal{X}(o)$. Since f is projective it follows that there is an open $W \subset U$ containing o such that

$$[f^*(f_*\mathcal{L})]_{|\mathcal{X}(W)} \to \mathcal{L}_W \qquad [IV.2.6]$$

is surjective and moreover $[f^*(f_*\mathcal{L})]_{|\mathcal{X}(W)}$ is locally free. The homomorphism [IV.2.6] defines a W-morphism

$$\begin{array}{ccccc} \mathcal{X}(W) & & \to & & \mathbb{P}(f^*(f_*\mathcal{L}))_{|\mathcal{X}(W)} \\ & \searrow & & \swarrow & \\ & & W & & \end{array} \qquad [IV.2.7]$$

whose restriction to $\mathcal{X}(o)$ is the embedding defined by the global sections of $\mathcal{L}(o)$. From Note 2 above it follows that there is an open subset $V \subset W$ containing o such that the restriction of [IV.2.7] to $\mathcal{X}(V)$ is an embedding. This implies the conclusion.

4. Let $\mathcal{E}$ be a locally free sheaf over $\mathbb{P}^1 \times S$, with S an algebraic integral scheme. Let $o \in S$ be a $\mathbf{k}$-rational point, and $\mathcal{E}(o) \cong \oplus_i \mathcal{O}(n_0^i)$ the fibre over o. Then

(i) there is an open set $U \subset S$ such that for each $s \in U$ we have

$$\mathcal{E}(s) \cong \oplus_i \mathcal{O}(n_s^i)$$

with

$$\max_i\{n_s^i\} \le \max_i\{n_o^i\} \ \text{ and } \min_i\{n_s^i\} \ge \min_i\{n_o^i\}$$

Moreover if $\mathcal{E}(o)$ is balanced (i.e. $n_o^i = n_o^j$ for all i, j) then $\mathcal{E}(s) \cong \mathcal{E}(o)$ for all $s \in U$.
(ii) For each $s \in S$ we have

$$\sum_i n_s^i = \sum_i n_0^i$$

Proof
(Caporaso-Sernesi(2003)) (i) By the structure theorem for locally free sheaves on $\mathbb{P}^1$ (see Okonek et al.(1980)) we know that for each $s \in S$ we have an isomorphism $\mathcal{E}(s) \cong \oplus_i \mathcal{O}(n_s^i)$ for some integers n_s^i. Let $M_0 = \max_i\{n_0^i\}$ and consider the sheaf $\bar{\mathcal{E}} := \mathcal{E} \otimes p^*\mathcal{O}(-M_0 - 1)$, where $p : \mathbb{P}^1 \times S \to \mathbb{P}^1$ is the projection. Since $h^0(\bar{\mathcal{E}}(0)) = 0$, from the Semicontinuity Theorem it follows that there is an open neighborhood U of 0 such that $h^0(\bar{\mathcal{E}}(s)) = 0$ for all $s \in U$; but this means that $\max_i\{n_s^i\} \leq M_0$ for all $s \in U$, which is the first statement of the Proposition. The statement about the minimum is proved similarly after replacing $\mathcal{E}$ by its dual. The last assertion is obvious.

(ii) Applying (i) to $\det(\mathcal{E})$ we find that every point $t \in S$ has an open neighborhood U_t where $\sum_i n_s^i = \sum_i n_t^i$ for all $s \in U_t$. Since S is connected we deduce that $\sum_i n_s^i$ is constant.

5. Let $f : \mathcal{X} \to S$ be a flat projective morphism with S an algebraic scheme, and let $o \in S$ be a **k**-rational point. Prove that:
(i) If $\mathcal{X}(o)$ is connected and $\mathcal{X}(s)$ is disconnected for all $s \neq o$ in an open neighborhood of o then $\mathcal{X}(o)$ is non-reduced.
In particular:
(ii) If $\mathcal{X}(o)$ is connected and reduced then $\mathcal{X}(s)$ is connected for all s in an open neighborhood of o.
(iii) If $\mathcal{X}(o)$ is disconnected then $\mathcal{X}(s)$ is disconnected for all s in an open neighborhood of o.

6. Let $f : X \to Y$ be a proper morphism of algebraic schemes with finite fibres. Let $g : Y' \to Y$ be an arbitrary morphism, $X' = X \times_Y Y'$, $f' : X' \to Y'$ and $g' : X' \to X$ the projections. Then for every quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ we have a canonical isomorphism

$$g^*(f_*\mathcal{F}) \cong f'_*(g'^*\mathcal{F})$$

Proof
Since it is proper and quasi-finite, f is finite, in particular it is affine. The conclusion follows from [EGA] Ch. II, 1.5.2. *q.e.d.*

7. Let $f : \mathcal{X} \to S$ be a morphism of finite type with S integral, and let $\mathcal{F}$ be a coherent sheaf on $\mathcal{X}$. There is a dense open subset $U \subset S$ such that the restriction of $\mathcal{F}$ to $f^{-1}(U)$ is flat over U.

Proof
The conclusion being local in S we may assume that $S = \mathrm{Spec}(A)$ where A is an integral **k**-algebra. Since $\mathcal{X}$ can be covered by finitely many affine

open sets it suffices to prove the assertion for each of them: therefore we may assume that $\mathcal{X} = \mathrm{Spec}(B)$, where B is an A-algebra of finite type and $\mathcal{F}$ corresponds to a B-module of finite type M. It will suffice to show that

$(*)$ there exists $a \in A$ such that M_a is a free A_a-module.

Note that if

$$0 \to L \to M \to N \to 0$$

is an exact sequence of B-modules such that L_a is A_a-free and N_c is A_c-free for some $a, c \in A$, then M_{ac} is A_{ac}-free. Moreover, since M is finitely generated, there is a composition series

$$M = M_0 \supset \cdots \supset M_n = (0)$$

such that every quotient M_i/M_{i+1} is isomorphic to B/p_i for some prime ideal $p_i \subset B$. Therefore it suffices to prove $(*)$ for modules of the form B/p where p is a prime ideal, i.e. we may assume that B is a domain and that $M = B$.

Let K be the quotient field of A. Then $B_K = B \otimes_A K$ is a K-algebra of finite type. We will prove $(*)$ by induction on the dimension d of $\mathrm{Spec}(B_K)$.

If $d = -1$, i.e. $B_K = (0)$, there exists $a \in A$ such that $a1_B = 0$: it follows that $B_a = (0)$ is A_a-free.

Let $d \geq 0$. By Nöther's Normalization Lemma there exist $b_1, \ldots, b_d \in B_K$ algebraically independent and such that B_K is integral over $K[b_1, \ldots, b_d]$. It follows easily that there exists $c \in A$ such that $b_1, \ldots, b_d \in B_c$ and B_c is integral over $A_c[b_1, \ldots, b_d]$; in particular B_c is an $A_c[b_1, \ldots, b_d]$-module of finite type. Therefore it is possible to find an exact sequence of $A_c[b_1, \ldots, b_d]$-modules

$$0 \to A_c[b_1, \ldots, b_d]^m \to B_c \to C \to 0$$

such that C is torsion. Since $A_c[b_1, \ldots, b_d]^m$ is A_c-free it suffices to prove $(*)$ for C. As before we may use a composition series for C to reduce to the case when C is an integral A_c-algebra and, since C is torsion, $\dim[\mathrm{Spec}(C_K)] < d$. We conclude by induction. *q.e.d.*

IV.3. THE HILBERT SCHEMES - INTRODUCTION

Consider a projective scheme Y, and a closed embedding $Y \subset \mathbb{P}^r$. Let's fix a *numerical polynomial* of degree $\leq r$, i.e. a polynomial $P(t) \in \mathbf{Q}[t]$ of the form:

$$P(t) = \sum_{i=0}^{r} a_i \binom{t+r}{i}$$

with $a_i \in \mathbb{Z}$ all i.

For every scheme S we let:

$$Hilb^Y_{P(t)}(S) = \left\{ \begin{array}{c} \text{flat families } \mathcal{X} \subset Y \times S \text{ of closed subschemes of } Y \\ \text{parametrized by } S \text{ with fibres having Hilbert polynomial } P(t) \end{array} \right\}$$

Since flatness is preserved under base change, this defines a contravariant functor

$$Hilb^Y_{P(t)} : (\text{schemes})^\circ \rightarrow (\text{sets})$$

called the *Hilbert functor of Y relative to* $P(t)$.

In case $Y = \mathbb{P}^r$ we will denote the Hilbert functor with the symbol $Hilb^r_{P(t)}$.

If the functor $Hilb^Y_{P(t)}$ is representable, the scheme representing it will be called the *Hilbert scheme of Y relative to* $P(t)$, and will be denoted $\mathrm{Hilb}^Y_{P(t)}$ (or $\mathrm{Hilb}^r_{P(t)}$ in case $Y = \mathbb{P}^r$). If $P(t) = n$ a constant polynomial then $\mathrm{Hilb}^Y_{P(t)}$ is usually denoted by $Y^{[n]}$.

If the Hilbert scheme $\mathrm{Hilb}^Y_{P(t)}$ exists then there is a universal element, i.e. there is a flat family of closed subschemes of Y having Hilbert polynomial equal to $P(t)$:

[*IV*.3.1] $$\mathcal{W} \subset Y \times \mathrm{Hilb}^Y_{P(t)}$$

parametrized by $\mathrm{Hilb}^Y_{P(t)}$ and possessing the following

UNIVERSAL PROPERTY: For each scheme S and for each flat family $\mathcal{X} \subset Y \times S$ of closed subschemes of Y having Hilbert polynomial $P(t)$ there is a unique morphism $S \rightarrow \mathrm{Hilb}^Y_{P(t)}$, called the *classifying map*, such that

$$\mathcal{X} = S \times_{\mathrm{Hilb}^Y_{P(t)}} \mathcal{W} \subset Y \times S$$

The family [IV.3.1] is called the *universal family*, and the pair $(\mathrm{Hilb}^Y_{P(t)}, \mathcal{W})$ represents the functor $Hilb^Y_{P(t)}$.

The family $\mathcal{W}$ is the universal element of $Hilb^Y_{P(t)}(\mathrm{Hilb}^Y_{P(t)})$, namely the element corresponding to the identity under the identification

$$\mathrm{Hom}(\mathrm{Hilb}^Y_{P(t)}, \mathrm{Hilb}^Y_{P(t)}) = Hilb^Y_{P(t)}(\mathrm{Hilb}^Y_{P(t)})$$

EXAMPLE Consider the constant polynomial $P(t) = 1$. Then we have a canonical identification $Y^{[1]} = Y$ and the universal family is the diagonal $\Delta \subset Y \times Y$.

To prove it consider an element of $Y^{[1]}(S)$ for some scheme S:

$$\begin{array}{lll} \Gamma & \subset & S \times Y \\ \downarrow f & & \\ S & & \end{array}$$

Then f is an isomorphism: infact it is a one-to-one morphism and $\mathcal{O}_S \to f_*\mathcal{O}_\Gamma$ is an isomorphism since $f_*\mathcal{O}_\Gamma$ is an $\mathcal{O}_S$-algebra which is locally free of rank one over $\mathcal{O}_S$. We therefore have the well defined morphism $g = qf^{-1} : S \to Y$ where $q : S \times Y \to Y$ is the projection. The morphism

$$(gf, q) : \Gamma \to Y \times Y$$

factors through Δ and induces a commutative diagram

$$\begin{array}{ccc} \Gamma & \to & \Delta \\ \downarrow & & \downarrow \\ S & \xrightarrow[g]{} & Y \end{array}$$

such that $\Gamma \cong g^*\Delta$. Therefore the family Γ is induced by Δ via the morphism g.

The existence of the Hilbert schemes in general will be proved in the next Section. We will now consider two important special cases.

* * * * * *

Hypersurfaces

If $X \subset \mathbb{P}^r$ is a hypersurface of degree d it has Hilbert polynomial

$$h(t) = \binom{t+r}{r} - \binom{t+r-d}{r} = \frac{d}{(r-1)!}\, t^{r-1} + \cdots$$

Conversely, if a closed subscheme Y of $\mathbb{P}^r$ has Hilbert polynomial $h(t)$ then it is a hypersurface of degree d.

Infact, since $h(t)$ has degree $r-1$, Y has dimension $r-1$, so $Y = Y_1 \cup Z$, with Y_1 a hypersurface and $\dim(Z) < r-1$. We have the exact sequence:

$$0 \to \mathcal{I}_{Y_1}/\mathcal{I}_Y \to \mathcal{O}_Y \to \mathcal{O}_{Y_1} \to 0$$

where $\mathcal{I}_Y, \mathcal{I}_{Y_1} \subset \mathcal{O}_{\mathbb{P}^r}$ are the ideal sheaves of Y and Y_1. We deduce that

$$h(t) = h_1(t) + k(t)$$

where $h_1(t)$ is the Hilbert polynomial of Y_1 and $k(t)$ is the Hilbert polynomial of $\mathcal{I}_{Y_1}/\mathcal{I}_Y$. Since this sheaf is supported on Z, we have $\deg(k(t)) < r-1$; therefore we see that

$$h_1(t) = \frac{d}{(r-1)!}\, t^{r-1} + \cdots$$

so Y_1 is a hypersurface of degree d, and therefore $h_1(t) = h(t)$. It follows that $k(t) = 0$, i.e. $\mathcal{I}_{Y_1} = \mathcal{I}_Y$, equivalently $Y = Y_1$.

Therefore $\mathrm{Hilb}^r_{h(t)}$, if it exists, parametrizes a universal family of hypersurfaces of degree d in $\mathbb{P}^r$. To prove its existence let $V := H^0(\mathbb{P}^r, \mathcal{O}(d))$ and in $\mathbb{P}(V)$ take homogeneous coordinates

$$(\ldots, c_{i(0),\ldots,i(r)}, \ldots)_{i(0)+\ldots+i(r)=d}$$

The hypersurface $\mathcal{H} \subset \mathbb{P}^r \times \mathbb{P}(V)$ defined by the equation

$$\sum c_{i(0),\ldots,i(r)} X_0^{i(0)} \cdots X_r^{i(r)} = 0$$

projects onto $\mathbb{P}(V)$ with fibres hypersurfaces of degree d. It follows from Proposition (IV.2.1) that $\mathcal{H}$ is flat over $\mathbb{P}(V)$. Let's denote by $p : \mathbb{P}^r \times \mathbb{P}(V) \to \mathbb{P}(V)$ the projection, and let $\mathcal{I}_{\mathcal{H}} \subset \mathcal{O}_{\mathbb{P}^r \times \mathbb{P}(V)}$ be the ideal sheaf of $\mathcal{H}$. For all $x \in \mathbb{P}(V)$ we have

$$1 = h^0(\mathbb{P}^r(x), \mathcal{I}_{\mathcal{H}(x)}(d)) = h^0(\mathbb{P}^r(x), \mathcal{I}_{\mathcal{H}}(d)(x))$$

and

$$0 = h^i(\mathbb{P}^r(x), \mathcal{I}_{\mathcal{H}(x)}(d)) = h^i(\mathbb{P}^r(x), \mathcal{I}_{\mathcal{H}}(d)(x))$$

$$0 = h^i(\mathcal{H}(x), \mathcal{O}_{\mathcal{H}(x)}(d))$$

for all $i \geq 1$. Applying (IV.2.5) and (IV.2.6) we deduce that

a) $R^1 p_* \mathcal{I}_{\mathcal{H}}(d) = 0$

b) $p_* \mathcal{I}_{\mathcal{H}}(d)$ is an invertible subsheaf of $p_* \mathcal{O}_{\mathbb{P}^r \times \mathbb{P}(V)}(d) = V \otimes_{\mathbf{k}} \mathcal{O}_{\mathbb{P}(V)}$

c) $p_* \mathcal{O}_{\mathbb{P}^r \times \mathbb{P}(V)}(d)/p_* \mathcal{I}_{\mathcal{H}}(d) = p_* \mathcal{O}_{\mathcal{H}}(d)$ is locally free.

It follows that

$$p_* \mathcal{I}_{\mathcal{H}}(d) = \mathcal{O}_{\mathbb{P}(V)}(-1)$$

the tautological invertible sheaf on $\mathbb{P}(V)$, and the natural map

$$p^* p_* \mathcal{I}_{\mathcal{H}}(d) \to \mathcal{I}_{\mathcal{H}}(d)$$

is an isomorphism. Therefore

$$\mathcal{I}_{\mathcal{H}} = [p^* \mathcal{O}_{\mathbb{P}(V)}(-1)](-d)$$

Let's prove that $\mathcal{H} \subset \mathbb{P}^r \times \mathbb{P}(V)$ is a universal family.
Suppose that

$$\begin{array}{ccc} \mathcal{X} & \subset & \mathbb{P}^r \times S \\ \downarrow f & & \\ S & & \end{array}$$

is a flat family of closed subschemes of $\mathbb{P}^r$ with Hilbert polynomial $h(t)$, i.e. hypersurfaces of degree d, and let $\mathcal{I}_\mathcal{X} \subset \mathcal{O}_{\mathbb{P}^r \times S}$ be the ideal sheaf of $\mathcal{X}$. Arguing as above we deduce that $f_*\mathcal{I}_\mathcal{X}(d)$ is an invertible subsheaf of $V \otimes_\mathbf{k} \mathcal{O}_S$ with locally free cokernel $f_*\mathcal{O}_\mathcal{X}(d)$, and that

$$\mathcal{I}_\mathcal{X} = [f^* f_* \mathcal{I}_\mathcal{X}(d)](-d)$$

We have an induced morphism $g : S \to \mathbb{P}(V)$ such that

$$g^*[\mathcal{O}_{\mathbb{P}(V)}(-1)] = f_*\mathcal{I}_\mathcal{X}(d)$$

The subscheme $S \times_{\mathbb{P}(V)} \mathcal{H} \subset \mathbb{P}^r \times S$ is defined by the ideal sheaf

$$(1 \times g)^* \mathcal{I}_\mathcal{H} = (1 \times g)^*[\mathcal{O}_{\mathbb{P}(V)}(-1)(-d)] =$$

$$= f^*[g^*\mathcal{O}_{\mathbb{P}(V)}(-1)](-d) = [f^* f_* \mathcal{I}_\mathcal{X}(d)](-d) = \mathcal{I}_\mathcal{X}$$

Hence $S \times_{\mathbb{P}(V)} \mathcal{H} = \mathcal{X}$. The proof of the uniqueness of g having this property is left to the reader. Therefore we see that $\mathcal{H} \subset \mathbb{P}^r \times \mathbb{P}(V)$ is a universal family, and $\mathrm{Hilb}^r_{h(t)} = \mathbb{P}(V)$.

