Appendix

In this Appendix we have collected some standard technical tools in the form in which they are used in the book. For some of them the reader could refer to Eisenbud(1995) and to Hartshorne(1977) as well.

A.1. DIFFERENTIALS

Let $A \to B$ be a ring homomorphism. As usual, we will denote by $\Omega_{B/A}$ the *module of differentials of* B *over* A, and by $d_{B/A} : B \to \Omega_{B/A}$ the canonical A-derivation. Recall that

$$\Omega_{B/A} := I/I^2$$

where $I = \ker(B \otimes_A B \xrightarrow{\mu} B)$ is the natural map, and for each $b \in B$

$$d_{B/A}(b) = b \otimes 1 - 1 \otimes b$$

is called the *differential of* b. We have a natural isomorphism of B-modules

$$\mathrm{Der}_A(B, M) \cong \mathrm{Hom}_B(\Omega_{B/A}, M)$$

Note that the exact sequence

$$[A.1.1] \qquad 0 \to \Omega_{B/A} \to (B \otimes_A B)/I^2 \xrightarrow{\mu'} B \to 0$$

where μ' is induced by μ, is an A-extension of B. The ring

$$P_{B/A} := (B \otimes_A B)/I^2$$

is called the *algebra of principal parts* of B over A. The A-extension [A.1.1] is trivial because we have splittings:

$$\lambda_1, \lambda_2 : B \to P_{B/A}$$

defined by $\lambda_1(b) = \overline{b \otimes 1}$, $\lambda_2(b) = \overline{1 \otimes b}$; note that $d_{B/A} = \lambda_1 - \lambda_2$. We will consider $P_{B/A}$ as a B-algebra via λ_1.

The following are some fundamental properties of the modules of differentials:

(A.1.1) PROPOSITION

(i) *If*

$$\begin{array}{ccc} B & & \\ \uparrow & & \\ A & \longrightarrow & A' \end{array}$$

are ring homomorphisms, then:

$$\Omega_{B/A} \otimes_A A' \cong \Omega_{B \otimes_A A'/A'}$$

(ii) If $A \to B$ is a ring homomorphism and $\Delta \subset B$ is a multiplicative system, then:

$$\Omega_{\Delta^{-1}B/A} \cong \Delta^{-1}\Omega_{B/A}$$

(iii) Let $K \to L$ be a finitely generated extension of fields. Then

$$\dim_L(\Omega_{L/K}) \geq \mathrm{trdeg}(L/K)$$

and equality holds if and only if L is separably generated over K. In particular $\Omega_{L/K} = (0)$ if and only if $K \subset L$ is a finite algebraic separable extension.

Proof
See Eisenbud(1995).

We have two standard exact sequences.

(A.1.2) THEOREM (Relative cotangent sequence) *Given ring homomorphisms*

$$A \xrightarrow{f} B \xrightarrow{g} C$$

there is an exact sequence of C-modules:

$$[A.1.2] \qquad \Omega_{B/A} \otimes_B C \xrightarrow{\alpha} \Omega_{C/A} \xrightarrow{\beta} \Omega_{C/B} \to 0$$

where the maps are given by:

$$\alpha(d_{B/A}(b) \otimes c) = c d_{C/A}(g(b)); \quad \beta(d_{C/A}(r)) = d_{C/B}(r) \qquad b \in B, \quad c \in C$$

Proof
See Eisenbud(1995), prop. 16.2.

When $B \to C$ is surjective we have $\Omega_{C/B} = (0)$ and the next theorem describes $\ker(\alpha)$.

(A.1.3) THEOREM (Conormal sequence) *Let*

$$A \xrightarrow{f} B \xrightarrow{g} C$$

be ring homomorphisms with g surjective, and let $J = \ker(g)$, so that $C = B/J$. Then:
(i) We have an exact sequence

$$[A.1.3] \qquad J/J^2 \xrightarrow{\delta} \Omega_{B/A} \otimes_B C \xrightarrow{\alpha} \Omega_{C/A} \to 0$$

where δ *is the* C*-linear map defined by* $\delta(\bar{x}) = d_{B/A}(x) \otimes 1$.
(ii) There is an isomorphism

$$\Omega_{(B/J^2)/A} \otimes_{(B/J^2)} C \cong \Omega_{B/A} \otimes_B C$$

In other words the conormal sequence [A.1.3] depends only on the first infinitesimal neighborhood of Spec(C) *in* Spec(B).
(iii) The map δ *is a split injection if and only if there is a map of* A*-algebras* $C \to B/J^2$ *splitting the projection* $B/J^2 \to C$.