* * * * * *

Grassmannians

The classical grassmannians are special cases of Hilbert schemes, since they parametrize linear spaces, which are the closed subschemes with linear Hilbert polynomials. Let's fix a $\mathbf{k}$-vector space V of dimension N, with and let $1 \leq n \leq N$. Letting

$$\mathbf{G}_{V,n}(S) = \{\text{loc. free rk } n \text{ quotients of the free sheaf } V^\vee \otimes_\mathbf{k} \mathcal{O}_S \text{ on } S\}$$

we define a contravariant functor:

$$\mathbf{G}_{V,n} : (\text{schemes}) \to (\text{sets})$$

called the *Grassmann functor*; we will denote it simply by $\mathbf{G}$ when no confusion is possible.

(IV.3.1) THEOREM *The Grassmann functor* $\mathbf{G}$ *is represented by a scheme* $G_n(V)$ *together with a locally free quotient of rank* n

$$V^\vee \otimes_\mathbf{k} \mathcal{O}_{G_n(V)} \to Q$$

called the universal quotient bundle.

Proof
Given a scheme S and an open cover $\{U_i\}$ of S, to give a locally free rank n quotient of $V^\vee \otimes_{\mathbf{k}} \mathcal{O}_S$ is equivalent to give one such quotient over each open set U_i so that they patch together on the intersections $U_i \cap U_j$. Therefore $\mathbf{G}$ is a sheaf.

Let's fix a basis $\{e_k\}$ of $V^\vee$ and choose a set J of n distinct indices in $\{1, \ldots, N\}$. We have an induced decomposition $V^\vee = E' \oplus E''$, with E' (resp. E'') a vector subspace of rank n (resp. $N - n$). We can define a subfunctor $\mathbf{G}_J$ of $\mathbf{G}$ letting:

$$\mathbf{G}_J(S) = \{\text{loc. free rk } n \text{ quotients } V^\vee \otimes_{\mathbf{k}} \mathcal{O}_S \to \mathcal{F} \text{ inducing } E' \otimes_{\mathbf{k}} \mathcal{O}_S \to \mathcal{F} \text{ surjective}\}$$

Let S be any scheme and $f : \mathrm{Hom}(-, S) \to \mathbf{G}$ a morphism of functors corresponding to a locally free rank n quotient

$$V^\vee \otimes_{\mathbf{k}} \mathcal{O}_S \to \mathcal{F}$$

The fibered product $S_J := \mathrm{Hom}(-, S) \times_{\mathbf{G}} \mathbf{G}_J$ is clearly represented by the open subscheme of S supported on the points where the map $E' \otimes_{\mathbf{k}} \mathcal{O}_S \to \mathcal{F}$ is surjective; this proves that $\mathbf{G}_J$ is an open subfunctor of $\mathbf{G}$. Since clearly the S_J's cover S, we also see that the family of subfunctors $\{\mathbf{G}_J\}$ is an open covering of $\mathbf{G}$.

To prove that $\mathbf{G}_J$ is representable note that if

$$q : V^\vee \otimes_{\mathbf{k}} \mathcal{O}_S \to \mathcal{F}$$

is an element of $\mathbf{G}(S)$ then the induced map

$$\eta : E' \otimes_{\mathbf{k}} \mathcal{O}_S \to \mathcal{F}$$

is surjective if and only if it is an isomorphism; in this case the composition

$$\eta^{-1} \circ q : V^\vee \otimes_{\mathbf{k}} \mathcal{O}_S \to E' \otimes_{\mathbf{k}} \mathcal{O}_S$$

restricts to the identity on $E' \otimes_{\mathbf{k}} \mathcal{O}_S$, hence it is determined by the composition

$$E'' \otimes_{\mathbf{k}} \mathcal{O}_S \to V^\vee \otimes_{\mathbf{k}} \mathcal{O}_S \to E' \otimes_{\mathbf{k}} \mathcal{O}_S$$

It follows that we can identify

$$\mathbf{G}_J(S) = \mathrm{Hom}(E'' \otimes_{\mathbf{k}} \mathcal{O}_S, E' \otimes_{\mathbf{k}} \mathcal{O}_S) = \mathrm{Hom}(E'', E') \otimes_{\mathbf{k}} \mathcal{O}_S$$

This proves that $\mathbf{G}_J$ is isomorphic to $\mathrm{Hom}(-, \mathbf{A}^{n(N-n)})$, hence it is representable. Now the theorem follows from proposition (A.4.0). *q.e.d.*

$G_n(V)$ is called the *grassmannian of n-dimensional subspaces of V*; it is also called the *grassmannian of $(n-1)$-dimensional projective subspaces of $\mathbb{P}(V)$*. When $V = \mathbf{k}^N$ the grassmannian $G_n(\mathbf{k}^N)$ is denoted $G(n, N)$.

When $n = 1$ the functor $\mathbf{G}_{V,1}$ is represented by $G_1(V) = I\!P(V) = \text{Proj}(\text{Sym}(V^\vee))$, the $(N-1)$-dimensional projective space associated to V. In this case $Q = \mathcal{O}_{I\!P(V)}(1)$.

From theorem (IV.3.1) it follows that for all schemes S the morphisms $f : S \to G_n(V)$ are in 1-1 correspondence with the locally free rank n quotients

$$V^\vee \otimes_{\mathbf{k}} \mathcal{O}_S \to \mathcal{F}$$

via $f \leftrightarrow f^*Q$. This is the *universal property of* $G_n(V)$.

The universal quotient bundle defines an exact sequence of locally free sheaves on $G_n(V)$:

$$0 \to K \to V^\vee \otimes_{\mathbf{k}} \mathcal{O}_{G_n(V)} \to Q \to 0$$

called the *tautological exact sequence*; K is called the *universal subbundle.* It is often useful to consider the dual subbundle:

$$\mathcal{T} := Q^\vee \subset V \otimes_{\mathbf{k}} \mathcal{O}_{G_n(V)}$$

called the *tautological bundle.*

Let S be a scheme. Associating to every locally free quotient of rank n

$$V^\vee \otimes_{\mathbf{k}} \mathcal{O}_S \to \mathcal{F}$$

the quotient

$$(\wedge^n V^\vee) \otimes_{\mathbf{k}} \mathcal{O}_S \to \wedge^n \mathcal{F}$$

we define a morphism of functors $\mathbf{G}_{V,n} \to \mathbf{G}_{\wedge^n V,1}$, which is induced by a morphism

$$\pi : G_n(V) \to I\!P(\wedge^n V)$$

π is called the *Plücker morphism.*

(IV.3.2) PROPOSITION *The Plücker morphism is a closed embedding. In particular $G_n(V)$ is a projective variety.*

Proof

As in the proof of (IV.3.1) we fix a basis of $V^\vee$ and we choose a set J of n distinct indices in $\{1, \ldots, N\}$. We obtain a decomposition $V^\vee = E' \oplus E''$ with $\dim(E') = n$, $\dim(E'') = N - n$, and an induced one:

$$\wedge^n V^\vee = \oplus_{i=0}^n (\wedge^{n-i} E') \otimes_{\mathbf{k}} \wedge^i E'' = \wedge^n E' \oplus F$$

where $F = \oplus_{i=1}^n (\wedge^{n-i} E') \otimes_{\mathbf{k}} \wedge^i E''$. For every scheme S let

$$I\!P_J(S) = \{\text{locally free rk 1 quotients } \wedge^n V^\vee \to \mathcal{L} \text{ s.t. the induced } \wedge^n E' \to \mathcal{L} \text{ is surjective}\}$$

We obtain a subfunctor $I\!P_J$ of $\mathbf{G}_{\wedge^n V,1}$. As in the proof of (A.4.10) we see that the $I\!P_J$'s form an open cover of $\mathbf{G}_{\wedge^n V,1}$ by functors representable by affine spaces.

Note that for every locally free rank n quotient $V^\vee \otimes_{\mathbf{k}} \mathcal{O}_S \to \mathcal{F}$ the induced homomorphism:

$$E' \otimes_{\mathbf{k}} \mathcal{O}_S \to \mathcal{F}$$

is surjective if and only if $\wedge^n E' \to \wedge^n \mathcal{F}$ is. Therefore $\mathbf{G}_J = \pi^{-1}(\mathbb{P}_J)$ and it suffices to prove that $\pi : \mathbf{G}_J \to \mathbb{P}_J$ is a closed embedding.

We may identify $\mathbf{G}_J$ with (the affine space associated to) $\mathrm{Hom}_{\mathbf{k}}(E', E'')$ and $\mathbb{P}_J$ with $\mathrm{Hom}_{\mathbf{k}}(F, \wedge^n E')$. Considering that

$$\mathrm{Hom}_{\mathbf{k}}(\wedge^{n-i} E', \wedge^n E') \cong \wedge^i E'$$

canonically via the perfect pairing:

$$\wedge^i E' \times \wedge^{n-i} E' \to \wedge^n E'$$

we have:

$$\mathrm{Hom}_{\mathbf{k}}(F, \wedge^n E') = \oplus_{i=1}^n \mathrm{Hom}_{\mathbf{k}}((\wedge^{n-i} E') \otimes_{\mathbf{k}} \wedge^i E'', \wedge^n E') =$$

$$= \oplus_{i=1}^n \mathrm{Hom}_{\mathbf{k}}(\wedge^i E'', \mathrm{Hom}_{\mathbf{k}}(\wedge^{n-i} E', \wedge^n E')) = \oplus_{i=1}^n \mathrm{Hom}_{\mathbf{k}}(\wedge^i E'', \wedge^i E')$$

and the map

$$\pi : \mathrm{Hom}_{\mathbf{k}}(E'', E') \to \mathrm{Hom}_{\mathbf{k}}(F, \wedge^n E')$$

is

$$\lambda \mapsto (\lambda, \wedge^2 \lambda, \ldots, \wedge^n \lambda)$$

This is the graph of a morphism of affine schemes, hence it is a closed embedding. *q.e.d.*

For some $1 \leq n \leq r$, let $G = G(n+1, r+1)$ be the grassmannian of n-dimensional projective subspaces of $\mathbb{P}^r$. Consider the *incidence relation*

$$\begin{array}{lcl} \mathbf{I} := \mathbb{P}(\mathcal{T}) & \subset & \mathbb{P}^r \times G \\ \quad\downarrow p & & \\ \quad G & & \end{array}$$

where $\mathcal{T} \subset p_* \mathcal{O}_{\mathbb{P}^r \times G} = \mathcal{O}_G^{r+1}$ is the tautological bundle; note that $\mathcal{T} = p_* \mathcal{I}_{\mathbf{I}}(1)$. For every closed point $v \in G$ the fibre $\mathbf{I}(v)$ is the projective subspace $\mathbb{P}(v) \subset \mathbb{P}^r$, hence it has Hilbert polynomial $\binom{t+n}{n}$. From Proposition (IV.2.1) it follows that p is a flat family. Suppose now that

$$\begin{array}{lcl} \Lambda & \subset & \mathbb{P}^r \times S \\ \downarrow q & & \\ S & & \end{array}$$

is another flat family whose fibres have Hilbert polynomial $\binom{t+n}{n}$. We have an inclusion of sheaves on S

$$q_* \mathcal{I}_\Lambda(1) \subset q_* \mathcal{O}_{\mathbb{P}^r \times S}(1) = \mathcal{O}_S^{r+1}$$

which has locally free cokernel $q_*\mathcal{O}_\Lambda(1)$. By the universal property of G the above inclusion induces a unique morphism

$$g : S \to G$$

such that $g^*(\mathcal{T}) = q_*\mathcal{I}_\Lambda(1)$. Since $\Lambda = I\!P(q_*\mathcal{I}_\Lambda(1))$ it follows that

$$\Lambda = S \times_G \mathbf{I}$$

namely the family q is obtained by base change from the incidence relation via the morphism g. This proves that

$$G(n+1, r+1) = \mathrm{Hilb}^r_{\binom{t+n}{n}}$$

NOTES

1. The construction of the grassmannian given here is taken from Kleiman(1969).

IV.4. THE HILBERT SCHEMES - EXISTENCE

(IV.4.1) THEOREM *For every projective scheme $Y \subset \mathbb{P}^r$ and every numerical polynomial $P(t)$, the Hilbert scheme $\mathrm{Hilb}^Y_{P(t)}$ exists and is a projective scheme.*

Proof
We will first prove the Theorem in the case $Y = \mathbb{P}^r$. From Theorem (IV.1.2) it follows that there is an integer m_0 such that for every closed subscheme $X \subset \mathbb{P}^r$ with Hilbert polynomial $P(t)$ the sheaf of ideals $\mathcal{I}_X$ is m_0-regular. It suffices to take

$$m_0 = F_r(-a_0, \dots, -a_{r-1}, 1 - a_r)$$

It follows that for every $k \geq m_0$

$$[IV.4.1] \qquad h^i(\mathbb{P}^r, \mathcal{I}_X(k)) = 0$$

for $i \geq 1$ and

$$h^0(\mathbb{P}^r, \mathcal{I}_X(k)) = \binom{k+r}{r} - P(k)$$

depends only on $P(k)$. Moreover by remark (IV.1.12)(ii) we have

$$[IV.4.2] \qquad h^i(X, \mathcal{O}_X(k)) = 0$$

all $k \geq m_0$ and all $i \geq 1$. Let

$$N = \binom{m_0+r}{r} - P(m_0)$$

$$V = H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(m_0))$$

Consider the grassmannian $G = G_N(V)$ of N-dimensional vector subspaces of V. Let $\mathcal{T} \subset V \otimes_{\mathbf{k}} \mathcal{O}_G$ be the tautological locally free sheaf of rank N and

$$p : \mathbb{P}^r \times G \to G$$

the projection. We may identify $V \otimes_{\mathbf{k}} \mathcal{O}_G = p_*[\mathcal{O}_{\mathbb{P}^r \times G}(m_0)]$. The image of the composition

$$\begin{array}{ccccc} p^*\mathcal{T}(-m_0) & \longrightarrow & V \otimes_{\mathbf{k}} \mathcal{O}_{\mathbb{P}^r \times G}(-m_0) & \longrightarrow & \mathcal{O}_{\mathbb{P}^r \times G} \\ & & \| & & \\ & & p^*p_*[\mathcal{O}_{\mathbb{P}^r \times G}(m_0)] \otimes \mathcal{O}_{\mathbb{P}^r \times G}(-m_0) & & \end{array}$$

is a sheaf of ideals: we will denote it by $\mathbf{J}$. It can also be obtained as the ideal sheaf in $\mathcal{O}_{\mathbb{P}^r \times G}$ corresponding to the subsheaf $\mathcal{T}\mathcal{O}_G[X_0, \ldots, X_r]$ of the sheaf of graded $\mathcal{O}_G$-modules $O_G[X_0, \ldots, X_r]$.

Let $\mathcal{Z} \subset \mathbb{P}^r \times G$ be the closed subscheme defined by $\mathbf{J}$ and denote by $q : \mathcal{Z} \to G$ the restriction of p to $\mathcal{Z}$.

Consider the flattening stratification

$$G^1 \coprod G^2 \coprod \cdots \subset G$$

for $\mathcal{O}_{\mathcal{Z}}$ and let H be the stratum relative to the polynomial $P(t)$. We will prove that $H = \mathrm{Hilb}^r_{P(t)}$ and that the universal family is the pullback of q to H:

$$\begin{array}{rccc} \mathcal{W} = & H \times_G \mathcal{Z} & \to & \mathcal{Z} \\ & \downarrow \pi & & \downarrow q \\ & H & \to & G \end{array}$$

By the choice of H, $\mathcal{W}$ defines a flat family of closed subschemes of $\mathbb{P}^r$ with Hilbert polynomial equal to $P(t)$.

Let's prove that $\mathcal{W}$ has the universal property.

Consider a flat family of closed subschemes of $\mathbb{P}^r$ with Hilbert polynomial $P(t)$:

$$\begin{array}{cc} \mathcal{X} \subset & \mathbb{P}^r \times S \\ & \downarrow f \\ & S \end{array}$$

From [IV.4.1] and [IV.4.2] and from Theorem (IV.2.5) it follows that

$$R^1 f_* \mathcal{I}_{\mathcal{X}}(m_0) = (0) = R^1 f_* \mathcal{O}_{\mathcal{X}}(m_0)$$

In particular we have an exact sequence on S:

$$\begin{array}{ccccc} 0 \to f_* \mathcal{I}_{\mathcal{X}}(m_0) \to & f_* \mathcal{O}_{\mathbb{P}^r \times S}(m_0) & \to f_* \mathcal{O}_{\mathcal{X}}(m_0) \to 0 \\ & \| & \\ & V \otimes_{\mathbf{k}} \mathcal{O}_S & \end{array}$$

If we apply Theorem (IV.2.5) for $j = -1$ we deduce that $f_* \mathcal{I}_{\mathcal{X}}(m_0)$ and $f_* \mathcal{O}_{\mathcal{X}}(m_0)$ are locally free and $f_* \mathcal{I}_{\mathcal{X}}(m_0)$ has rank N.

From the universal property of G it follows that there exists a unique morphism $g : S \to G$ such that

$$f_* \mathcal{I}_{\mathcal{X}}(m_0) = g^* \mathcal{T}$$

Claim: For all $m \gg m_0$ we have $f_* \mathcal{O}_{\mathcal{X}}(m) = g^* p_* \mathcal{O}_{\mathcal{Z}}(m)$.

Proof of the Claim: For all $m \gg m_0$ we have exact sequences:

$$0 \to p_* \mathbf{J}(m) \to q_* \mathcal{O}_{\mathbb{P}^r \times G}(m) \to p_* \mathcal{O}_{\mathcal{Z}}(m) \to 0$$

on G and

$$0 \to f_*\mathcal{I}_{\mathcal{X}}(m) \to f_*\mathcal{O}_{I\!P^r \times S}(m) \to f_*\mathcal{O}_{\mathcal{X}}(m) \to 0$$

on S; since $g^*p_*\mathcal{O}_{I\!P^r\times G}(m) = f_*\mathcal{O}_{I\!P^r\times S}(m)$ it suffices to show that:

$$f_*\mathcal{I}_{\mathcal{X}}(m) \cong g^*p_*\mathbf{J}(m)$$

for all $m \gg m_0$. For all such m we have the equality on G:

$$p_*\mathbf{J}(m) = \mathrm{Im}[\mathcal{T} \otimes p_*\mathcal{O}(m - m_0) \to p_*\mathcal{O}_{I\!P^r\times G}(m)]$$

induced by the surjections $p^*\mathcal{T}(m - m_0) \to \mathbf{J}(m)$ of sheaves on $I\!P^r \times G$. Hence for all $m \gg m_0$ we have:

$$g^*p_*\mathbf{J}(m) = g^*\mathrm{Im}[\mathcal{T} \otimes p_*\mathcal{O}_{I\!P^r\times G}(m - m_0) \to p_*\mathcal{O}_{I\!P^r\times G}(m)] =$$

$$= \mathrm{Im}[g^*\mathcal{T} \otimes f_*\mathcal{O}_{I\!P^r\times S}(m - m_0) \to f_*\mathcal{O}_{I\!P^r\times S}(m)] =$$

$$= \mathrm{Im}[f_*\mathcal{I}_{\mathcal{X}}(m_0) \otimes f_*\mathcal{O}_{I\!P^r\times S}(m - m_0) \to f_*\mathcal{O}_{I\!P^r\times S}(m)] = f_*\mathcal{I}_{\mathcal{X}}(m)$$

and this proves the Claim.

From the Claim it follows that

(i) g factors through H. Indeed from (IV.2.4) it follows that for all $m \gg m_0$:

$$g^*q_*\mathcal{O}_{\mathcal{Z}}(m) = f_*(1 \times g)^*\mathcal{O}_{\mathcal{Z}}(m)$$

Since $g^*p_*\mathcal{O}_{\mathcal{Z}}(m) = f_*\mathcal{O}_{\mathcal{X}}(m)$ is locally free of rank $P(m)$ for all such m Proposition (IV.2.1) implies that $(1\times g)^*\mathcal{O}_{\mathcal{Z}}$ is flat over S with Hilbert polynomial $P(t)$. Hence g factors by the definition of H.

(ii) $\mathcal{X} = S \times_H \mathcal{W}$. Indeed

$$\mathcal{X} = \mathrm{Proj}[\textstyle\bigoplus_{m\gg 0} f_*\mathcal{O}_{\mathcal{X}}(m)] = \mathrm{Proj}[\textstyle\bigoplus_{m\gg 0} g^*p_*\mathcal{O}_{\mathcal{Z}}(m)] =$$

$$= \mathrm{Proj}[\textstyle\bigoplus_{m\gg 0} g^*\pi_*\mathcal{O}_{\mathcal{W}}(m)] = S \times_H \mathrm{Proj}[\textstyle\bigoplus_{m\gg 0} \pi_*\mathcal{O}_{\mathcal{W}}(m)] = S \times_H \mathcal{W}$$

Properties (i) and (ii) imply that $H = \mathrm{Hilb}^r_{P(t)}$ and that π is the universal family.

By construction $\mathrm{Hilb}^r_{P(t)}$ is a quasi-projective scheme. To prove that it is projective it suffices to show that it is proper over $\mathbf{k}$. We will use the valuative criterion of properness. Let A be a discrete valuation $\mathbf{k}$-algebra with quotient field L and residue field K, and let

$$\varphi : \mathrm{Spec}(L) \to \mathrm{Hilb}^r_{P(t)}$$

be any morphism. We must show that φ extends to a morphism

$$\tilde{\varphi} : \mathrm{Spec}(A) \to \mathrm{Hilb}^r_{P(t)}$$

Pulling back the universal family by φ we obtain a flat family

$$\mathcal{X} \subset \mathbb{P}^r \times \mathrm{Spec}(L)$$

of closed subschemes of $\mathbb{P}^r$ with Hilbert polynomial $P(t)$. Since $\mathrm{Spec}(A)$ is nonsingular of dimension one and

$$\mathrm{Spec}(L) = \mathrm{Spec}(A)\backslash\{\text{closed point}\}$$

Proposition III.9.8 of Hartshorne(1977) implies the existence of a flat family

$$\mathcal{X}' \subset \mathbb{P}^r \times \mathrm{Spec}(A)$$

which extends $\mathcal{X}$. By the universal property of $\mathrm{Hilb}^r_{P(t)}$ this family corresponds to a morphism $\tilde{\varphi} : \mathrm{Spec}(A) \to \mathrm{Hilb}^r_{P(t)}$ which extends φ. This concludes the proof of the Theorem in the case $Y = \mathbb{P}^r$.

Let's now assume that Y is an arbitrary closed subscheme of $\mathbb{P}^r$. It will suffice to show that the functor $Hilb^Y_{P(t)}$ is represented by a closed subscheme of $\mathrm{Hilb}^r_{P(t)}$.

Applying Theorem (IV.1.2) twice we can find an integer μ such that $\mathcal{I}_Y \subset \mathcal{O}_{\mathbb{P}^r}$ is μ-regular and such that for every closed subscheme $X \subset \mathbb{P}^r$ with Hilbert polynomial $P(t)$ the ideal sheaf $\mathcal{I}_X \subset \mathcal{O}_{\mathbb{P}^r}$ is μ-regular. Let

$$V = H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(\mu)), \quad U = H^0(\mathbb{P}^r, \mathcal{I}_Y(\mu))$$

It follows from (IV.2.5) and (IV.2.6) that $\pi_*\mathcal{I}_{\mathcal{W}}(\mu)$ is a locally free subsheaf of $V \otimes_{\mathbf{k}} \mathcal{O}_{\mathrm{Hilb}}$ with locally free cokernel.

On $\mathrm{Hilb}^r_{P(t)}$ consider the composition

$$\Psi : U \otimes_{\mathbf{k}} \mathcal{O}_{\mathrm{Hilb}} \to V \otimes_{\mathbf{k}} \mathcal{O}_{\mathrm{Hilb}} \to V \otimes_{\mathbf{k}} \mathcal{O}_{\mathrm{Hilb}}/\pi_*\mathcal{I}_{\mathcal{W}}(\mu)$$

Let $Z \subset \mathrm{Hilb}^r_{P(t)}$ be the closed subscheme defined by the condition $\Psi = 0$, or equivalently by the condition

$$U \otimes_{\mathbf{k}} \mathcal{O}_Z \subset \pi_*\mathcal{I}_{\mathcal{W}}(\mu) \otimes \mathcal{O}_Z \qquad [IV.4.3]$$

Letting $j : Z \to \mathrm{Hilb}^r_{P(t)}$ be the inclusion, one easily sees that condition [IV.4.3] implies that

$$\mathcal{I}_{Y \times Z} \subset (1 \times j)^*\mathcal{I}_{\mathcal{W}} \subset \mathcal{O}_{\mathbb{P}^r \times Z}$$

hence that

$$Z \times_{\mathrm{Hilb}} \mathcal{W} \subset Y \times Z \subset \mathbb{P}^r \times Z \qquad [IV.4.4]$$

It is straightforward to check that $Z = \mathrm{Hilb}^Y_{P(t)}$ and that [IV.4.4] is the universal family. This concludes the proof of Theorem (IV.4.1). *q.e.d.*

For any projective scheme $Y \subset \mathbb{P}^r$ it is often convenient to consider the functor:

$$Hilb^Y : (\text{schemes}) \to (\text{sets})$$

defined as:

$$Hilb^Y(S) = \coprod_{P(t)} Hilb^Y_{P(t)}(S)$$

This functor is represented by the disjoint union

$$\mathrm{Hilb}^Y = \coprod_{P(t)} \mathrm{Hilb}^Y_{P(t)}$$

which is a scheme locally of finite type (but not of finite type because it has infinitely many connected components unless $\dim(Y) = 0$). It is *the Hilbert scheme of* Y. One convenient feature of Hilb^Y is that it is independent on the projective embedding of Y, even though the indexing of its components $Hilb^Y_{P(t)}$ by Hilbert polynomials does depend on the embedding. For this reason, when considering Hilb^Y we will not need to specify a projective embedding of Y.

Let's fix a projective scheme Y, and in the Hilbert scheme Hilb^Y let's consider a $\mathbf{k}$-rational point $[X]$ which parametrizes a closed subscheme $X \subset Y$. Denote by $\mathcal{I} \subset \mathcal{O}_Y$ the ideal sheaf of X in Y. The local Hilbert functor H^Y_X is a subfunctor of the restriction to $\mathcal{A}$ of the Hilbert functor; since Hilb^Y represents the Hilbert functor we have, with the notation introduced in §III.1:

$$H^Y_X(A) = \mathrm{Hom}(\mathrm{Spec}(A), \mathrm{Hilb}^Y)_{[X]}$$

for every A in $\mathrm{ob}(\mathcal{A})$. In particular H^Y_X is prorepresented by the local ring $\hat{\mathcal{O}}_{\mathrm{Hilb},[X]}$. We can therefore apply the results proved in §III.3 to obtain information about the local properties of Hilb^Y at $[X]$. In particular we have the following:

(IV.4.2) THEOREM *(i) There is a canonical isomorphism of* $\mathbf{k}$*-vector spaces:*

$$T_{[X]}\mathrm{Hilb}^Y \cong H^0(X, N_{X/Y})$$

where $N_{X/Y} = Hom_{\mathcal{O}_X}(\mathcal{I}/\mathcal{I}^2, \mathcal{O}_X)$ *is the normal sheaf of* X *in* Y.
(ii) If $X \subset Y$ *is a regular embedding then the obstruction space of* $\mathcal{O}_{\mathrm{Hilb}^Y,[X]}$ *is a subspace of* $H^1(X, N_{X/Y})$.

Consider a flat family of closed subschemes of Y:

$$\begin{array}{rc} \mathcal{X} \subset & Y \times S \\ & \downarrow f \\ & S \end{array}$$

It induces a functorial morphism $S \to \mathrm{Hilb}^Y$ whose differential at a $\mathbf{k}$-rational point $s \in S$ is a linear map

$$\chi_s : T_s S \to H^0(\mathcal{X}(s), N_{\mathcal{X}(s)/Y})$$

called the *characteristic map* of the family f.

The simplest illustration of Theorem (IV.4.2) is for $Y^{[1]} = Y$. In this case (IV.4.2)(i) simply says that $\mathrm{Hom}(m_p/m_p^2, \mathbf{k})$ is the Zariski tangent space of Y at a $\mathbf{k}$-rational point $p \in Y$. The obstruction space is $o(\mathcal{O}_{Y^{[1]},[p]}) = o(\mathcal{O}_{Y,p})$. Of course if p is a singular point then it is not regularly embedded in Y, and $H^1(p, N_{p/Y}) = 0$ is not an obstruction space for the local Hilbert functor.

NOTES

1. It is a classical result of Hartshorne that $\mathrm{Hilb}^r_{P(t)}$ is connected for all r and $P(t)$ (see Hartshorne(1966)). For general Y this is no longer true: for example, if $Q \subset \mathbb{P}^3$ is a nonsingular quadric then Hilb^Q_{t+1} has two connected components.

2. Let $X \subset Y$ be a closed embedding of projective schemes. It can be easily verified that for any closed subscheme $Z \subset X$, the induced injective linear map

$$H^0(Z, N_{Z/X}) \to H^0(Z, N_{Z/Y})$$

coincides with the differential at $[Z]$ of the closed embedding

$$\mathrm{Hilb}^X \subset \mathrm{Hilb}^Y$$

3. The Hilbert schemes, for their definition and construction, require the representability of the grassmannians functors. Infact the projective space $\mathbb{P}^r$ itself is a special case of grassmannian, and the proof of the existence of the Hilbert schemes relies heavily on the fact that the grassmannians functor is representable.

IV.5. THE QUOT SCHEMES

We will now introduce an important class of schemes, the so called *Quot schemes*, which generalize the Hilbert schemes. As special cases we will obtain the *relative Hilbert schemes*.

Let $p : X \to S$ be a projective morphism of algebraic schemes, and let $\mathcal{O}_X(1)$ be a line bundle on X very ample with respect to p. Fix a coherent sheaf $\mathcal{H}$ on X and a numerical polynomial $P(t) \in \mathbf{Q}[t]$. We define a functor

$$Quot^{X/S}_{\mathcal{H},P(t)} : (\text{schemes}/S)^\circ \to (\text{sets})$$

called the *Quot functor of X/S relative to $\mathcal{H}$ and $P(t)$*, in the following way:

$$Quot^{X/S}_{\mathcal{H},P(t)}(Z \to S) = \left\{ \begin{array}{l} \text{coherent quotients } \mathcal{H}_Z \to F, \text{ flat over } Z, \\ \text{having Hilbert polynomial } P(t) \text{ on the fibres of } X_Z \to Z \end{array} \right\}$$

where we have denoted $X_Z = Z \times_S X$ and $\mathcal{H}_Z$ the pullback of $\mathcal{H}$ on X_Z, as usual. When $S = \mathrm{Spec}(\mathbf{k})$ we write $Quot^X_{\mathcal{H},P(t)}$ instead of $Quot^{X/\mathrm{Spec}(\mathbf{k})}_{\mathcal{H},P(t)}$.

This Definition generalizes the Hilbert functors which are obtained in the case $S = \mathrm{Spec}(\mathbf{k})$ and $\mathcal{H} = \mathcal{O}_X$.

(IV.5.1) THEOREM *The functor $Quot = Quot^{X/S}_{\mathcal{H},P(t)}$ is represented by a projective S-scheme*

$$\mathrm{Quot}^{X/S}_{\mathcal{H},P(t)} \to S$$

Proof

We first consider the case $S = \mathrm{Spec}(\mathbf{k})$ and $X = I\!P^r$. From Proposition (IV.1.14) it follows that there is an integer m such that for each scheme Z and for each $(\varphi : \mathcal{H}_Z \to F) \in Quot(Z)$, letting $N = \ker(\varphi)$, all the sheaves $N(z), \mathcal{H}(z) = \mathcal{H}, F(z)$, $z \in Z$, are m-regular. Therefore, letting $p_Z : I\!P^r \times Z \to Z$ be the projection, we obtain an exact sequence of locally free sheaves on Z:

$$0 \to p_{Z*}N(m) \to H^0(I\!P^r, \mathcal{H}(m)) \otimes_{\mathbf{k}} \mathcal{O}_Z \to p_{Z*}F(m) \to 0$$

Moreover, for each $m' \geq m$ there is an exact sequence

$$H^0(I\!P^r, \mathcal{O}(m'-m)) \otimes_{\mathbf{k}} p_{Z*}N(m) \to H^0(I\!P^r, \mathcal{H}(m')) \otimes_{\mathbf{k}} \mathcal{O}_Z \to p_{Z*}F(m') \to 0$$

where the first map is given by multiplication of sections. This shows that $p_{Z*}F(m)$ uniquely determines the sheaf of graded $\mathcal{O}_Z[X_0, \ldots, X_r]$-modules $\bigoplus_{k \geq m} p_{Z*}F(k)$, which in turn determines F. Therefore, letting $H^0(\mathbb{P}^r, \mathcal{H}(m)) = V$, we have an injective morphism of functors:

$$Quot \to \mathbf{G}_{\check{V},P(m)}$$

given by:

$$\begin{array}{ccc} Quot(Z) & \to & \mathbf{G}_{\check{V},P(m)}(Z) \\ (\mathcal{H}_Z \to F) & \mapsto & [V \otimes_{\mathbf{k}} \mathcal{O}_Z \to p_{Z*}F(m)] \end{array}$$

On $G = G_{P(m)}(\check{V})$ consider the tautological exact sequence

$$0 \to K \to V \otimes_{\mathbf{k}} \mathcal{O}_G \to Q \to 0$$

Let moreover $p_2 : \mathbb{P}^r \times G \to G$ and $p_1 : \mathbb{P}^r \times G \to \mathbb{P}^r$ be the projections. On G we have $\Gamma_*(\mathcal{H}) \otimes_{\mathbf{k}} \mathcal{O}_G$, which is a sheaf of graded $\mathcal{O}_G[X_0, \ldots, X_r]$-modules, and determines $p_1^*(\mathcal{H})$. Consider the subsheaf $K\mathcal{O}_G[X_0, \ldots, X_r]$ and the sheaf $\mathcal{F}$ on $G \times \mathbb{P}^r$ corresponding to the quotient $\Gamma_*(\mathcal{H}) \otimes_{\mathbf{k}} \mathcal{O}_G/K\mathcal{O}_G[X_0, \ldots, X_r]$, and let $G_P \subset G$ be the stratum corresponding to P of the flattening stratification of $\mathcal{F}$. Then we claim that a morphism of schemes $f : Z \to G$ defines an element of $Quot(Z)$ if and only if f factors through G_P, and therefore $Quot$ is represented by G_P. The proof of this fact is similar to the one given for the proof of Theorem (IV.4.1) and will be left to the reader.

Since G_P is quasi-projective, to prove that it is projective amounts to prove that it is proper over $\mathbf{k}$, and this can be done using the valuative criterion of properness. Let A be a discrete valuation $\mathbf{k}$-algebra with quotient field L and residue field K, and let

$$\varphi : \mathrm{Spec}(L) \to G_P$$

be any morphism. We must show that φ extends to a morphism

$$\tilde{\varphi} : \mathrm{Spec}(A) \to G_P$$

The datum of φ corresponds to an element $(\varphi_L : \mathcal{H}_L \to F_L)$ of $Quot(\mathrm{Spec}(L))$. The existence of $\tilde{\varphi}$ will be proved if there is a quotient $\varphi_A : \mathcal{H}_A \to F_A$ on $\mathbb{P}^r \times \mathrm{Spec}(A)$ which is flat over $\mathrm{Spec}(A)$ and which restricts to F_L over $\mathbb{P}^r \times \mathrm{Spec}(L)$. Let $i : \mathbb{P}^r \times \mathrm{Spec}(L) \to \mathbb{P}^r \times \mathrm{Spec}(A)$ be the inclusion, and take $F_A = i_*(F_L)$. Obviously F_A restricts to F_L. Moreover, if $K_L = \ker(\varphi_L)$, we have $R^1 i_*(K_L) = 0$ and therefore a surjection $\mathcal{H}_A = i_*(\mathcal{H}_L) \to F_A$. We need the following

LEMMA *Let X be a scheme, U an open subset of X and $i : U \to X$ the inclusion. Then for every coherent sheaf F on U we have*

$$\mathrm{Ass}(i_*(F)) = \mathrm{Ass}(F)$$

Proof

Since $i_*(F)_{|U} = F$ we have $\mathrm{Ass}(i_*(F)) \cap U = \mathrm{Ass}(F)$. Therefore we only need to prove that $\mathrm{Ass}(i_*(F)) \subset U$.

We may assume that $X = \mathrm{Spec}(A)$ and $U = \mathrm{Spec}(B)$ are affine. The inclusion i corresponds to an injective homomorphism $A \to B$ and $F = M^\sim$ for a f.g. B-module M. Let $x \in \mathrm{Ass}(i_*(F))$ and assume that $x \in X \backslash U$. Then the ideal $p_x \subset A$ annihilates an element $m_x \in i_*(F)_x$ which corresponds to a section $m \in \Gamma(V, i_*(F))$ for some open neighborhood V of x. Up to shrinking X we may assume $V = X$, so that $m \in \Gamma(X, i_*(F)) = \Gamma(U, F) = M$ is annihilated by the ideal $p_x B$. But $p_x B = B$ because $x \notin U$ and therefore $m = 0$: this is a contradiction. The Lemma is proved.

From the Lemma it follows that $\mathrm{Ass}(F_A) = \mathrm{Ass}(F_L)$: therefore, using the fact that F_L is flat over $\mathrm{Spec}(L)$ and (Hartshorne(1977), Prop. III.9.7), we deduce that F_A is flat over $\mathrm{Spec}(A)$. This concludes the proof of the Theorem in the case $S = \mathrm{Spec}(\mathbf{k})$ and $X = \mathbb{P}^r$.