Proof
(i) see e.g. Eisenbud(1995), prop. 16.3.

(ii) Comparing the exact sequence [A.1.3] with the analogous sequence associated to $A \to B/J^2 \to C$ we get a commutative diagram:

$$\begin{array}{ccccccc} J/J^2 & \to & \Omega_{B/A} \otimes_B C & \to & \Omega_{C/A} & \to 0 \\ \| & & \downarrow & & \| & \\ J/J^2 & \to & \Omega_{(B/J^2)/A} \otimes_{(B/J^2)} C & \to & \Omega_{C/A} & \to 0 \end{array}$$

and the vertical arrow, which is induced by $B \to B/J^2$, must be an isomorphism.

(iii) By (ii) we may assume that $J^2 = 0$, i.e. that $0 \to J \to B \to C \to 0$ is an A-extension. Assume that $\delta : J \to \Omega_{B/A} \otimes_B C$ is a split injection, and let $\sigma : \Omega_{B/A} \otimes_B C \to J$ be a splitting. Then the composition

$$B \xrightarrow{\bar{d}} \Omega_{B/A} \otimes_B C \xrightarrow{\sigma} J$$

is an A-derivation. It follows that $1 - \sigma\bar{d} : B \to B$ is an A-homomorphism such that $(1 - \sigma\bar{d})(J) = 0$ and therefore it induces an A-homomorphism $C \to B$ which splits g.

Conversely assume that $g : B \to C$ has a section $\tau : C \to B$. Then we have a derivation

$$D : B \to J \oplus \Omega_{C/A}$$

given by $D(b) = (b - (\tau g)(b), d_{C/A}(g(b)))$. One easily checks that D induces an isomorphism $\Omega_{B/A} \otimes_B C \cong J \oplus \Omega_{C/A}$, thus proving the assertion. *q.e.d.*

As an application of (A.1.3) we have the following:

(A.1.4) PROPOSITION *Let* K *be a field and* (B, m) *a local* K*-algebra with residue field* $B/m = K'$. *Then the map*

$$\delta : m/m^2 \to \Omega_{B/K} \otimes_B K'$$

in the exact sequence [A.1.2] relative to $K \to B \to K'$ *is injective if and only if* $K \subset K'$ *is a separable field extension.*
In particular, if $B/m = K$ *then*

$$\delta : m/m^2 \to \Omega_{B/K} \otimes_B K$$

is an isomorphism. Therefore

$$\dim(B) \leq \dim_K(\Omega_{B/K} \otimes_B K)$$

Proof

See Eisenbud(1995), cor. 16.13. The last assertion follows from the conormal sequence relative to $K \to B \to K$. *q.e.d.*

The following Theorem describes the module of differentials for regular local rings.

(A.1.5) THEOREM *Assume that K is a field and B is a local noetherian K-algebra with residue field $B/m = K$. If $\Omega_{B/K}$ is a free B-module of rank equal to dim(B) then B is a regular local ring. If K is perfect (e.g. algebraically closed) and B is e.f.t. over K then the converse is also true.*

Proof

Assume first that $\Omega_{B/K}$ is free of rank equal to $\dim(B)$. Then $\dim_K(m/m^2) = \dim(B)$ by (A.1.6), so B is a regular local ring.