Assume now that S and X are arbitrary. Consider the closed embedding $j : X \to \mathbb{P}^r \times S$ determined by $\mathcal{O}_X(1)$. Replacing $\mathcal{H}$ by $j_*\mathcal{H}$ we can assume that $X = \mathbb{P}^r \times S$. Let $h, h' \gg 0$ be such that we have an exact sequence:

$$\mathcal{O}_{\mathbb{P}^r \times S}(-h')^{M'} \to \mathcal{O}_{\mathbb{P}^r \times S}(-h)^M \to \mathcal{H} \to 0$$

for some M, M'. Then for each S-scheme $Z \to S$ and for each

$$(\mathcal{H}_Z \to F) \in Quot^{X/S}_{\mathcal{H},P(t)}(Z \to S)$$

we obtain that the composition

$$\mathcal{O}_{\mathbb{P}^r \times Z}(-h)^M \to \mathcal{H}_Z \to F \to 0$$

is a surjection, i.e. is an element of $Quot^{X/S}_{\mathcal{O}(-h)^M,P(t)}(Z \to S)$. This proves that the functor $Quot^{X/S}_{\mathcal{H},P(t)}$ is a subfunctor of the functor $Quot^{X/S}_{\mathcal{O}(-h)^M,P(t)}$, and this functor is evidently represented by $\mathrm{Quot}^{\mathbb{P}^r}_{\mathcal{O}(-h)^M,P(t)} \times S$.
Conversely, a quotient

$$(\mathcal{O}_{\mathbb{P}^r \times Z}(-h)^M \to F) \in Quot^{X/S}_{\mathcal{O}_X(-h)^M,P(t)}(Z \to S)$$

is in $Quot^{X/S}_{\mathcal{H},P(t)}(Z \to S)$ if and only if the composition

$$\mathcal{O}_{\mathbb{P}^r \times Z}(-h')^{M'} \to \mathcal{O}_{\mathbb{P}^r \times Z}(-h)^M \to F$$

is zero. This means that the condition for an S-morphism

$$Z \to \mathrm{Quot}^{X/S}_{\mathcal{O}(-h)^M,P(t)} = \mathrm{Quot}^{\mathbb{P}^r/\mathbf{k}}_{\mathcal{O}(-h)^M,P(t)} \times S$$

to define an element of $Quot^{X/S}_{\mathcal{H},P(t)}(Z \to S)$ is that it factors through the closed subscheme defined by the entries of the matrix of the homomorphism:

$$\mathcal{O}_{\mathbb{P}^r \times \mathrm{Quot}}(-h')^{M'} \to \mathcal{O}_{\mathbb{P}^r \times \mathrm{Quot}}(-h)^M$$

and this is a closed condition. This proves that $\mathcal{Q}uot^{X/S}_{\mathcal{H},P(t)}$ is represented by a closed subscheme of $\mathrm{Quot}^{\mathbb{P}^r/\mathbf{k}}_{\mathcal{O}(-h)^M,P(t)} \times S$. *q.e.d.*

From the fact that $\mathrm{Quot}^{X/S}_{\mathcal{H},P(t)}$ represents the functor $\mathcal{Q}uot^{X/S}_{\mathcal{H},P(t)}$ it follows that there is a *universal quotient*

$$(\mathcal{H}_{\mathrm{Quot}} \to \mathcal{F}) \in \mathcal{Q}uot^{X/S}_{\mathcal{H},P(t)}(\mathrm{Quot}^{X/S}_{\mathcal{H},P(t)})$$

corresponding to the identity morphism under the identification

$$\mathrm{Hom}(\mathrm{Quot}, \mathrm{Quot}) = \mathcal{Q}uot(\mathrm{Quot})$$

In case $\mathcal{H} = \mathcal{O}_X$ the scheme $\mathrm{Quot}^{X/S}_{\mathcal{O}_X,P(t)}$ is denoted $\mathrm{Hilb}^{X/S}_{P(t)}$ and called the *relative Hilbert scheme* of X/S with respect to the polynomial $P(t)$.

It will be sometimes convenient to consider the functor

$$\mathcal{Q}uot^{X/S}_{\mathcal{H}} : (\mathrm{schemes}/S)^{\circ} \to (\mathrm{sets})$$

defined as:

$$\mathcal{Q}uot^{X/S}_{\mathcal{H}}(Z \to S) = \coprod_{P(t)} \mathcal{Q}uot^{X/S}_{\mathcal{H},P(t)}(Z \to S)$$

This functor is represented by the disjoint union

$$\mathrm{Quot}^{X/S}_{\mathcal{H}} = \coprod_{P(t)} \mathrm{Quot}^{X/S}_{\mathcal{H},P(t)}$$

which is a scheme locally of finite type, called *the Quot scheme of X over S relative to $\mathcal{H}$*; it carries a universal quotient $\mathcal{H}_{\mathrm{Quot}} \to \mathcal{F}$.

Similarly we will consider the *relative Hilbert scheme of X over S*:

$$\mathrm{Hilb}^{X/S} = \coprod_{P(t)} \mathrm{Hilb}^{X/S}_{P(t)}$$

The construction of the Quot scheme commutes with base change; this is a result which follows quite directly from the definition, but it is worth pointing it out:

(IV.5.2) PROPOSITION (base change property) *Given a projective morphism $X \to S$, a coherent sheaf $\mathcal{H}$ on X, and a morphism $T \to S$, there is a natural identification:*

$$\mathrm{Quot}^{X_T/T}_{\mathcal{H}_T} = T \times_S \mathrm{Quot}^{X/S}_{\mathcal{H}}$$

Proof

Consider the product diagram

$$\begin{array}{ccc} T \times \mathrm{Quot}^{X/S}_{\mathcal{H}} & \to & \mathrm{Quot}^{X/S}_{\mathcal{H}} \\ \downarrow & & \downarrow \\ T & \to & S \end{array}$$

The universal quotient $\mathcal{H}^{X/S} \to \mathcal{F}$ on $\mathrm{Quot}_{\mathcal{H}}^{X/S}$ pullsback to a quotient $\mathcal{H}^{X_T/T} \to \mathcal{F}_T$ on $T \times \mathrm{Quot}_{\mathcal{H}}^{X/S}$. It is immediate that the T-scheme $T \times \mathrm{Quot}_{\mathcal{H}}^{X/S}$ endowed with this quotient represents the functor $Quot_{\mathcal{H}_T}^{X_T/T}$. *q.e.d.*

* * * * * *

Local properties

(IV.5.3) PROPOSITION *Let $X \to S$ be a projective morphism of algebraic schemes, $\mathcal{H}$ a coherent sheaf on X, flat over S, and $\pi : Q = \mathrm{Quot}_{\mathcal{H}}^{X/S} \to S$ the associated Quot scheme over S. Let $s \in S$ be a* **k***-rational point and $q \in \pi^{-1}(s) = Q(s)$ corresponding to a coherent quotient $f : \mathcal{H} \to \mathcal{F}$ with kernel $\mathcal{K}$. Let*

$$f_s : \mathcal{H}(s) \to \mathcal{F}(s)$$

be the restriction of f to the fibre $X(s)$, whose kernel is $\mathcal{K}(s) = \mathcal{K} \otimes \mathcal{O}_{X(s)}$ (by the flatness of $\mathcal{F}$). Then there is an exact sequence

$$0 \to \mathrm{Hom}(\mathcal{K}(s), \mathcal{F}(s)) \to t_q Q \xrightarrow{d\pi_q} t_s S \to \mathrm{Ext}^1_{\mathcal{O}_{X(s)}}(\mathcal{K}(s), \mathcal{F}(s))$$

and an inclusion:

$$\ker[o(\pi_q^\sharp)] \subset \mathrm{Ext}^1_{\mathcal{O}_{X(s)}}(\mathcal{K}(s), \mathcal{F}(s))$$

where $o(\pi_q^\sharp) : o(\mathcal{O}_{Q,q}) \to o(\mathcal{O}_{S,s})$ is the obstruction map of the local homomorphism $\pi_q^\sharp : \mathcal{O}_{S,s} \to \mathcal{O}_{Q,q}$. In particular π is smooth at q if $\mathrm{Ext}^1_{\mathcal{O}_{X(s)}}(\mathcal{K}(s), \mathcal{F}(s)) = 0$.

Proof

A vector in $\ker(d\pi_q)$ corresponds to a commutative diagram:

$$\begin{array}{ccc} \mathrm{Spec}(\mathbf{k}[\epsilon]) & \to & Q \\ \downarrow & & \downarrow \pi \\ \mathrm{Spec}(\mathbf{k}) & \to & S \end{array}$$

such that the upper horizontal arrow has image $\{q\}$. The above diagram corresponds to an exact and commutative diagram of sheaves on $X(s)$:

$$\begin{array}{ccccccc} & & & 0 & & 0 & \\ & & & \downarrow & & \downarrow & \\ & & & \mathcal{K}(s)_\epsilon & \longrightarrow & \mathcal{K}(s) & \to 0 \\ & & & \downarrow & & \downarrow & \\ 0 \to & \epsilon\mathcal{H}(s) & \to & \mathcal{H}(s)[\epsilon] & \longrightarrow & \mathcal{H}(s) & \to 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \to & \epsilon\mathcal{F}(s) & \to & \mathcal{F}(s)_\epsilon & \longrightarrow & \mathcal{F}(s) & \to 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ & 0 & & 0 & & 0 & \end{array}$$

where the middle row is exact by the flatness of $\mathcal{H}$. Replacing the middle row by its pushout under $\epsilon\mathcal{H}(s) \to \epsilon\mathcal{F}(s)$ we see that this diagram is equivalent to the following one:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \mathcal{K}(s) & = & \mathcal{K}(s) & \to 0 & \\
 & & & & \downarrow & & \downarrow & & \\
0 \to & \epsilon\mathcal{F}(s) & \to & & \mathcal{P} & \longrightarrow & \mathcal{H}(s) & \to 0 & \\
 & \| & & & \downarrow & & \downarrow & & \\
0 \to & \epsilon\mathcal{F}(s) & \to & & \mathcal{F}(s)_\epsilon & \longrightarrow & \mathcal{F}(s) & \to 0 & \\
 & \downarrow & & & \downarrow & & \downarrow & & \\
 & 0 & & & 0 & & 0 & &
\end{array}$$

and therefore we deduce that $\ker(d\pi_q) = \mathrm{Hom}(\mathcal{K}(s), \mathcal{F}(s))$.

Now consider A in $\mathcal{A}$ and a commutative diagram

$$[IV.5.1] \qquad \begin{array}{ccc}
A & \stackrel{\varphi}{\longleftarrow} & \mathcal{O}_{Q,q} \\
\uparrow \eta & & \uparrow \pi_q^\sharp \\
B & \underset{\tilde{\varphi}}{\longleftarrow} & \mathcal{O}_{S,s}
\end{array}$$

where η is a small extension in $\mathcal{A}$. This diagram corresponds to an exact diagram of sheaves on X:

$$[IV.5.2] \qquad \begin{array}{ccccccc}
 & & & & 0 & \\
 & & & & \downarrow & \\
 & & & & \mathcal{K}_A & \\
 & & & & \downarrow & \\
\gamma: \quad 0 \to & \mathcal{H}(s) & \to & \mathcal{H} \otimes_{\mathbf{k}} B & \longrightarrow & \mathcal{H} \otimes_{\mathbf{k}} A & \to 0 \\
 & & & & \downarrow & \\
 & & & & \mathcal{F}_A & \\
 & & & & \downarrow & \\
 & & & & 0 &
\end{array}$$

where the row is exact by the flatness of $\mathcal{H}$ over S. By pushing out by the quotient $f_s : \mathcal{H}(s) \to \mathcal{F}(s)$ and then pulling back by $\alpha : \mathcal{K}_A \to \mathcal{H} \otimes_{\mathbf{k}} A$ we obtain an element

$$[\alpha^* f_{s*}(\gamma)] \in \mathrm{Ext}^1_{\mathcal{O}_X \otimes A}(\mathcal{K}_A, \mathcal{F}(s)) = \mathrm{Ext}^1_{\mathcal{O}_{X(s)}}(\mathcal{K}(s), \mathcal{F}(s))$$

By construction this element vanishes if and only if the previous diagram can be embedded in a commutative diagram with exact rows and columns

$$\begin{array}{cccccccc}
 & 0 & & 0 & & 0 & \\
 & \downarrow & & \downarrow & & \downarrow & \\
0 \to & \mathcal{K}(s) & \to & \mathcal{K}_B & \to & \mathcal{K}_A & \to 0 \\
 & \downarrow & & \downarrow & & \downarrow & \\
0 \to & \mathcal{H}(s) & \to & \mathcal{H} \otimes_{\mathbf{k}} B & \longrightarrow & \mathcal{H} \otimes_{\mathbf{k}} A & \to 0 \\
 & \downarrow & & \downarrow & & \downarrow & \\
0 \to & \mathcal{F}(s) & \to & \mathcal{F}_B & \to & \mathcal{F}_A & \\
 & \downarrow & & \downarrow & & \downarrow & \\
 & 0 & & 0 & & 0 &
\end{array}$$

The middle column of this diagram is an element of $Quot_{\mathcal{H}}^{X/S}(\mathrm{Spec}(B))$, which corresponds to a homomorphism $\varphi' : \mathcal{O}_{Q,q} \to B$ making the diagram

$$\begin{array}{ccc} A & \overset{\varphi}{\longleftarrow} & \mathcal{O}_{Q,q} \\ \uparrow \eta & \swarrow \varphi' & \uparrow \pi_q^\sharp \\ B & \underset{\tilde{\varphi}}{\longleftarrow} & \mathcal{O}_{S,s} \end{array}$$

commutative. Therefore we have associated an element of $\mathrm{Ext}^1_{\mathcal{O}_{X(s)}}(\mathcal{K}(s), \mathcal{F}(s))$ to each diagram [IV.5.2]. It is straightforward to check that this correspondence is linear.

Taking $\eta : \mathbf{k}[\epsilon] \to \mathbf{k}$ we get an inclusion

$$\mathrm{coker}(d\pi_q) \subset \mathrm{Ext}^1_{\mathcal{O}_{X(s)}}(\mathcal{K}(s), \mathcal{F}(s))$$

Taking any small extension η in $\mathcal{A}$ we can apply Proposition (I.3.7) to yield the conclusion. *q.e.d.*

(IV.5.4) COROLLARY *Under the same assumptions of (IV.5.3), if*

$$\mathrm{Ext}^1_{\mathcal{O}_{X(s)}}(\mathcal{K}(s), \mathcal{F}(s)) = 0$$

then $\pi : Q \to S$ *is smooth at* q *of relative dimension* $\dim\left[\mathrm{Hom}(\mathcal{K}(s), \mathcal{F}(s))\right]$.

When $S = \mathrm{Spec}(\mathbf{k})$ we obtain the following "absolute" version of Proposition (IV.5.3).

(IV.5.5) COROLLARY *If* X *is a projective scheme,* $\mathcal{H}$ *a coherent sheaf on* X *and* $f : \mathcal{H} \to \mathcal{F}$ *a coherent quotient of* $\mathcal{H}$ *with* $\ker(f) = \mathcal{K}$ *then, letting* $Q = \mathrm{Quot}^X_{\mathcal{H}}$, *we have:*

$$T_{[f]}Q = \mathrm{Hom}(\mathcal{K}, \mathcal{F})$$

and the obstruction space of $\mathcal{O}_{Q,[f]}$ *is a subspace of* $\mathrm{Ext}^1(\mathcal{K}, \mathcal{F})$.
In particular, if $\mathrm{Ext}^1(\mathcal{K}, \mathcal{F}) = 0$ *then* Q *is nonsingular of dimension* $\dim\big(\mathrm{Hom}(\mathcal{K}, \mathcal{F})\big)$ *at* $[f]$.

A special case of Proposition (IV.5.3) is the following:

(IV.5.6) PROPOSITION *Let* $p : \mathcal{X} \to S$ *be a projective flat morphism of algebraic schemes, and* $\pi : \mathrm{Hilb}^{\mathcal{X}/S} \to S$ *the relative Hilbert scheme. For a closed point* $s \in S$ *let* $X = \mathcal{X}(s)$ *be the fibre over* s *and let* $Z \subset X$ *be a closed subscheme with ideal sheaf* $\mathcal{I} \subset \mathcal{O}_X$. *Then there is an exact sequence:*

$$0 \to H^0(Z, N_{Z/X}) \to T_{[Z]}\mathrm{Hilb} \overset{d\pi_{[Z]}}{\longrightarrow} T_sS \to \mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{I}, \mathcal{O}_Z)$$

If moreover $Z \subset X$ *is a regular embedding then the above exact sequence becomes:*

$$[IV.5.3] \qquad 0 \to H^0(Z, N_{Z/X}) \to T_{[Z]}\mathrm{Hilb} \overset{d\pi_{[Z]}}{\longrightarrow} T_sS \to H^1(Z, N_{Z/X})$$

If $\mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{I}, \mathcal{O}_Z) = (0)$ *(resp.* $H^1(Z, N_{Z/X}) = (0)$ *in case* $Z \subset X$ *is a regular embedding) then* π *is smooth at* $[Z]$ *of relative dimension* $h^0(Z, N_{Z/X})$*.*

NOTES

1. One should compare the statement of (IV.5.6) with (III.3.11), since the local relative Hilbert functor of Z in X relative to $X \to S$ is prorepresented by the local ring $\hat{\mathcal{O}}_{\mathrm{Hilb}^{X/S}_{[Z]}}$.

2. Our presentation of the Quot schemes is an adaptation of the one given in Huybrechts-Lehn(1997). For a description of the sheaf of differentials of the Quot schemes see Lehn(1998).

IV.6. EXAMPLES

Complete intersections

We have already discussed some properties of the local Hilbert functor of a complete intersection $X \subset I\!P^r$, which of course correspond to local properties of the Hilbert scheme Hilb^r at $[X]$. It is easy to check that, despite the fact that $H^1(X, N_X) \neq (0)$ in general, every complete intersection X is unobstructed in $I\!P^r$.

We may assume $\dim(X) > 0$. Let's suppose that $X \subset I\!P^r$, $r \geq 2$, is the complete intersection of $r-n$ hypersurfaces $f_1, \dots, f_{r-n}$ of degrees $d_1 \leq d_2 \leq \dots \leq d_{r-n}$ respectively, $n < r$.

Consider a basis $\Phi^{(1)}, \dots, \Phi^{(m)}$ of $\oplus_j H^0(I\!P^r, \mathcal{O}(d_j))$ where

$$\Phi^{(h)} = (\phi_1^{(h)}, \dots, \phi_{r-n}^{(h)})$$

$h = 1, \dots, m$, and the $\phi_j^{(h)} \in \mathbf{k}[X_0, \dots, X_r]$. Consider indeterminates $u_1, \dots, u_m$ and the $(r-n)$-tuple

$[IV.6.1]$

$$\mathbf{f} + u_1\Phi^{(1)} + \dots + u_m\Phi^{(m)} = (f_1 + u_1\phi_1^{(1)} + \dots + u_m\phi_1^{(m)}, \dots\dots, f_{r-n} + u_1\phi_{r-n}^{(1)} + \dots + u_m\phi_{r-n}^{(m)})$$

of elements of the polynomial ring $\mathbf{k}[\underline{u}, \underline{x}] = \mathbf{k}[u_1, \dots, u_m, X_0, \dots, X_r]$.