Assume conversely that K is perfect and that B is a regular local ring, e.f.t. over K. Then we have

$$\dim_K(\Omega_{B/K} \otimes_B K) = \dim_K(m/m^2) = \dim(B)$$

Let L be the quotient field of B. Then, by (A.1.1)(iii), we have

$$\Omega_{B/K} \otimes_B L = \Omega_{L/K}$$

and

$$\dim_L(\Omega_{L/K}) = \mathrm{trdeg}(L/K) = \dim(B)$$

because L is separably algebraic over K, since K is perfect. Therefore we have

$$\dim_K(\Omega_{B/K} \otimes_B K) = \dim(B) = \dim_L \Omega_{B/K} \otimes_B L$$

Since B is e.f.t. over K, $\Omega_{B/K}$ is a finitely generated B-module, and from Lemma (A.2.7) it follows that it is free of rank equal to $\dim(B)$. *q.e.d.*

In particular we have the following:

(A.1.6) COROLLARY *Let k be an algebraically closed field, and let B be an integral k-algebra of finite type. Then B is a regular ring if and only if $\Omega_{B/k}$ is a projective B-module of rank equal to dim(B).*

Proof

Both conditions are satisfied if and only if they are satisfied after localizing at the maximal ideals of B. For every maximal ideal $m \subset B$ the local ring B_m is a k-algebra e.f.t. with residue field k. By (A.1.5) B_m is a regular local ring if and only if $\Omega_{B_m/k} = (\Omega_{B/k})_m$ is free of rank equal to $\dim(B)$. The conclusion follows. *q.e.d.*

(A.1.7) PROPOSITION *If the ring homomorphism $A \to B$ is e.f.t. then $\Omega_{B/A}$ is a B-module of finite type.*

If in particular $B = S^{-1}A[X_1, \dots, X_n]$ for some multiplicative system S, then $\Omega_{B/A}$ is a free B-module of rank n with basis $\{d_{B/A}(X_1), \dots, d_{B/A}(X_n)\}$.

Proof

The last assertion is elementary (see Eisenbud(1995)). To prove the first, let $B = (S^{-1}P)/J$, where $P = A[X_1, \dots, X_n]$ and $S \subset P$ is a multiplicative system. Then $\Omega_{B/A}$ is a quotient of $\Omega_{S^{-1}P/A} \otimes_{S^{-1}P} B$, by the conormal sequence. *q.e.d.*

(A.1.8) REMARK If A and B are only assumed to be noetherian then $\Omega_{B/A}$ is not necessarily a B-module of finite type even if A is a field. An example is given by $\Omega_{\mathbf{Q}[[X]]/\mathbf{Q}}$ (see [EGA] ch. $\mathbf{0}_{IV}$, n. 20.7.16).

(A.1.9) EXAMPLES

(i) Assume that $B = S^{-1}A[X_1, \dots, X_n]$ for some multiplicative system S. Then $\mathrm{Der}_A(B, B) = \mathrm{Hom}_B(\Omega_{B/A}, B)$ is a free module of rank n with basis

$$\left\{\frac{\partial}{\partial X_1}, \dots, \frac{\partial}{\partial X_n}\right\}$$

which is the dual of the basis

$$\{d_{B/A}(X_1), \dots, d_{B/A}(X_n)\}$$

of $\Omega_{B/A}$, and where $\frac{\partial}{\partial X_j} : B \to B$ is the partial A-derivation with respect to X_j.

Let $Y_1, \dots, Y_n \in B$ be such that the jacobian determinant

$$\det\left(\frac{\partial Y_i}{\partial X_j}\right)$$

is a unit in B. Then

$$\{d_{B/A}(Y_1), \dots, d_{B/A}(Y_n)\}$$

is another basis of $\Omega_{B/A}$ and we have:

$$d_{B/A}(X_j) = \frac{\partial X_j}{\partial Y_1} d_{B/A}(Y_1) + \cdots + \frac{\partial X_j}{\partial Y_n} d_{B/A}(Y_n)$$

Dually:

$$\frac{\partial}{\partial X_j} = \frac{\partial Y_1}{\partial X_j}\frac{\partial}{\partial Y_1} + \cdots + \frac{\partial Y_n}{\partial X_j}\frac{\partial}{\partial Y_n} \qquad [A.1.4]$$

The proof of these statements is straightforward.