Let $K_\bullet(\mathbf{f} + \sum_h u_h\Phi^{(h)})$ be the Koszul complex relative to [IV.6.1] and

$$\Delta := \mathrm{Supp}[H_1(K_\bullet(\mathbf{f} + \sum_h u_h\Phi^{(h)}))] \subset \mathbf{A}^{m+r+1} = \mathrm{Spec}(\mathbf{k}[\underline{u}, \underline{x}])$$

Denoting by $p : \mathbf{A}^{m+r+1} \to \mathbf{A}^m$ the projection, $U := \mathbf{A}^m \backslash p(\Delta)$ is the set of points $\underline{u} \in \mathbf{A}^m$ such that $K_\bullet(\mathbf{f} + \sum_h t_h\Phi^{(h)})$ is exact; U is an open set containing the origin.

In $I\!P^r \times \mathbf{A}^m$ consider the closed subscheme

$$\mathcal{X} = \mathrm{Proj}\Big(\mathbf{k}[\underline{u}, \underline{x}]/(f_1 + u_1\phi_1^{(1)} + \dots + u_m\phi_1^{(m)}, \dots\dots, f_{r-n} + u_1\phi_{r-n}^{(1)} + \dots + u_m\phi_{r-n}^{(m)})\Big)$$

the projection $\pi : \mathcal{X} \to \mathbf{A}^m$ and its restriction $\pi_U : \mathcal{X}_U \to U$, where $\mathcal{X}_U := \pi^{-1}(U)$. All the fibres of π_U are complete intersections of multidegree $(d_1, \dots, d_{r-n})$ and $\mathcal{X}(\underline{0}) = X$. The Hilbert polynomial of a complete intersection depends only on its multidegree because it can be computed using the Koszul complex: it follows that all the fibres of π_U have the same Hilbert polynomial $P(t)$ and therefore π_U is a flat family of deformations of X in $I\!P^r$. In an obvious way the tangent space of U

at $\underline{0}$ can be identified with $\oplus_j H^0(\mathbb{P}^r, \mathcal{O}(d_j))$, and the characteristic map with the restriction

$$\varphi : \oplus_j H^0(\mathbb{P}^r, \mathcal{O}(d_j)) \to \oplus_j H^0(X, \mathcal{O}_X(d_j))$$

Since $\dim(X) > 0$ the map φ is surjective, as one easily verifies using the Koszul complex; since moreover U is nonsingular at $\underline{0}$, we see that the family π_U is complete at $\underline{0}$. From Proposition (I.8.3)(iii) it follows that $\mathrm{Hilb}^r_{P(t)}$ is smooth at $[X]$. From the completeness of π_U it also follows that complete intersections are parametrized by an open subset of $\mathrm{Hilb}^r_{P(t)}$.

It is interesting to observe that the closure of this open set may contain points parametrizing nonsingular subschemes of $\mathbb{P}^r$ which are not complete intersections. An example of such a subscheme is given by a trigonal canonical curve $C \subset \mathbb{P}^4$: the quadrics containing C intersect in a rational cubic surface S, so it is not a complete intersection since it has degree 8; but $[C]$ is in the closure of the family of complete intersections of three quadrics. It is apparently unknown whether a similar phenomenon may occur in $\mathbb{P}^3$, namely whether there are nonsingular curves in $\mathbb{P}^3$ which are flat limits of complete intersections without being complete intersections.

The Kodaira-Spencer map of the families π_U has been studied in Sernesi(1975) in the case of complete intersections of dimension ≥ 2: π_U has general moduli except for the cases of surfaces of multidegrees $(4), (2,3), (2,2,2)$ (respectively in $\mathbb{P}^3$, in $\mathbb{P}^4$ and in $\mathbb{P}^5$), i.e. for complete intersection K3-surfaces.

An obstructed nonsingular curve in $\mathbb{P}^3$ (Mumford(1962))

We will show that the Hilbert scheme $\mathrm{Hilb}^{\mathbb{P}^3}$ has an everywhere nonreduced component Σ which generically parametrizes nonsingular curves of degree 14 and genus 24. It will follow that every curve parametrized by a general point of Σ is obstructed in $\mathbb{P}^3$.

A general element of Σ is constructed as follows. Let $F \subset \mathbb{P}^3$ be a nonsingular cubic surface, $E, H \subset F$ respectively a line and a plane section in F. Let $C \subset F$ be a general member of the linear system $|4H + 2E|$. Using Bertini's Theorem one easily checks that C is irreducible and nonsingular; its degree and genus are $(C \cdot H) = 14$ and $\frac{1}{2}(C - H \cdot C) + 1 = 24$. From the exact sequence:

$$0 \to K_C(H) \to N_C \to \mathcal{O}_C(3H) \to 0$$

we see that

$$h^1(C, N_C) = h^1(C, \mathcal{O}_C(3H)) = h^0(C, K_C(-3H)) = h^0(C, \mathcal{O}_C(2E)) = 1$$

where the last equality follows easily from the exact sequence

$$0 \to \mathcal{O}_F(-4H) \to \mathcal{O}_F(2E) \to \mathcal{O}_C(2E) \to 0$$

and from $h^0(\mathcal{O}_F(-4H)) = 0 = h^1(\mathcal{O}_F(-4H))$ and $h^0(\mathcal{O}_F(2E)) = h^0(\mathcal{O}_F(E)) = 1$. Moreover the linear system $|C| = |4H + 2E|$ has dimension

$$\dim(|C|) = 1 + \dim(|C|_C) = h^0(C, K_C(H)) = 37$$

and therefore, since every curve C is contained in a unique cubic surface (because $9 < 14$), the dimension of the family W of all curves C we are considering is $19 + 37 = 56 = 4 \cdot 14$ but they satisfy $h^0(C, N_C) = 56 + h^1(C, N_C) = 57$. We will prove that W is an open set of a component Σ of $\mathrm{Hilb}^{\mathbb{P}^3}$ and this will imply that Σ is everywhere nonreduced. Our assertion will be proved if we show that our curves C are not contained in a family whose general member is a curve D not contained in a cubic surface. But on every such D the line bundle $\mathcal{O}_D(4)$ is nonspecial and therefore, by Riemann-Roch, $h^0(D, \mathcal{O}_D(4)) = 33$, hence D is contained in a pencil of quartic surfaces. Let G_1, G_2 be two linearly independent quartics containing D: they are both irreducible because otherwise D would be either contained in a plane or in a conic, which is not the case because there are no nonsingular curves of degree 14 and genus 24 on such surfaces. We have $G_1 \cap G_2 = D \cup q$ where q is a conic; since q has at most double points D has at most triple points and therefore G_1 and G_2 cannot be simultaneously singular at any point of D, thus the general quartic surface G containing D is nonsingular along D. By applying Riemann-Roch on G we obtain $\dim(|D|_G) = 24$. Therefore, since G is not a general quartic surface (because D is not a complete intersection), we see that the family of pairs (D, G) has dimension $\leq 33 + 24 = 57$ so that the family Z of curves D has dimension ≤ 56. This shows that the family W, which has dimension 56, cannot be in the closure of Z and this proves the assertion.

It is instructive to observe that we can write the linear system $|C|$ on a nonsingular cubic surface F as $|4H + 2E| = |6H - 2(H - E)|$ and this means that we can find a sextic surface F_6 such that $F \cap F_6 = C \cup q_1 \cup q_2$ where q_1 and q_2 are disjoint conics; if $[C] \in \Sigma$ is general then one can show that q_1, q_2 and F_6 can be chosen to be nonsingular.

There is another component R of $\mathrm{Hilb}^{\mathbb{P}^3}$ which generically parametrizes nonsingular curves C' of degree 14 and genus 24 such that

$$C' \cup E \cup \Gamma = F_3 \cap F_6$$

where E is a line and Γ is a rational normal cubic which are disjoint. We have in this case $|C'| = |6H - E - \Gamma|$ and

$$\begin{aligned} h^1(C', N_{C'}) &= h^1(C', \mathcal{O}_{C'}(3H)) = h^0(C', K_{C'}(-3H)) = \\ &= h^0(C', \mathcal{O}_{C'}(2H - E - \Gamma)) = h^0(F_3, \mathcal{O}_{F_3}(2H - E - \Gamma)) = 0 \end{aligned}$$

so that general curves of R are unobstructed.

We refer the reader to Curtin(1981) for another point of view about this example.

This example was the first published of this kind. Many others have appeared in the literature since thereafter (see Gruson-Peskine(1978), Gruson-Peskine(1982), Sernesi(1981), Ellia-Fiorentini(1984), Kleppe(1987), Walter(1992), Bolondi-Kleppe-Miro Roig(1991), Martin Deschamps-Perrin(1996), Guffroy(2003)).

A line is obstructed inside a cone

Let $Q \subset \mathbb{P}^3$ be a quadric cone with vertex v, and $L \subset Q$ a line. Then

$$N_{L/Q} = \mathcal{O}_L(1) \subset N_{L/\mathbb{P}^3} = \mathcal{O}_L(1) \oplus \mathcal{O}_L(1)$$

(see example (II.3.5)(iii)); in particular $H^0(L, N_{L/Q}) = 2$ and $H^1(L, N_{L/Q}) = 0$. On the other hand the Hilbert scheme Hilb^Q is 1-dimensional at the point $[L]$ since L moves in a 1-dimensional family. It follows that L is obstructed in Q (see Di Gennaro(1990) for generalizations of this example).

An obstructed (non reduced) scheme

In $\mathbb{P}^3$ consider the scheme

$$X = \mathrm{Proj}\Big(\mathbf{k}[X_0, \ldots, X_3]/J\Big)$$

where

$$J = (X_1X_2, X_1X_3, X_2X_3, X_3^2)$$

X is supported on the reducible conic defined by the equations

$$X_1X_2 = 0, \quad X_3 = 0$$

has an embedded point at $(1,0,0,0)$ and has Hilbert polynomial $2(t+1)$ (see example (IV.3.4)). As in (IV.3.4) we consider the flat family parametrized by $\mathbf{A}^1$:

$$\mathcal{X} = \mathrm{Proj}\Big(\mathbf{k}[u, X_0, \ldots, X_3]/(X_1X_2, X_1X_3, X_2(X_3 - uX_0), X_3(X_2 - uX_0))\Big) \subset \mathbb{P}^3 \times \mathbf{A}^1$$

We have $X = \mathcal{X}(0)$. If $u \neq 0$ then $\mathcal{X}(u)$ is a pair of disjoint lines. Let

$$g : \mathbf{A}^1 \to \mathrm{Hilb}^3_{2(t+1)}$$

be the classifying map. If $u \neq 0$ we have

$$h^1(\mathcal{X}(u), N_{\mathcal{X}(u)}) = 0; \quad h^0(\mathcal{X}(u), N_{\mathcal{X}(u)}) = 8$$

Therefore $g(u)$ is a smooth point and the tangent space has dimension 8.

In order to show that X is obstructed it suffices to show that

$$[IV.6.3] \qquad h^0(X, N_X) > 8$$

because $g(0)$ and $g(u)$ belong to the same irreducible component of $\mathrm{Hilb}^3_{2(t+1)}$.

Consider the surjection

$$\mathbf{f} : \mathcal{O}_{\mathbb{P}^3}(-2)^{\oplus 4} \to \mathcal{I}_X \to 0$$

determined by the four equations of degree two which define X. Elementary computations, based on the fact that the generators of the ideal are monomials, lead to the following resolution of $\mathcal{O}_X$ which extends $\mathbf{f}$:

$$[IV.6.4] \qquad 0 \to \mathcal{O}_{\mathbb{P}^3}(-4) \xrightarrow{B} \mathcal{O}_{\mathbb{P}^3}(-3)^{\oplus 4} \xrightarrow{A} \mathcal{O}_{\mathbb{P}^3}(-2)^{\oplus 4} \xrightarrow{\mathbf{f}} \mathcal{O}_{\mathbb{P}^3} \to \mathcal{O}_X \to 0$$

A and B being given by the following matrices:

$$B = \begin{pmatrix} X_3 \\ -X_3 \\ X_2 \\ -X_1 \end{pmatrix} \qquad A = \begin{pmatrix} X_3 & X_3 & 0 & 0 \\ -X_2 & 0 & X_3 & 0 \\ 0 & -X_1 & 0 & X_3 \\ 0 & 0 & -X_1 & -X_2 \end{pmatrix}$$

By taking $Hom(-, \mathcal{O}_X)$ we obtain the following exact sequence:

$$0 \to N_X \to \mathcal{O}_X(2)^{\oplus 4} \xrightarrow{{}^tA} \mathcal{O}_X(3)^{\oplus 4}$$

from which we deduce that

$$[IV.6.5] \qquad H^0(X, N_X) = \ker[H^0(\mathcal{O}_X(2))^{\oplus 4} \xrightarrow{{}^tA} H^0(X, \mathcal{O}_X(3))^{\oplus 4}]$$

Using resolution [IV.6.4] it is easy to show that the restriction maps

$$\varphi_n : H^0(\mathbb{P}^3, \mathcal{O}(n)) \to H^0(X, \mathcal{O}_X(n))$$

are surjective if $n \geq 2$. This allows us to identify $H^0(X, \mathcal{O}_X(2))$ and $H^0(X, \mathcal{O}_X(3))$ with the homogeneous parts of degree 2 and 3 respectively of $\mathbf{k}[X_0, \ldots, X_3]/J$. Hence using [IV.6.5] we can represent $H^0(X, N_X)$ by 4-tuples of polynomials. Precisely $H^0(X, N_X)$ is, modulo J, the vector space of 4-tuples

$$\underline{q} = (q_1, q_2, q_3, q_4)$$

of homogeneous polynomials of degree 2 such that $A\ {}^t\underline{q} \in (J_3)^4$. It is easy to find all of them because J is generated by monomials. Computing one finds that a basis of $H^0(X, N_X)$ is defined by the following column vectors:

$$\begin{matrix} X_1^2 & X_1X_0 & X_2^2 & X_2X_0 & X_3X_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & X_1^2 & X_1X_0 & X_3X_0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & X_2^2 & X_2X_0 & X_3X_0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & X_3X_0 \end{matrix}$$

In particular we see that $h^0(X, N_X) = 12$, and this proves [IV.6.3].

A little extra work shows that $[X] = g(0)$ belongs to two irreducible components of $\mathrm{Hilb}^3_{2(t+1)}$. We already know one of them of dimension 8: it contains $g(u)$, $u \neq 0$, and a general point of it parametrizes a pair of disjoint lines.

The other component has dimension 11 and a general point of it parametrizes the disjoint union $Y = Q \cup \{p\}$ of a conic Q and a point p. Note that

$$h^0(Y, N_Y) = h^0(Q, N_Q) + h^0(p, N_p) = 8 + 3 = 11$$

and $h^1(Y, N_Y) = 0$. Hence Y is a smooth point of a component of dimension 11 of $\mathrm{Hilb}^3_{2(t+1)}$. Therefore to conclude it suffices to produce a flat family parametrized by an irreducible curve, e.g. $\mathbf{A}^1$,

$$\mathcal{Y} \subset \mathbb{P}^3 \times \mathbf{A}^1$$

such that $\mathcal{Y}(0) = X$, $\mathcal{Y}(1) = Y$. Here it is:

$$\mathcal{Y} = \mathrm{Proj}\Big(\mathbf{k}[v, X_0, X_1, X_2, X_3]/\mathbf{I}\Big)$$

where

$$\mathbf{I} = (X_1X_2, X_1X_3 + vX_1X_0, X_2X_3 + vX_2X_0, X_3^2 - v^2X_0^2)$$

Clearly $\mathcal{Y}(0) = X$; since

$$\mathbf{I} = (X_1, X_2, X_3 - vX_0) \cap (X_3 + vX_0, X_1X_2)$$

it follows that for all $v \neq 0$ $\mathcal{Y}(v)$ is the disjoint union of a conic and a point. The flatness of $\mathcal{Y}$ follows from A.2(XV).

This example shows that in general the Hilbert schemes are reducible and not equidimensional.

A reducible Hilbert scheme of divisors

Another example of reducible Hilbert scheme is the following, which appears in Severi(1916). Let $Y = C \times C'$ where C and C' are projective nonsingular connected curves of genera g and g' respectively and let

$$\begin{array}{ccc} Y & \xrightarrow{p'} & C' \\ \downarrow p & & \\ C & & \end{array}$$

be the projections; assume $g, g' \geq 2$. Consider an effective divisor $D = x_1 + \cdots + x_g$ of degree g on C, and an effective divisor $D' = x'_1 + \cdots + x'_{g'}$ of degree g' on C', both consisting of distinct points, and let

$$\Gamma = p^{-1}(D) + p'^{-1}(D') = C'_{x_1} + \cdots + C'_{x_g} + C_{x'_1} + \cdots + C_{x'_{g'}}$$

where $C'_x = p^{-1}(x)$ and $C_{x'} = p'^{-1}(x')$. Γ is a reduced divisor, has gg' nodes and no other singularity. If either D or D' is non-special the curve Γ belongs to an irreducible component H_1 of Hilb^Y of dimension $g + g'$ generically consisting of curves of the same form, obtained by moving D and D'. When both D and D' are special divisors the curve Γ belongs to a linear system of dimension ≥ 3 whose general member is a nonsingular curve and therefore belongs to another irreducible component H_2 of Hilb^Y which has dimension $g + g' - 1$. The intersection $H_1 \cap H_2$ is irreducible of dimension $g + g' - 2$.

Elementary transformations

Let $\mathcal{H}$ be a coherent sheaf on the projective scheme X, and let $P(t) = n$ be a constant polynomial, where n is a positive integer. Then we have two different Quot schemes associated to these data.

The first one is $\text{Quot}^{X/X}_{\mathcal{H},n}$, whose $\mathbf{k}$-rational point are quotients $\mathcal{H} \to \mathcal{F}$ which are locally free of rank n. It will be considered later in this Section.

The other one is $\text{Quot}^X_{\mathcal{H},n}$. A $\mathbf{k}$-rational point of this scheme is nothing but a quotient $\mathcal{H} \to \mathcal{F}$ such that $\mathcal{F}$ is a torsion sheaf with finite support and $h^0(\mathcal{F}) = n$. When $n = 1$ then $\mathcal{F} \cong \mathbf{k}(x)$ for some closed point $x \in X$: therefore we have a natural morphism

$$\begin{array}{rccc} q: & \text{Quot}^X_{\mathcal{H},1} & \to & X \\ & & & \\ & (\mathcal{H} \to \mathcal{F}) & \mapsto & \text{Supp}(\mathcal{F}) \end{array}$$

and $\text{Quot}^X_{\mathcal{H},1}$ is a scheme over X. Let $\mathcal{H} \to \mathcal{F}$ be a $\mathbf{k}$-rational point of $\text{Quot}^X_{\mathcal{H},1}$; then $\ker[\mathcal{H} \to \mathcal{F}]$ is called an *elementary transform* of $\mathcal{H}$. The process of passing from $\mathcal{H}$ to $\ker[\mathcal{H} \to \mathcal{F}]$ is called an *elementary transformation* centered at x. This construction is classical when X is a projective nonsingular curve and $\mathcal{H}$ is locally free. For generalizations of it see Maruyama(1986).

(IV.6.1) PROPOSITION *Assume that* $\text{Supp}(\mathcal{H})$ *is connected. Then* $\text{Quot}^X_{\mathcal{H},1}$ *is connected.*

Proof
The natural morphism

$$\begin{array}{rccc} q: & \text{Quot}^X_{\mathcal{H},1} & \to & X \\ & & & \\ & (\mathcal{H} \to \mathcal{F}) & \mapsto & \text{Supp}(\mathcal{F}) \end{array}$$

has image $\text{Supp}(\mathcal{H})$. Every $\mathcal{H} \to \mathcal{F}$ factors as

$$\begin{array}{ccccc} \mathcal{H} & \to & \mathcal{H} \otimes \mathcal{O}_x & & \\ & & \downarrow & & \\ & & \mathcal{F} & & \cong O_x \\ & & \downarrow & & \\ & & 0 & & \end{array}$$

and therefore the fibre $q^{-1}(x)$ is identified with $I\!P(H^0(\mathcal{H} \otimes \mathcal{O}_x))^\vee$ which is connected. The conclusion follows. *q.e.d.*

Hilbert schemes of points

Consider a projective scheme Y and, for a positive integer n, the Hilbert scheme $Y^{[n]}$. We have already seen in §IV.3 that $Y^{[1]} \cong Y$: we will therefore assume $n \geq 2$.

Let $Z \subset Y$ be a closed subscheme of length n. Then

$$H^0(Z, N_Z) = \bigoplus_{y \in \mathrm{Supp}(Z)} \mathrm{Hom}(\mathcal{I}_{Z_y}, \mathcal{O}_{Z_y})$$

and

$$H^1(Z, N_Z) = (0)$$

Moreover from the exact sequence [IV.3.9] applied to $Z \subset Y$ we see that

$$\mathrm{Ext}^1(\mathcal{I}_Z, \mathcal{O}_Z) = H^0(Y, Ext^1(\mathcal{I}_Z, \mathcal{O}_Z)) = \bigoplus_{y \in \mathrm{Supp}(Z)} \mathrm{Ext}^1(\mathcal{I}_{Z_y}, \mathcal{O}_{Z_y})$$

It follows that the local properties of $Y^{[n]}$ are determined by the independent contributions from each of its points. The following properties follow at once:

a) *If Z is reduced and supported at n distinct points of Y then $[Z]$ is a nonsingular point of $Y^{[n]}$ if and only if it is supported at nonsingular points of Y.*

b) *If Y is reduced then the set of $[Z]$'s with Z supported at n nonsingular points of Y is an open set of dimension* $n \dim(Y)$ *contained in the nonsingular locus of $Y^{[n]}$.*

Another important property is the following:

c) (Fogarty(1968)) *If Y is connected then $Y^{[n]}$ is also connected.*

Proof

Let $n \geq 1$ and let $\mathcal{I} \subset \mathcal{O}_{Y \times Y^{[n]}}$ be the ideal sheaf of the universal family in $Y \times Y^{[n]}$. Then we have a diagram of morphisms:

$$\begin{array}{ccc} & \mathrm{Quot}^{Y \times Y^{[n]}}_{\mathcal{I},1} & \\ \swarrow p & & \searrow q \\ Y^{[n+1]} & & Y \times Y^{[n]} \end{array}$$

where q is the natural morphism, which is surjective because $\mathrm{Supp}(\mathcal{I}) = Y \times Y^{[n]}$. The morphism p is defined as follows.

Let $(y, [Z]) \in Y \times Y^{[n]}$ be a $\mathbf{k}$-rational point and let $\gamma : \mathcal{I} \to \mathbf{k}(y, [Z])$ be a quotient, which is a $\mathbf{k}$-rational point of $\mathrm{Quot}^{Y \times Y^{[n]}}_{\mathcal{I},1}$. Then $\ker(\gamma) \subset \mathcal{O}_{Y \times Y^{[n]}}$ is an ideal sheaf such that $\ker(\gamma)\mathcal{O}_{Y \times [Z]}$ has colength 1 in $\mathcal{I} \otimes \mathcal{O}_{Y \times [Z]}$. Therefore $\ker(\gamma)\mathcal{O}_{Y \times [Z]}$ defines a subscheme $W \subset Y$ of length $n+1$ containing Z and x; we define $p(\gamma) = [W]$. The morphism p is clearly sutjective. Since $Y \times Y^{[n]}$ is connected by induction, we conclude that $Y^{[n+1]}$ is connected by Proposition (IV.6.1). *q.e.d.*

In general $Y^{[n]}$ is singular even if Y is nonsingular. Notable exceptions are the cases $\dim(Y) = 1, 2$.

If C is a projective nonsingular curve and $n \geq 1$ an integer, every closed subscheme $D \subset C$ of length n is a Cartier divisor, therefore regularly embedded in C. It follows that $C^{[n]}$ is nonsingular of dimension

$$h^0(D, N_D) = h^0(D, \mathcal{O}_D) = n$$

The case of surfaces is more subtle.

(IV.6.2) THEOREM (Fogarty(1968)) *If Y is a projective nonsingular connected surface then $Y^{[n]}$ is nonsingular connected of dimension $2n$.*

Proof
Let $[Z] \in Y^{[n]}$. We then have:

$$\mathrm{Ext}^i(\mathcal{I}_Z, \mathcal{O}_Z) = (0) \qquad i \geq 3$$

Moreover from the exact sequence:

$$0 \to \mathcal{I}_Z \to \mathcal{O}_Y \to \mathcal{O}_Z \to 0$$

we obtain the sequence:

$$\begin{array}{ll} 0 & \to \mathrm{Hom}(\mathcal{O}_Z, \mathcal{O}_Z) \to \mathrm{Hom}(\mathcal{O}_Y, \mathcal{O}_Z) \to \mathrm{Hom}(\mathcal{I}_Z, \mathcal{O}_Z) \to \\ & \to \mathrm{Ext}^1(\mathcal{O}_Z, \mathcal{O}_Z) \to \mathrm{Ext}^1(\mathcal{O}_Y, \mathcal{O}_Z) \to \mathrm{Ext}^1(\mathcal{I}_Z, \mathcal{O}_Z) \to \\ & \to \mathrm{Ext}^2(\mathcal{O}_Z, \mathcal{O}_Z) \to \mathrm{Ext}^2(\mathcal{O}_Y, \mathcal{O}_Z) \to \mathrm{Ext}^2(\mathcal{I}_Z, \mathcal{O}_Z) \to \quad 0 \end{array}$$

Since $\mathrm{Ext}^i(\mathcal{O}_Y, \mathcal{O}_Z) = H^i(Y, \mathcal{O}_Z) = (0)$ for $i \geq 1$ we see that

$$\mathrm{Ext}^2(\mathcal{I}_Z, \mathcal{O}_Z) = (0)$$

and

$$\mathrm{Ext}^1(\mathcal{I}_Z, \mathcal{O}_Z) \cong \mathrm{Ext}^2(\mathcal{O}_Z, \mathcal{O}_Z) = \mathrm{Hom}(\mathcal{O}_Z, \mathcal{O}_Z \otimes \omega_Y)^\vee = \mathrm{Hom}(\mathcal{O}_Z, \mathcal{O}_Z)^\vee = H^0(Z, \mathcal{O}_Z)^\vee$$

Therefore

$$\sum_{i=0}^{2} (-1)^i \dim[\mathrm{Ext}^i(\mathcal{I}_Z, \mathcal{O}_Z)] = h^0(Z, N_Z) - h^0(Z, \mathcal{O}_Z) = h^0(Z, N_Z) - n$$

Since the left hand side is independent of Z, it follows that $h^0(Z, N_Z)$ is also independent of Z. But $Y^{[n]}$ is connected and has an open set which is nonsingular and of dimension $2n$: the conclusion follows. *q.e.d.*

To see that $Y^{[n]}$ is singular if $\dim(Y) = 3$ consider $\mathbb{P}^3$ with homogeneous coordinates $X_0, \ldots, X_3$ and the subscheme $Z = V(X_1^2, X_2^2, X_3^2, X_1X_2, X_1X_3, X_2X_3)$.

Then $[Z] \in (\mathbb{P}^3)^{[4]}$. A computation similar to that of the example of the previous subsection shows that the Zariski tangent space of $(\mathbb{P}^3)^{[4]}$ at $[Z]$ has dimension 18. But $(\mathbb{P}^3)^{[4]}$ is irreducible and its general point is nonsingular of dimension 12: it follows that $[Z]$ is a singular point.

Relative grassmannians and projective bundles

Consider a coherent sheaf $\mathcal{E}$ on an algebraic scheme S, and let $P(t) = n$, where n is a positive integer, be a constant polynomial. Then $\mathrm{Quot}^{S/S}_{\mathcal{E},n}$ is a projective S-scheme which will be denoted $\mathrm{Quot}_n(\mathcal{E})$ in what follows. We will denote by

$$\rho : \mathrm{Quot}_n(\mathcal{E}) \to S$$

the structural projective morphism and by

$$\rho^*\mathcal{E} \to \mathcal{Q}$$

the universal quotient sheaf; $\mathcal{Q}$ is locally free of rank n. The pair $(\mathrm{Quot}_n(\mathcal{E})/S, \mathcal{Q})$ represents the functor

$$Quot_n(\mathcal{E}) : (\mathrm{schemes}/S)^\circ \to (\mathrm{sets})$$

defined by:

$$Quot_n(\mathcal{E})(f : T \to S) = \{\text{locally free rk } n \text{ quotients } f^*\mathcal{E} \to \mathcal{F} \text{ on } T\}$$

On $\mathrm{Quot}_n(\mathcal{E})$ we have a *tautological exact sequence*

$$0 \to \mathcal{K} \to \rho^*(\mathcal{E}) \to \mathcal{Q} \to 0$$

If $\mathcal{E}$ is locally free we define

$$G_n(\mathcal{E}) := \mathrm{Quot}_n(\mathcal{E}^\vee)$$

and call it the *grassmannian bundle* of subbundles of rank n of $\mathcal{E}$. For $n = 1$ and $\mathcal{E}$ locally free we obtain

$$G_1(\mathcal{E}) =: \mathbb{P}(\mathcal{E})$$

the *projective bundle* associated to $\mathcal{E}$. The tautological exact sequence on $\mathbb{P}(\mathcal{E})$ is:

$$0 \to \mathcal{K} \to \rho^*(\mathcal{E}^\vee) \to \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1) \to 0$$

Note that for a finite dimensional $\mathbf{k}$-vector space V we have

$$\mathbb{P}(V \otimes_{\mathbf{k}} \mathcal{O}_S) = \mathbb{P}(V) \times S$$

(IV.6.3) PROPOSITION *Let $\mathcal{E}$ be a locally free sheaf on the algebraic scheme S, and let*

$$0 \to \mathcal{K} \to \rho^*(\mathcal{E}) \to \mathcal{Q} \to 0 \qquad [IV.6.2]$$

be the tautological exact sequence on $\mathrm{Quot}_n(\mathcal{E})$ *for some* $1 \leq n \leq \mathrm{rk}(\mathcal{E})$. *Then there is a natural isomorphism*

$$\Omega^1_{\mathrm{Quot}_n(\mathcal{E})/S} \cong Hom(\mathcal{Q}, \mathcal{K})$$

and therefore

$$T_{\mathrm{Quot}_n(\mathcal{E})/S} \cong Hom(\mathcal{K}, \mathcal{Q})$$

Proof

Letting $B = \mathrm{Quot}_n(\mathcal{E})$ consider the product $B \times_S B$ with projections $\mathrm{pr}_i : B \times_S B \to B$, $i = 1, 2$, and let $\mathcal{E}_{B\times_S B}$ be the pullback of $\mathcal{E}$ on $B \times_S B$. Denote by $\mathcal{I}_\Delta \subset \mathcal{O}_{B\times_S B}$ the ideal sheaf of the diagonal $\Delta \subset B \times_S B$. The tautological exact sequence [IV.6.2] pulls back to two exact sequences:

$$0 \to \mathrm{pr}_i^*\mathcal{K} \to \mathcal{E}^\vee_{B\times_S B} \to \mathrm{pr}_i^*\mathcal{Q} \to 0$$

on $B \times_S B$ whose restrictions to Δ coincide, and Δ is characterized by this property. This can be also expressed by saying that Δ is the vanishing scheme of the composition

$$\mathrm{pr}_1^*\mathcal{K} \to \mathcal{E}^\vee_{B\times_S B} \to \mathrm{pr}_2^*\mathcal{Q}$$

Therefore we have a surjective homomorphism:

$$Hom(\mathrm{pr}_2^*\mathcal{Q}, \mathrm{pr}_1^*\mathcal{K}) \to \mathcal{I}_\Delta$$

(see (IV.2.2)) which, restricted to Δ, gives a surjective homomorphism:

$$Hom(\mathcal{Q}, \mathcal{K}) \to \mathcal{I}_\Delta/\mathcal{I}_\Delta^2 = \Omega^1_{B/S}$$

Since both sheaves are locally free and of the same rank, it has to be an isomorphism. *q.e.d.*

(IV.6.4) PROPOSITION *Let*

$$0 \to \mathcal{E} \xrightarrow{\alpha} \mathcal{F} \xrightarrow{\beta} \mathcal{G} \to 0$$

be an exact sequence of locally free sheaves on the algebraic scheme S, and $n \geq 1$ an integer. Then there is a closed immersion

$$\mathrm{Quot}_n(\mathcal{G}) \subset \mathrm{Quot}_n(\mathcal{F})$$

and a natural identification:

$$N_{\mathrm{Quot}_n(\mathcal{G})/\mathrm{Quot}_n(\mathcal{F})} = \rho^*\mathcal{E}^\vee \otimes \mathcal{Q} \otimes \mathcal{O}_{\mathrm{Quot}_n(\mathcal{G})}$$

where $\rho : \mathrm{Quot}_n(\mathcal{F}) \to S$ *is the structure morphism and* $\rho^*\mathcal{F} \to \mathcal{Q}$ *is the universal quotient.*

Proof

Let $f : T \to S$ be a morphism. For every locally free rank n quotient

$$(f^*\mathcal{G} \to \mathcal{H}) \in Quot_n(\mathcal{G})(T)$$

there is associated, by composition with the surjective homomorphism $f^*(\beta) : f^*\mathcal{F} \to f^*\mathcal{G}$, an element

$$(f^*\mathcal{F} \to \mathcal{H}) \in Quot_n(\mathcal{F})(T)$$

Therefore $Quot_n(\mathcal{G})$ is a subfunctor of $Quot_n(\mathcal{F})$. Consider the diagram of homomorphisms on $\mathrm{Quot}_n(\mathcal{F})$:

$$\begin{array}{ccccc} \rho^*(\mathcal{E}) & & & & \\ \downarrow \rho^*(\alpha) & & & & \\ \rho^*(\mathcal{F}) & \xrightarrow{\gamma} & \mathcal{Q} & \to 0 \\ \downarrow \rho^*(\beta) & & & & \\ \rho^*(\mathcal{G}) & & & & \\ \downarrow & & & & \\ 0 & & & & \end{array}$$

Given a morphism $f : T \to S$, an element of

$$Quot_n(\mathcal{F})(T) = \mathrm{Hom}_S(T, \mathrm{Quot}_n(\mathcal{F}))$$

belongs to $Quot_n(\mathcal{G})(T)$ if and only if it factors through the closed subscheme $\mathrm{D}_0(\gamma\rho^*(\alpha))$ of $\mathrm{Quot}_n(\mathcal{F})$. This proves that $Quot_n(\mathcal{G})$ is a closed subfunctor of $Quot_n(\mathcal{F})$, and therefore the embedding $\mathrm{Quot}_n(\mathcal{G}) \subset \mathrm{Quot}_n(\mathcal{F})$ is closed. More precisely, this analysis shows that $\mathrm{Quot}_n(\mathcal{G}) = \mathrm{D}_0(\gamma\rho^*(\alpha))$. According to Example (IV.2.8) we therefore have a surjective homomorphism:

$$Hom(\mathcal{Q}, \rho^*(\mathcal{E})) \to \mathcal{I}$$

where $\mathcal{I} \subset \mathcal{O}_{\mathrm{Quot}_n(\mathcal{F})}$ is the ideal sheaf of $\mathrm{Quot}_n(\mathcal{G})$. By restricting to $\mathrm{Quot}_n(\mathcal{G})$ we obtain a surjective homomorphism:

$$Hom(\mathcal{Q}, \rho^*\mathcal{E}) \otimes \mathcal{O}_{\mathrm{Quot}_n(\mathcal{G})} \to \mathcal{I}/\mathcal{I}^2 \to 0$$

which is an isomorphism because both are locally free and of the same rank. The conclusion follows. *q.e.d.*

The following is immediate:

(IV.6.5) COROLLARY *Let*

$$0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{G} \to 0$$

be an exact sequence of locally free sheaves on the algebraic scheme S. Then there is a closed immersion

$$\mathbb{P}(\mathcal{E}) \subset \mathbb{P}(\mathcal{F})$$

and a natural identification:

$$N_{\mathbb{P}(\mathcal{E})/\mathbb{P}(\mathcal{F})} = \rho^*\mathcal{G} \otimes \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$$

(IV.6.6) REMARKS: Let $\mathcal{E}$ be a locally free sheaf on an algebraic scheme S, and let

$$[IV.6.3] \qquad 0 \to \mathcal{Q}^\vee \to \rho^*(\mathcal{E}) \to \mathcal{K}^\vee \to 0$$

be the dual of the tautological exact sequence [IV.6.2] on $G_n(\mathcal{E}) = \mathrm{Quot}_n(\mathcal{E}^\vee)$. Tensoring with $\mathcal{Q}$ we obtain the exact sequence:

$$[IV.6.4] \qquad \begin{array}{ccccccc} 0 \to & \mathcal{Q}^\vee \otimes \mathcal{Q} & \to & \rho^*(\mathcal{E}) \otimes \mathcal{Q} & \to & \mathcal{K}^\vee \otimes \mathcal{Q} & \to 0 \\ & & & & & \| & \\ & & & & & T_{G_n(\mathcal{E})} & \end{array}$$

In the case $n = 1$ and $S = \mathrm{Spec}(\mathbf{k})$ we have $\mathcal{E} = V$ a vector space and $G_1(V) = \mathbb{P}(V) = \mathbb{P}$; the dual of the tautological sequence is

$$0 \to \mathcal{O}_{\mathbb{P}}(-1) \to V \otimes \mathcal{O}_{\mathbb{P}} \to T_{\mathbb{P}}(-1) \to 0$$

and the sequence [IV.6.4] is the Euler sequence

$$0 \to \mathcal{O}_{\mathbb{P}} \to V \otimes \mathcal{O}_{\mathbb{P}}(1) \to T_{\mathbb{P}} \to 0$$

Therefore [IV.6.4] is a generalization of the Euler sequence.