(ii) Let K be a field and let $B = K[X, Y]/(XY)$, where X, Y are indeterminates. Then, since $\Omega_{K[X,Y]/K} \otimes B \cong BdX \oplus BdY$, using the conormal sequence we deduce that

$$\Omega_{B/K} \cong \frac{BdX \oplus BdY}{(YdX \oplus XdY)}$$

It follows that the element $YdX = -XdY$ is killed by the maximal ideal (X,Y) and therefore it generates a torsion submodule

$$T := (YdX) \cong K \subset \Omega_{B/K}$$

The quotient is

$$\frac{\Omega_{B/K}}{T} = \frac{BdX \oplus BdY}{(YdX, XdY)} \cong K[X]dX \oplus K[Y]dY \cong (X,Y)$$

where the last isomorphism is given by $f(X)dX \oplus g(Y)dY \mapsto f(X)X + g(Y)Y$.

(iii) Let K be a field and let $B = K[t,X,Y]/(f)$ where t, X, Y are indeterminates and $f = XY + t$. Then arguing as before we see that

$$\Omega_{B/K[t]} \cong \frac{BdX \oplus BdY}{(YdX \oplus XdY)}$$

The element $YdX = -XdY$ is not killed by any $b \in B$; therefore $\Omega_{B/K[t]}$ is torsion free of rank one.

(iv) Let k be a field and let $k[\epsilon] := k[t]/(t^2)$, where we have denoted by ϵ the class of t mod (t^2). Then the conormal sequence of $k \to k[t] \to k[\epsilon]$ is

$$(t^2)/(t^4) \to \Omega_{k[t]/k} \otimes_{k[t]} k[\epsilon] \to \Omega_{k[\epsilon]/k} \to 0$$

and the middle term is isomorphic to $k[\epsilon]$. The first map acts as

$$\begin{array}{ccc} \bar{t}^2 & \mapsto & 2\epsilon \\ \bar{t}^3 & \mapsto & 0 \end{array}$$

Therefore

$$\Omega_{k[\epsilon]/k} = \begin{cases} kd\epsilon & \text{if } \mathrm{char}(k) \neq 2; \\ k[\epsilon]d\epsilon & \text{if } \mathrm{char}(k) = 2 \end{cases}$$

and $d : k[\epsilon] \to \Omega_{k[\epsilon]/k}$ acts as $d(\alpha + \epsilon\beta) = \beta d\epsilon$.

An obvious generalization of the above computation shows that if $A = k[t]/(t^n)$, $n \geq 2$ and $\mathrm{char}(k) = 0$ or $\mathrm{char}(k) > n$ then

$$\Omega_{A/k} = A/(\bar{t}^{n-1})$$

(v) Assume $\mathrm{char}(k) = 0$. Let

$$0 \to (t) \to R' \to R \to 0$$

be a small extension in $\mathcal{A}$. Then the conormal sequence

$$\eta : 0 \to (t) \xrightarrow{\delta} \Omega_{R'/k} \otimes_{R'} R \to \Omega_{R/k} \to 0$$

is exact also on the left. To prove it note that δ is given by the following composition:

$$\begin{array}{ll} \delta: \quad (t) \subset R' \stackrel{d_{R'/k}}{\longrightarrow} \Omega_{R'/k} & \to \Omega_{R/k} \\ & \quad \| \\ & \Omega_{R'/k}/t\Omega_{R'/k} \end{array}$$

But $dt := d_{R'/k}(t) \neq 0$ because $t \notin k$ and $dt \notin t\Omega_{R'/k}$ because (t) is a principal ideal and $\mathrm{char}(k) = 0$.

(vi) If $B \in \mathrm{ob}(\mathcal{A}^*)$ then $t_B^\vee := m_B/m_B^2$ and $t_B := (m_B/m_B^2)^\vee$ are the (Zariski) *cotangent space* respectively *tangent space* of B. We have $m_B/m_B^2 \cong \Omega_{B/\mathbf{k}} \otimes_B \mathbf{k}$ by Prop. (A.1.6), and therefore

$$\mathrm{Der}_{\mathbf{k}}(B, \mathbf{k}) = \mathrm{Hom}_B(\Omega_{B/\mathbf{k}}, \mathbf{k}) = \mathrm{Hom}_{\mathbf{k}}(\Omega_{B/\mathbf{k}} \otimes_B \mathbf{k}, \mathbf{k}) = (m_B/m_B^2)^\vee$$

Moreover there is a natural identification

$$\mathrm{Der}_{\mathbf{k}}(B, \mathbf{k}) = \mathrm{Hom}_{\mathbf{k}-alg}(B, \mathbf{k}[\epsilon])$$

which we leave to the reader to verify.