The scheme $\underline{\mathrm{Hom}}(X, Y)$.

Let X and Y be schemes, with X projective and Y quasi-projective. For every scheme S let:

$$F(S) = \mathrm{Hom}_S(X \times S, Y \times S)$$

This defines a contravariant functor:

$$F : (\text{schemes})^\circ \to (\text{sets})$$

called *the functor of morphisms from X to Y.*

For every $\Phi \in F(S)$ let $\Gamma_\Phi \subset X \times Y \times S$ be its graph. Then $\Gamma_\Phi \cong X \times S$ is flat over S and therefore defines a flat family of closed subschemes of $X \times Y$ parametrized by S. This means that F is a subfunctor of $Hilb^{X \times Y}$.

If $G \subset X \times Y \times S$ is a flat family of closed subschemes of $X \times Y$, proper over S, then the projection $\pi : G \to Y \times S$ is a family of morphisms into Y and the locus of points $s \in S$ such that $\pi(s)$ is an isomorphism is open (Note 2 of §IV.2). This means that F is an open subfunctor of $Hilb^{X \times Y}$, represented by an open subscheme of $\mathrm{Hilb}^{X \times Y}$, which we denote $\underline{\mathrm{Hom}}(X, Y)$. It is called *the scheme of morphisms of X into Y.*

Let X and Y be as above, and consider the contravariant functor

$$G : (\text{schemes})^\circ \to (\text{sets})$$

defined as follows:

$$G(S) = \{S\text{-isomorphisms } X \times S \to Y \times S\}$$

Clearly G is a subfunctor of F. It is easy to prove that G is represented by an open subscheme $\underline{\mathrm{Isom}}(X, Y)$ of $\underline{\mathrm{Hom}}(X, Y)$, called *the scheme of isomorphisms from X to Y.* When $X = Y$ it is denoted $\underline{\mathrm{Aut}}(X)$ and called *the scheme of automorphisms of X.*

The following result follows immediately from Proposition (II.3.8) and Corollary (II.3.9):

(IV.6.7) PROPOSITION *Let $f : X \to Y$ be a morphism of algebraic schemes, with X reduced and projective and Y nonsingular and quasiprojective. Then*

$$T_{[f]}(\underline{\mathrm{Hom}}(X, Y)) \cong H^0(X, f^*T_Y)$$

and the obstruction space of $\underline{\mathrm{Hom}}(X, Y)$ at $[f]$ is contained in $H^1(X, f^*T_Y)$. If X is nonsingular then the tangent space to $\underline{\mathrm{Aut}}(X)$ at 1_X is $H^0(X, T_X)$.

Let $j : X \subset Y$ be a closed embedding of projective nonsingular schemes. Then j induces an inclusion

$$J : \underline{\mathrm{Aut}}(X) \subset \underline{\mathrm{Hom}}(X, Y)$$

such that $J(1_X) = j$ and which is induced by the closed embedding $1_X \times j : X \times X \subset X \times Y$. It follows that J is a closed embedding. Its differential at 1_X is the injective linear map

$$H^0(T_X) \to H^0(T_{Y|X})$$

coming from the natural inclusion $T_X \subset T_{Y|X}$. In fact from the diagram of inclusions:

$$\begin{array}{ccc} X \times X & \subset & X \times Y \\ \cup & & \cup \\ \Delta & \cong & \Gamma_j \end{array}$$

we deduce the commutative diagram:

$$\begin{array}{ccc} N_{\Delta/X\times X} & \subset & N_{\Gamma_j/X\times Y} \\ \| & & \| \\ T_X & \subset & T_{Y|X} \end{array}$$

and we conclude according to §IV.4, Note 2.

As an example consider $X = \mathbb{P}^1$, $Y = \mathbb{P}^r$ and $j : \mathbb{P}^1 \to \mathbb{P}^r$ the r-th Veronese embedding. Locally around $[j]$ we have a well defined morphism

$$M : \underline{\mathrm{Hom}}(\mathbb{P}^1, \mathbb{P}^r) \to \mathrm{Hilb}^{\mathbb{P}^r}$$

sending $[j] \mapsto [j(\mathbb{P}^1)]$ with fibre $M^{-1}([j(\mathbb{P}^1)])$ an open neighborhood of the identity in $\mathrm{Aut}(\mathbb{P}^1)$. Consider the following diagram consisting of two exact sequences:

$$\begin{array}{ccccccc} & & & 0 & & & \\ & & & \uparrow & & & \\ 0 \to & T_{\mathbb{P}^1} & \to & j^*T_{\mathbb{P}^r} & \to & N_j & \to 0 \\ & & & \uparrow & & & \\ & & & \mathcal{O}_{\mathbb{P}^1(r)^{r+1}} & & & \\ & & & \uparrow & & & \\ & & & \mathcal{O}_{\mathbb{P}^1} & & & \\ & & & \uparrow & & & \\ & & & 0 & & & \end{array}$$

From the vertical sequence (the Euler sequence restricted to $\mathbb{P}^1$) we get

$$h^1(j^*T_{\mathbb{P}^r}) = 0, \quad h^0(j^*T_{\mathbb{P}^r}) = r(r+2)$$

Since $h^1(T_{\mathbb{P}^1}) = 0$ from the other sequence we obtain $h^1(N_j) = 0$ and the exact sequence

$$\begin{array}{ccccccc} (0) \to & H^0(T_{\mathbb{P}^1}) & \to & H^0(j^*T_{\mathbb{P}^r}) & \xrightarrow{*} & H^0(N_j) & \to (0) \\ & \| & & \| & & \| & \\ & T_{1_{\mathbb{P}^1}}\mathrm{Aut}(\mathbb{P}^1) & & T_{[j]}\underline{\mathrm{Hom}}(\mathbb{P}^1, \mathbb{P}^r) & & T_{j(\mathbb{P}^1)]}\mathrm{Hilb}^{\mathbb{P}^r} & \end{array}$$

Since the map $*$ can be identified with $dM_{[j]}$ we see that M and $\underline{\mathrm{Hom}}(\mathbb{P}^1, \mathbb{P}^r)$ are smooth at $[j]$ and $\mathrm{Hilb}^{\mathbb{P}^r}$ is smooth at $[j(\mathbb{P}^1)]$; moreover

$$\dim_{[j]}(\underline{\mathrm{Hom}}(\mathbb{P}^1, \mathbb{P}^r) = r(r+2) = \dim_{j(\mathbb{P}^1)}(\mathrm{Hilb}^{\mathbb{P}^r}) + 3$$

For more on the schemes $\underline{\mathrm{Hom}}(\mathbb{P}^1, X)$ and applications to uniruledness see Debarre(2001).

IV.7. SEVERI VARIETIES

An important refinement of the Hilbert functors derives from the consideration of flat families of closed subschemes of a projective scheme Y having prescribed singularities, i.e. of families all of whose members have the same type of singularity in some specified sense. This leads to the notion of *equisingularity* and to a related vast area of research. In this Section we will only concentrate on the specific case of families of plane curves with assigned number of nodes and cusps: we will show how to construct universal families of such curves, whose parameter schemes are called *Severi varieties* for historical reasons. This is a subject with a long history and a wealth of important results, both classical and modern. Here we will limit ourselves to prove a few basic results and to indicate some of their generalizations and the main references in the literature. We will assume $\mathrm{char}(\mathbf{k}) = 0$ in this Section.

* * * * * *

Equisingular infinitesimal deformations

Let Y be a projective nonsingular variety and $X \subset Y$ a closed subscheme whose ideal sheaf $\mathcal{I}_X \subset \mathcal{O}_Y$ we will sometimes simply denote by $\mathcal{I}$ in this subsection. Recall ([II.4.4]) that on X we have an exact sequence of coherent sheaves:

$$0 \to T_X \to T_{Y|X} \to N_{X/Y} \to T^1_X \to 0$$

The sheaf

$$N'_{X/Y} := \ker[N_{X/Y} \to T^1_X]$$

is called the *equisingular normal sheaf* of X in Y. Clearly $N'_{X/Y} = N_{X/Y}$ if X is nonsingular (see §II.4). By definition sections of the equisingular normal sheaf parametrize first order deformations of X in Y which are locally trivial, because they induce trivial deformations around every point of X.

An alternative description of the equisingular normal sheaf can be given as follows. Let

$$T_Y(-\log X) \subset T_Y$$

be the inverse image of T_X under the natural restriction homomorphism $\mathrm{T}_Y \to T_{Y|X}$. Then $T_Y(-\log X)$ is called *the sheaf of germs of tangent vectors to* Y *which are tangent to* X. We clearly have an inclusion $\mathcal{I}T_Y \subset T_Y(-\log X)$ such that

$$T_X = T_Y(-\log X)/\mathcal{I}T_Y$$

and an exact sequence

$$[IV.7.1] \qquad 0 \to T_Y(-\log X) \to T_Y \to N'_{X/Y} \to 0$$

From the definition it follows that, for every open set $U \subset Y$, $\Gamma(U, T_Y(-\log X))$ consists of those **k**-derivations $D \in \Gamma(U, T_Y)$ such that $D(g) \in \Gamma(U, \mathcal{I})$ for every $g \in \Gamma(U, \mathcal{I})$.

(IV.7.1) EXAMPLES

(i) Assume that X is a hypersurface in Y; then $N_{X/Y} \cong \mathcal{O}_X(X)$. Locally on an affine open set $U \subset Y$ we have $(N_{X/Y})_{|U\cap X} \cong \mathcal{O}_{U\cap X}$. Assume that X can be represented by an equation $f(x_1, \dots, x_n) = 0$ in local coordinates on U; then from the definition of T^1_X it follows that $(N'_{X/Y})_{|U\cap X} \subset \mathcal{O}_{U\cap X}$ is the image of the ideal sheaf $(\partial f/\partial x_1, \dots, \partial f/\partial x_n) \subset \mathcal{O}_U$. We deduce that an equisingular first order deformation of X in Y corresponding to a local section $\bar{g}$ of $N'_{X/Y}$ can be written locally as

$$f(\underline{x}) + \epsilon g(\underline{x}) = 0$$

where

$$g(\underline{x}) = a_1(\underline{x})\frac{\partial f}{\partial x_1} + \cdots + a_n(\underline{x})\frac{\partial f}{\partial x_n}$$

restricts to $\bar{g}$. Therefore if $Y = \mathbb{P}^n$ and X is a hypersurface of degree d we have an exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^n} \to \mathcal{I}(d) \to N'_{X/\mathbb{P}^n} \to 0$$

where $\mathcal{I} \subset \mathcal{O}_{\mathbb{P}^n}$ is the ideal sheaf locally generated by the partial derivatives of a local equatio of X. In the special case of a curve X in a surface Y assume that $p \in X$ is a singular point and let $f(x, y) = 0$ be a local equation of X around p. If p is a node then $(\partial f/\partial x, \partial f/\partial y) = m_p$ is just the maximal ideal of p; if p is an ordinary cusp with principal tangent say $y = 0$ then $(\partial f/\partial x, \partial f/\partial y) = (x, y^2)$ is an ideal of colength 2.

(ii) Let $X \subset \mathbb{P}^2$ be a (possibly reducible) plane curve of degree d of equation $F(X_0, X_1, X_2) = 0$, having δ nodes $p_1, \dots, p_\delta$ and no other singularity. This case is important because every nonsingular projective curve is birationally equivalent to a nodal plane curve. Denote by $\Delta = \{p_1, \dots, p_\delta\} \subset \mathbb{P}^2$ the 0-dimensional reduced scheme of the nodes of X and by $\nu : C \to X$ the normalization map. The above analysis shows that sections of $H^0(\mathcal{I}_\Delta(d))$, i.e. curves of degree d which are adjoint to X, cut on X sections of $N'_{X/\mathbb{P}^2}$. This means that

$$\nu^*(N'_{X/\mathbb{P}^2}) = \nu^*[\mathcal{O}_X(d) \otimes \mathcal{I}_\Delta] =$$

$$= \nu^*\mathcal{O}_X(d)(-p'_1 - p''_1 - \cdots - p'_\delta - p''_\delta) = \omega_C \otimes \nu^*\mathcal{O}(3)$$

where $\nu^{-1}(p_i) = \{p'_i, p''_i\}$, $i = 1, \dots, \delta$, and therefore we have

$$h^0(C, \nu^*(N'_{X/\mathbb{P}^2})) = 3d + g - 1, \quad h^1(C, \nu^*(N'_{X/\mathbb{P}^2})) = 0$$

where g is the geometric genus of X. Moreover, since

$$\nu_*[\nu^*(N'_{X/I\!P^2})] = \nu_*[\nu^*(\mathcal{O}_X(d) \otimes \mathcal{I}_\Delta] =$$

$$\mathcal{O}_X(d) \otimes \nu_*\mathcal{O}_C(-p'_1 - p''_1 - \cdots - p'_\delta - p''_\delta) = \mathcal{O}_X(d) \otimes \mathcal{I}_\Delta = N'_{X/I\!P^2}$$

we have

[*IV*.7.2]
$$h^0(X, N'_{X/I\!P^2}) = h^0(C, \nu^*(N'_{X/I\!P^2})) = 3d + g - 1 = \tbinom{d+2}{2} - \delta - 1$$
$$h^1(X, N'_{X/I\!P^2}) = h^1(C, \nu^*(N'_{X/I\!P^2})) = 0$$

Finally, since

$$h^0(\mathcal{I}_\Delta(d)) \geq \binom{d+2}{2} - \delta = 3d + g$$

and $H^0(\mathcal{I}_\Delta(d))/(F) \subset H^0(N'_{X/I\!P^2})$, comparing with [IV.7.2] we see that Δ imposes independent conditions to curves of degree d and that

$$H^0(N'_{X/I\!P^2}) = H^0(\mathcal{I}_\Delta(d))/(F)$$

or, equivalently, the restriction map

$$H^0(\mathcal{I}_\Delta(d)) \to H^0(N'_{X/I\!P^2})$$

is surjective. Note that $N'_{X/I\!P^2}$ is a *non invertible* subsheaf of $N_{X/I\!P^2}$.

(iii) Another interesting case is obtained by taking an irreducible curve $X \subset I\!P^2$ of degree d having δ nodes $p_1, \ldots, p_\delta$ and κ ordinary cusps $q_1, \ldots, q_\kappa$ as its only singularities. This case is important because branch curves of generic projection on $I\!P^2$ of projective nonsingular surfaces are curves of this type.

Let $\nu : C \to X$ be the normalization map. Denoting $\bar{q}_j = \nu^{-1}(q_j)$, $j = 1, \ldots, \kappa$, we have in this case, according to the above description

$$\nu^*(N'_{X/P^2}) = \mathcal{O}_C(d)(-p'_1 - p''_1 - \cdots - p'_\delta - p''_\delta - 3\bar{q}_1 - \cdots - 3\bar{q}_\kappa) =$$

$$= \omega_C \otimes \nu^*\mathcal{O}_X(3)(-\bar{q}_1 - \cdots - \bar{q}_\kappa)$$

As before one shows that $\nu_*[\nu^*(N'_{X/P^2})] = N'_{X/P^2}$ and therefore

[*IV*.7.3]
$$h^0(X, N'_{X/I\!P^2}) = h^0(C, \omega_C \otimes \nu^*\mathcal{O}_X(3)(-\bar{q}_1 - \cdots - \bar{q}_\kappa)) \geq \binom{d+2}{2} - \delta - 2\kappa - 1$$

and in general we may have strict inequality and $h^1(X, N'_{X/I\!P^2}) \neq 0$ because the invertible sheaf $\omega_C \otimes \nu^*\mathcal{O}_X(3)(-\bar{q}_1 - \cdots - \bar{q}_\kappa)$ can be special. But if $\kappa < 3d$ then it is certainly non special and therefore in such a case we have

$$h^0(X, N'_{X/I\!P^2}) = \tbinom{d+2}{2} - \delta - 2\kappa - 1 = 3d + g - 1 - \kappa$$

$$h^1(X, N'_{X/I\!P^2}) = 0$$

* * * * * *

The Severi varieties

Given an integer $d > 0$ consider the complete linear system $|\mathcal{O}(d)|$ of plane curves of degree d. It is a flat family of closed subschemes (precisely of Cartier divisors) of $\mathbb{P}^2$ parametrized by the projective space $\Sigma_d = \mathbb{P}[H^0(\mathbb{P}^2, \mathcal{O}(d))]$:

$$|\mathcal{O}(d)| : \begin{array}{c} \mathcal{H} \subset \mathbb{P}^2 \times \Sigma_d \\ \downarrow \\ \Sigma_d \end{array}$$

The linear system $|\mathcal{O}(d)|$ has a universal property with respect to families of plane curves of degree d because The pair $(\Sigma_d, |\mathcal{O}(d)|)$ represents the Hilbert functor:

$$\Lambda_d : (\text{algschemes})^\circ \to (\text{sets})$$

$$\Lambda_d(S) = \{\text{flat families } \mathcal{C} \subset \mathbb{P}^2 \times S \text{ of plane curves of degree } d \text{ parametrized by } S\}$$

(see §IV.3). In this subsection we want to consider the problem of constructing a universal family of reduced curves in $\mathbb{P}^2$ having degree d, an assigned number δ of nodes and κ of ordinary cusps and no other singularity. If such a universal family exists it is parametrized by a scheme which we denote by $\mathcal{V}_d^{\delta,\kappa}$. These schemes have been studied classically: the foundations of their theory were given in Severi(1921) and they are therefore called *Severi schemes* or *Severi varieties*.

If the Severi scheme $\mathcal{V}_d^{\delta,\kappa}$ exists then, by the universal property, there is a functorially defined morphism

$$[IV.7.4] \qquad \mathcal{V}_d^{\delta,\kappa} \to \Sigma_d$$

We start from the definition of the functor we want to represent.

(IV.7.2) DEFINITION *Let d, δ, κ as above. Then*

$$\mathbf{V}_d^{\delta,\kappa} : (\mathit{algschemes})^\circ \to (\mathit{sets})$$

is defined as follows. For each algebraic scheme S

$$\mathbf{V}_d^{\delta,\kappa}(S) = \left\{ \begin{array}{c} \textit{flat families } \mathcal{C} \subset \mathbb{P}^2 \times S \textit{ of plane curves of degree } d \\ \textit{formally locally trivial at each } \mathbf{k}\textit{-rational } s \in S \textit{ whose geometric} \\ \textit{fibres are curves with } \delta \textit{ nodes and } \kappa \textit{ cusps as singularities} \end{array} \right\}$$

Obviously $\mathbf{V}_d^{\delta,\kappa}$ is a subfunctor of Λ_d. The main result about $\mathbf{V}_d^{\delta,\kappa}$ is the following

(IV.7.3) THEOREM *For each d, δ, κ as above the functor $\mathbf{V}_d^{\delta,\kappa}$ is represented by an algebraic scheme $\mathcal{V}_d^{\delta,\kappa}$ which is a (possibly empty) locally closed subscheme of Σ_d.*

In case $\kappa = 0$ we write $\mathcal{V}_d^\delta$ instead of $\mathcal{V}_d^{\delta,0}$. The first published proof of this result is in Wahl(1974). We will not reproduce it in full generality here, but we will only consider the case $\kappa = 0$, i.e. the case of nodal curves. This assumption allows a technically simpler argument without changing the structure of the original proof. We need some Lemmas.

(IV.7.4) LEMMA *Let $p \in \mathbb{P}^2$, $\mathcal{O}$ the local ring of $\mathbb{P}^2$ at p, $B_0 = \mathcal{O}/(f_0)$ the local ring of a plane curve having a node at p. Assume that for some A in $\mathcal{A}$ we have a deformation $A \to B$ of B_0 over A such that $T^1_{B/A}$ is A-flat. Then B is trivial and $T^1_{B/A} \cong A$.*

Proof

By induction on $\dim_{\mathbf{k}}(A)$. The case $A = \mathbf{k}$ is trivial because

$$T^1_{B_0} = \mathcal{O}/(f_0, f_{0X}, f_{0Y}) = \mathcal{O}/(f_{0X}, f_{0Y}) \cong \mathbf{k}$$

(see (II.4.2)). In the general case consider a small extension

$$0 \to (\epsilon) \to A \to A' \to 0$$

and the induced deformation $A' \to B'$. We have $B = (A \otimes_{\mathbf{k}} \mathcal{O})/(f)$ for some f which reduces to f_0 modulo m_A, and $T^1_{B/A} = B/(f_X, f_Y)$. Therefore $B' = (A' \otimes_{\mathbf{k}} \mathcal{O})/(f')$ where f' is obtained from f by reducing the coefficients to A', and

$$T^1_{B'/A'} = B'/(f'_X, f'_Y) = B/(f_X, f_Y) \otimes_A A' = T^1_{B/A} \otimes_A A'$$

It follows that $T^1_{B'/A'}$ is A'-flat and, by induction, we have

$$B' = (A' \otimes_{\mathbf{k}} \mathcal{O})/(f_0)$$

and

$$T^1_{B'/A'} = A' \otimes_{\mathbf{k}} [\mathcal{O}/(f_0, f_{0X}, f_{0Y})] = A'$$

Thus $f = f_0 + \epsilon g$ where $g \in \mathbf{k}$. We have:

$$T^1_{B/A} = (A \otimes_{\mathbf{k}} \mathcal{O})/(f_0 + \epsilon g, f_{0X}, f_{0Y}) = A/(\epsilon g)$$

where the last equality follows from the fact that $f_0 \in (f_{0X}, f_{0Y})$. Since $A/(\epsilon g)$ is A-flat if and only if $g = 0$ it follows that $B = (A \otimes_{\mathbf{k}} \mathcal{O})/(f_0) = A \otimes_{\mathbf{k}} (\mathcal{O}/(f_0))$ is the trivial deformation and $T^1_{B/A} = A$. *q.e.d.*

(IV.7.5) LEMMA *Let $f : X \to S$ be a flat morphism of algebraic schemes which factors as*

$$\begin{array}{ccc} X & \xrightarrow{j} & Y \\ & \searrow & \downarrow q \\ & & S \end{array}$$

where j is a regular embedding of codimension 1 and q is smooth. Then for every morphism of algebraic schemes $\varphi : S' \to S$ we have

$$T^1_{S' \times_S X/S'} \cong \Phi^* T^1_{X/S}$$

where $\Phi : S' \times_S X \to X$ is the projection (i.e. $T^1_{X/S}$ commutes with base change).

Proof

Since the question is local we may reduce to a diagram of **k**-algebras of the form:

$$\begin{array}{ccc} B = P/(f) & \to & B' = P_{A'}/(f') \\ \uparrow & & \uparrow \\ A & \to & A' \end{array}$$

where P is a smooth A-algebra, $f \in P$ is a regular element, and f' is the image of f in $P_{A'} = P \otimes_A A'$. Then we have

$$\begin{array}{ccc} (f)/(f^2) \cong B & \xrightarrow{\delta} & \Omega_{P/A} \otimes_P B \\ \bar{f} & \mapsto & df \otimes 1 \end{array}$$

and

$$\begin{array}{ccc} (f')/(f'^2) \cong B' & \xrightarrow{\delta'} & \Omega_{P'/A'} \otimes_{P'} B' \\ \bar{f}' & \mapsto & df' \otimes 1 \end{array}$$

But since

$$\Omega_{P'/A'} \otimes_{P'} B' = (\Omega_{P/A} \otimes_P B) \otimes_B B'$$

we have $\delta' = \delta \otimes_B B'$ and

$$T^1_{B/A} \otimes_B B' = \text{coker}(\delta^\vee) \otimes_B B' = \text{coker}(\delta'^\vee) = T^1_{B'/A'}$$

q.e.d.

(IV.7.6) LEMMA *Let S be an algebraic scheme and $\mathcal{C} \subset I\!P^2 \times S$ a flat family of plane curves of degree d. Let $s \in S$ be a **k**-rational point such that the fibre $\mathcal{C}(s)$ is a curve having at most nodes as singularities. Then $T^1_{\mathcal{C}/S}$ is S-flat at a point $p \in \mathcal{C}(s)$ if and only if the family is formally locally trivial at s around p.*

Proof

If p is a nonsingular point of $\mathcal{C}(s)$ then $\mathcal{C} \to S$ is smooth at p, hence $T^1(\mathcal{C}/S, \mathcal{O}_\mathcal{C})_p = 0$ and the assertion is obvious.

Let's assume that p is a node of $\mathcal{C}(s)$. Let $A = \mathcal{O}_{S,s}$, $A_\alpha := A/m^\alpha$, $S_\alpha = \text{Spec}(A_\alpha)$ and $\mathcal{C}_\alpha \to S_\alpha$ the induced infinitesimal deformation of $\mathcal{C}_1 = \mathcal{C}(s)$; denote by $B = \mathcal{O}_{\mathcal{C},p}$ and by $B_\alpha = \mathcal{O}_{\mathcal{C}_\alpha,p}$. By Lemma (IV.7.5) we have:

$$[IV.7.5] \qquad T^1_{\mathcal{C}_\alpha/S_\alpha,p} = T^1_{B_\alpha/A_\alpha} \cong T^1(\mathcal{C}/S, \mathcal{O}_\mathcal{C})_p \otimes_{\mathcal{O}_{\mathcal{C},p}} B_\alpha = T^1_{B/A} \otimes_B B_\alpha$$

Assume that $T^1(\mathcal{C}/S, \mathcal{O}_\mathcal{C})_p = T^1_{B/A}$ is A-flat. Then $T^1_{B_\alpha/A_\alpha}$ is A_α-flat by [IV.7.5] and therefore B_α is the trivial deformation of $B_0 = B/m_A B = \mathcal{O}_{\mathcal{C}(s),p}$, by Lemma

(IV.7.4). Since this is true for every α we conclude that the family $\mathcal{C} \to S$ is formally locally trivial at p.

Conversely, assume that $\mathcal{C} \to S$ is formally locally trivial at p. Then $B_\alpha \cong B_0 \otimes_{\mathbf{k}} A_\alpha$ for all α and $T^1_{B_\alpha/A_\alpha}$ is A_α-flat by Lemma (IV.7.4). But by [IV.7.5] we have

$$T^1_{B_\alpha/A_\alpha} = T^1_{B/A} \otimes_B B_\alpha = T^1_{B/A} \otimes_A A_\alpha$$

and from the local criterion of flatness we deduce that $T^1_{B/A} = T^1(\mathcal{C}/S, \mathcal{O}_{\mathcal{C}})_p$ is A-flat. *q.e.d.*

Proof of Theorem (IV.7.3)

Consider the universal family $\mathcal{H} \subset I\!P^2 \times \Sigma_d$ of plane curves of degree d and let $\{W_i\}$ be the flattening stratification of $T^1_{\mathcal{H}/\Sigma_d}$. Let $W = W_i$ be a stratum containing a $\mathbf{k}$-rational point $s \in \Sigma_d$ parametrizing a reduced curve $\mathcal{H}(s)$ having δ nodes and no other singularity, and let $\mathcal{H}' \subset I\!P^2 \times W$ be the induced family of degree d curves. By Lemma (IV.7.5) we have

$$T^1_{\mathcal{H}'/W} \cong T^1_{\mathcal{H}/\Sigma_d} \otimes \mathcal{O}_{\mathcal{H}'}$$

and therefore, by construction, $T^1_{\mathcal{H}'/W}$ is flat over W. Moreover, since $\mathcal{H}' \subset I\!P^2 \times W$ is a regular embedding of codimension 1, $T^1_{\mathcal{H}'/W}$ is of the form $\mathcal{O}_V$ for some closed subscheme $V \subset I\!P^2 \times W$. By applying Lemma (IV.7.5) again we deduce

$$\mathcal{O}_{V(s)} = T^1_{\mathcal{H}'/W} \otimes \mathbf{k} \cong T^1_{\mathcal{H}(s)}$$

which is a reduced scheme of length δ supported at $\mathrm{Sing}(\mathcal{H}(s))$. This implies that $V \to W$ is etale at the δ points of $V(s)$.Therefore there is an open neighborhood U of $s \in W$ such that $V(U) \to U$ is etale of degree δ. If $u \in U$ is a $\mathbf{k}$-rational point then $\mathcal{H}(u)$ is a curve such that $T^1_{\mathcal{H}(u)} \cong V(u)$, hence $\mathcal{H}(u)$ has δ singular points $p_1, \dots, p_\delta$ and no other singularity, such that $T^1_{\mathcal{O}_{p_j}} \cong \mathbf{k}$. From Proposition (II.4.6) it follows that $\mathcal{H}(u)$ is a δ-nodal curve. Therefore by applying Lemma (IV.7.6) we see that the family $\mathcal{H}'(U) \to U$ is an element of $\mathbf{V}^{\delta,0}_d(U)$.

Putting together all these open sets we obtain a locally closed subset $U_i \subset W_i$ such that the induced family $\mathcal{H}'_i \subset I\!P^2 \times U_i$ defines an element of $\mathbf{V}^{\delta,0}_d(U_i)$. Now let

$$\mathcal{V}^\delta_d = \cup_i U_i$$

and let $\overline{\mathcal{H}} \subset I\!P^2 \times \mathcal{V}^\delta_d$ be the induced family. If $\mathcal{C} \subset I\!P^2 \times S$ is an element of $\mathbf{V}^{\delta,0}_d(S)$ for an algebraic scheme S then by the universal property of $|\mathcal{O}(d)|$ we obtain a unique morphism $S \to \Sigma_d$ inducing the given family by pullback. By Lemma (IV.7.6) and the defining property of the flattening stratification such morphism factors through $\mathcal{V}^\delta_d$. Thus $(\mathcal{V}^\delta_d, \overline{\mathcal{H}})$ represents the functor $\mathbf{V}^{\delta,0}_d$. *q.e.d.*

We now consider the local properties of the Severi varieties.

(IV.7.7) PROPOSITION *Let $C \subset \mathbb{P}^2$ be a reduced curve having degree d, δ nodes and κ ordinary cusps and no other singularity. Let $[C] \in \mathcal{V} = \mathcal{V}_d^{\delta,\kappa}$ be the point parametrizing C. Then there is a natural identification:*

$$T_{[C]}\mathcal{V} = H^0(C, N'_{C/\mathbb{P}^2})$$

and $H^1(C, N'_{C/\mathbb{P}^2})$ is an obstruction space for $\mathcal{O}_{\mathcal{V},[C]}$.

Proof
$T_{[C]}\mathcal{V}$ is the subspace of $T_{[C]}\Sigma_d = H^0(C, \mathcal{O}_C(d))$ corresponding to locally trivial first order deformations, and these are the elements of $H^0(C, N'_{C/\mathbb{P}^2})$ by the very definition of $N'_{C/\mathbb{P}^2}$. From the proof of Proposition (II.3.3) it is obvious that obstructions to deforming locally trivial deformations lie in the space $H^1(C, N'_{C/\mathbb{P}^2})$. *q.e.d.*

According to the classical terminology, we call $\mathcal{V}_d^{\delta,\kappa}$ *regular* at a point $[C]$ if $H^1(C, N'_{C/\mathbb{P}^2}) = 0$; otherwise $\mathcal{V}_d^{\delta,\kappa}$ is called *superabundant* at $[C]$. An irreducible component W of $\mathcal{V}_d^{\delta,\kappa}$ is called regular (resp. superabundant) if it is regular (resp. superabundant) on an open subset. $\mathcal{V}_d^{\delta,\kappa}$ is called regular if all its components are regular; otherwise it is called superabundant. From (IV.7.7) and from example (IV.7.1)(iii) it follows that if a component W of $\mathcal{V}_d^{\delta,\kappa}$ is regular then it is generically nonsingular of dimension $3d + g - 1 - \kappa$, where g is the geometric genus of C, i.e. of pure codimension $\delta + 2\kappa$ in Σ_d.

(IV.7.8) COROLLARY *If $\kappa < 3d$ then $\mathcal{V}_d^{\delta,\kappa}$, if non-empty, is regular at every point. In particular $\mathcal{V}_d^{\delta}$ is regular at every point, in particular has pure dimension*

$$3d + g - 1 = \binom{d+2}{2} - 1 - \delta$$

provided $\delta \le \binom{d}{2}$.

Proof
The first part follows from Proposition (IV.7.7) and from Example (IV.7.1)(iii). The condition $\delta \le \binom{d}{2}$ comes from the fact that $\binom{d}{2}$ is the maximum possible number of double points for a plane curve of degree d and it is attained by nodal unions of d lines. *q.e.d.*

EXAMPLE If $\kappa > 3d$ then $\mathcal{V}_d^{\delta,\kappa}$ can be superabundant. The following classical example is due to B. Segre (see Segre(1929a) and Zariski(1971), p. 220). Consider plane curves of the following type:

$$C : [f_{2m}(x, y)]^3 + [f_{3m}(x, y)]^2 = 0$$

where $f_{2m}(x, y)$ and $f_{3m}(x, y)$ are general polynomials of the indicated degrees, and $m > 2$. Then $d = \deg(C) = 6m$, $\delta = 0$ and $\kappa = 6m^2$ because the only singularities of C are the points of intersection of the curves $f_{2m} = 0$ and $f_{3m} = 0$ and clearly they are cusps. C is irreducible of geometric genus

$$g = \binom{6m-1}{2} - 6m^2$$

The dimension of the family of curves C is

$$R := \binom{2m+2}{2} + \binom{3m+2}{2} - 1 = \frac{1}{2}(13m+2)(m+1)$$

which is larger than

$$r = \frac{6m(6m+3)}{2} - 2\kappa = 6m^2 + 9m$$

Infact $R - r = \binom{m-1}{2}$. Therefore $\mathcal{V}_{6m}^{0,6m^2}$ is superabundant at all points $[C]$.

Let's compute $h^1(C, N'_{C/\mathbb{P}^2})$. By the analysis of example (IV.7.1)(iii) we know that $h^1(C, N'_{C/\mathbb{P}^2})$ equals the index of speciality ι of the linear system cut on the normalization $\tilde{C}$ of C by the curves of degree $6m$ passing through the cusps and tangent there to the cuspidal tangents. It is an easy computation (see Zariski(1971) p. 220 for details) that

$$\iota = R - r$$

The conclusion is that each $[C]$ is a nonsingular point of a superbundant component of $\mathcal{V}_{6m}^{0,6m^2}$ of dimension R.

For a modern treatment of this example see Tannenbaum(1984). It is more difficult to construct obstructed points of $\mathcal{V}_d^{\delta,\kappa}$; see Note 1 below.

NOTES

1. The Severi varieties $\mathcal{V}_d^{\delta,\kappa}$ may have a complicated structure. If there are too many cusps then in general a $[C] \in \mathcal{V}_d^{\delta,\kappa}$ satisfies $H^1(C, N'_{C/\mathbb{P}^2}) \neq (0)$ and in fact $\mathcal{V}_d^{\delta,\kappa}$ can be singular at such a $[C]$. To decide whether this effectively happens has been a long standing classical problem (see Zariski(1971), ch. VIII, where a discussion of this topic is given). The first published example of a singular point of a Severi variety is in Wahl(1974): it is a plane irreducible curve of degree 104 with 3636 nodes and 900 cusps. For other examples see Luengo(1987) and Guffroy(2003).

2. An important classical problem, known as the "Severi problem", has been to decide about the irreducibility of the open set of $\mathcal{V}_d^{\delta}$ parametrizing irreducible nodal curves. This problem has been solved affermatively in Harris(1986) and, independently, in Ran(1986) and Treger(1988). See also Loeser(1987). It is known that the open set of $\mathcal{V}_d^{\delta,\kappa}$ parametrizing irreducible curves is reducible in general if $\kappa > 0$. For examples see Segre(1929b) (such examples are also reported in Zariski(1971)).

3. (** to be expanded **) Corollary (IV.7.8) does not say anything about the non-emptyness of $\mathcal{V}_d^{\delta,\kappa}$. For nodal curves Severi claimed the following:

For each $0 \leq \delta \leq \binom{d}{2}$ *there exist plane curves of degree d having exactly δ nodes and no other singularities, i.e.* $\mathcal{V}_d^{\delta} \neq \emptyset$. *If* $0 \leq \delta \leq \binom{d-1}{2}$ *then there exist irreducible plane curves of degree d having exactly δ nodes and no other singularities.*

This statement and generalizations have been reconsidered in Tannenbaum(1980), Fulton(1983). A similar statement for curves with nodes and ordinary cusps has been made in Segre(1929a), but apparently it has not been reconsidered from a modern point of view.

4. Corollary (IV.7.8) should be compared with Theorem (III.6.7). Infact the two results can be shown to be closely related because they give two different descriptions of the local structure of families of plane nodal curves. The clue is given by Lemma (III.6.6).

5. The proof of Theorem (IV.7.6) can be easily modified to prove the existence of universal families of curves with nodes and cusps (generalized Severi varieties) on a projective nonsingular surface Y. In such a proof one replaces Σ_d by Hilb^Y and uses the existence and the universal property of Hilb^Y.

Such generalized Severi varieties behave in a way relatively similar to the $\mathcal{V}_d^{\delta,\kappa}$'s as long as Y has Kodaira dimension ≤ 0 (see Lange-Sernesi(2002), Tannenbaum(1980)). On surfaces of general type the situation changes radically. On such a surface Y the generalized Severi varieties are in general superabundant even when $\kappa = 0$ and it is not clear in which range of δ they are not empty. A systematic study of them has started relatively recently. We refer the reader to Chiantini-Sernesi(1997), Greuel-Lossen-Shustin(1997), Chiantini-Ciliberto(1999), Flamini(2001), Flamini(2002) for details.