If $\mu : \Lambda \to B$ is a homomorphism in $\mathcal{A}^*$, the induced homomorphism

$$d\mu^\vee : m_\Lambda/m_\Lambda^2 \to m_B/m_B^2$$

is the *codifferential* of μ, while its transpose

$$d\mu : t_B \to t_\Lambda$$

is the *differential* of μ. We define the *relative cotangent space of B over Λ* to be

$$t_{B/\Lambda}^\vee := \mathrm{coker}(d\mu^\vee) = m_B/(m_B^2 + m_\Lambda B)$$

and the *relative tangent space of B over Λ* as its dual:

$$t_{B/\Lambda} = \ker(d\mu) = \left[m_B/(m_B^2 + m_\Lambda B)\right]^\vee$$

From the exact sequence

$$\Omega_{\Lambda/\mathbf{k}} \otimes_\Lambda B \to \Omega_{B/\mathbf{k}} \to \Omega_{B/\Lambda} \to 0$$

tensored by $\mathbf{k}$ we deduce an identification $t_{B/\Lambda}^\vee = \Omega_{B/\Lambda} \otimes_B \mathbf{k}$ and therefore

$$t_{B/\Lambda} = \mathrm{Hom}_B(\Omega_{B/\Lambda}, \mathbf{k}) = \mathrm{Der}_\Lambda(B, \mathbf{k}) = \mathrm{Hom}_{\Lambda-alg}(B, \mathbf{k}[\epsilon])$$

where the Λ-algebra structure on $\mathbf{k}[\epsilon]$ is defined by the composition $\Lambda \to \mathbf{k} \to \mathbf{k}[\epsilon]$ (the last equality is straightforward to verify).

* * * * * *

If $f : X \to Y$ is a morphism of schemes, we denote by $\Omega^1_{X/Y}$ the *sheaf of relative differentials*, or the *relative cotangent sheaf*, on X. It satisfies

$$\Omega^1_{X/Y,x} = \Omega_{\mathcal{O}_{X,x}/\mathcal{O}_{Y,f(x)}}$$

for all $x \in X$. If $f : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is a morphism of affine schemes then

$$\Omega^1_{\mathrm{Spec}(B)/\mathrm{Spec}(A)} = (\Omega_{B/A})^\sim$$

We denote by

$$T_{X/Y} := Hom(\Omega^1_{X/Y}, \mathcal{O}_X)$$

the *sheaf of relative derivations*, or the *relative tangent sheaf* of f.

We will write Ω^1_X and T_X instead of $\Omega^1_{X/\mathrm{Spec}(\mathbf{k})}$ and $T_{X/\mathrm{Spec}(\mathbf{k})}$ respectively; they are the *cotangent sheaf* and the *tangent sheaf* of X, respectively (cotangent and tangent *bundles* if locally free).

If X is algebraic and $x \in X$ is closed then, by (A.1.6):

$$\Omega^1_{X,x} \otimes \mathbf{k}(x) = \frac{m_{X,x}}{m^2_{X,x}}$$

is the cotangent space of X at x, and

$$T_xX := T_{X,x} \otimes \mathbf{k}(x) = \left(\frac{m_{X,x}}{m^2_{X,x}}\right)^\vee \cong \mathrm{Der}_{\mathbf{k}}(\mathcal{O}_{X,x}, \mathbf{k})$$

is the Zariski tangent space of X at x.

Let S be a scheme and

$$X \xrightarrow{g} Y$$

a morphisms of S-schemes. The induced homomorphism of sheaves on X:

$$g^*\Omega^1_{Y/S} \to \Omega^1_{X/S}$$

is called the *relative codifferential* of g. The dual homomorphism:

$$T_{X/S} \to Hom(g^*\Omega^1_{Y/S}, \mathcal{O}_X)$$

is the *relative differential* of g. When $S = \mathrm{Spec}(\mathbf{k})$ we have $g^*\Omega^1_Y \to \Omega^1_X$, which is the *codifferential* of g, while its dual

$$dg : T_X \to Hom(g^*\Omega^1_Y, \mathcal{O}_X)$$

is the *differential* of g. Note that if $\Omega^1_{Y/S}$ is locally free then

$$Hom(g^*\Omega^1_{Y/S}, \mathcal{O}_X) = g^*Hom(\Omega^1_{Y/S}, \mathcal{O}_Y) = g^*T_{Y/S}$$

but in general the first and the second sheaf are different.

Given $f : Y \to S$ and a closed embedding $i : X \subset Y$, we have an exact sequence of sheaves on X:

$$[A.1.5] \qquad \mathcal{I}/\mathcal{I}^2 \to i^*\Omega^1_{Y/S} \to \Omega^1_{X/S} \to 0$$

where $\mathcal{I} \subset \mathcal{O}_Y$ is the ideal sheaf of X in Y. [A.1.5] is called the *relative conormal sequence*. When $S = \mathrm{Spec}(\mathbf{k})$ we obtain the *conormal sequence*

$$\mathcal{I}/\mathcal{I}^2 \to i^*\Omega^1_Y \to \Omega^1_X \to 0$$

(A.1.10) EXAMPLES: In the following examples we will describe the global vector fields on the given schemes by exhibiting their restrictions to an affine open set. All will be done by explicit computation.

(i) $H^0(T_{\mathbb{P}^1})$ can be described explicitely as follows. Consider $\mathbb{P}^1 = U_0 \cup U_1$ where $U_0 = \mathrm{Spec}(\mathbf{k}[\xi])$ and $U_1 = \mathrm{Spec}(\mathbf{k}[\eta])$ with $\eta = \xi^{-1}$ on $U_0 \cap U_1$. We have

$$\frac{\partial}{\partial\eta} = \frac{\partial\xi}{\partial\eta}\frac{\partial}{\partial\xi} = -\frac{1}{\eta^2}\frac{\partial}{\partial\xi} = -\xi^2\frac{\partial}{\partial\xi}$$

on $U_0 \cap U_1$. Let $\theta \in H^0(T_{\mathbb{P}^1})$; then

$$\theta_{|U_0} = g(\xi)\frac{\partial}{\partial\xi} \qquad g(\xi) \in \mathbf{k}[\xi]$$

and

$$\theta_{|U_1} = h(\eta)\frac{\partial}{\partial\eta} \qquad h(\eta) \in \mathbf{k}[\eta]$$

On $U_0 \cap U_1$ we have

$$g(\xi)\frac{\partial}{\partial\xi} = h(\eta)\frac{\partial}{\partial\eta} = -h(\xi^{-1})\xi^2\frac{\partial}{\partial\xi}$$

and therefore $g(\xi) = -h(\xi^{-1})\xi^2$. It follows that $g(\xi) = a_0 + a_1\xi + a_2\xi^2$ and $h(\eta) = -(a_0\eta^2 + a_1\eta + a_2)$, with $a_0, a_1, a_2 \in \mathbf{k}$. In particular $H^0(T_{\mathbb{P}^1}) \cong \mathbf{k}^3$.

Moreover $H^i(T_{\mathbb{P}^1}) = 0$ if $i \geq 1$. For $i \geq 2$ it is obvious. Let $\theta \in H^1(T_{\mathbb{P}^1})$ be represented by a Chech 1-cocycle defined by $\theta_{01} \in \Gamma(U_0 \cap U_1, T_{\mathbb{P}^1})$. It can be written as

$$\theta_{01} = \sum_{i=-m}^{n} a_i\xi^i$$

Letting $\theta_1 = \sum_{i=-m}^{-1} a_i\eta^{-i}$ and $\theta_0 = -\sum_{i=0}^{n} a_i\xi^i$ we obtain:

$$\theta_{01} = \theta_1 - \theta_0$$

so $\{\theta_{01}\}$ is a coboundary.

(ii) We want to describe $H^0(T_{\mathbf{A}^1\times I\!P^1})$. Let $\mathbf{A}^1 \times I\!P^1 = V_0 \cup V_1$ where

$$V_0 = \mathbf{A}^1 \times U_0 = \mathrm{Spec}[z,\xi])$$

$$V_1 = \mathbf{A}^1 \times U_1 = \mathrm{Spec}[z,\eta])$$

and $\eta = \xi^{-1}$ on $V_0 \cap V_1 = \mathrm{Spec}(\mathbf{k}[z,\xi,\xi^{-1}])$. We have

$$\frac{\partial}{\partial \eta} = \frac{\partial \xi}{\partial \eta}\frac{\partial}{\partial \xi} = -\frac{1}{\eta^2}\frac{\partial}{\partial \xi} = -\xi^2\frac{\partial}{\partial \xi}$$

on $V_0 \cap V_1$. Let $\theta \in H^0(T_{\mathcal{A}^1\times I\!P^1})$; then

$$\theta_{|V_0} = g(z,\xi)\frac{\partial}{\partial z} + h(z,\xi)\frac{\partial}{\partial \xi} \qquad g(z,\xi), h(z,\xi) \in \mathbf{k}[z,\xi]$$

$$\theta_{|V_1} = \gamma(z,\eta)\frac{\partial}{\partial z} + \chi(z,\eta)\frac{\partial}{\partial \eta} \qquad \gamma(z,\eta), \chi(z,\eta) \in \mathbf{k}[z,\eta]$$

On $V_0 \cap V_1$ we have:

$$g(z,\xi) = \gamma(z,\xi^{-1})$$

and therefore $g(z,\xi) = g(z)$ is constant with respect to ξ. Moreover

$$h(z,\xi)\frac{\partial}{\partial \xi} = \chi(z,\eta)\frac{\partial}{\partial \eta} = -\chi(z,\xi^{-1})\xi^2\frac{\partial}{\partial \xi}$$

and therefore

$$h(z,\xi) = -\chi(z,\xi^{-1})\xi^2$$

It follows that $h(z,\xi) = a(z) + b(z)\xi + c(z)\xi^2$, with $a(z), b(z), c(z) \in \mathbf{k}[z]$. In conclusion every $\theta \in H^0(T_{\mathbf{A}^1\times I\!P^1})$ restricts to V_0 as a vector field of the form

$$[A.1.6] \qquad \theta_{|V_0} = g(z)\frac{\partial}{\partial z} + (a(z) + b(z)\xi + c(z)\xi^2)\frac{\partial}{\partial \xi}$$

with $g(z), a(z), b(z), c(z) \in \mathbf{k}[z]$, and conversely every such vector field is the restriction of a global section of $T_{\mathbf{A}^1\times I\!P^1}$. As in example (i) we also deduce that $H^i(T_{\mathbf{A}^1\times I\!P^1}) = 0$ if $i \geq 1$.

In a similar way one describes $H^0(T_{(\mathbf{A}^1\setminus\{0\})\times I\!P^1})$ by showing that the image of the restriction

$$H^0(T_{(\mathbf{A}^1\setminus\{0\})\times I\!P^1}) \to H^0(T_{(\mathbf{A}^1\setminus\{0\})\times U_0})$$

consists of the vector fields of the form [A.1.6] with $g(z), a(z), b(z), c(z) \in \mathbf{k}[z,z^{-1}]$.

(iii) We now consider, for a given integer $m \geq 0$, the rational ruled surface

$$F_m = I\!P(\mathcal{O}_{I\!P^1}(m) \oplus \mathcal{O}_{I\!P^1})$$

Let $\pi : F_m \to \mathbb{P}^1$ be the projection. Then F_m can be represented as

$$F_m = \pi^{-1}(U) \cup \pi^{-1}(U') = (U \times \mathbb{P}^1) \cup (U' \times \mathbb{P}^1)$$

where $U = \mathrm{Spec}(\mathbf{k}[z])$, $U' = \mathrm{Spec}(\mathbf{k}[z'])$ and $z' = z^{-1}$ on $U \cap U'$. We consider the affine open sets

$$V_0 = \mathrm{Spec}(\mathbf{k}[z, \xi]) \subset U \times \mathbb{P}^1$$
$$V_0' = \mathrm{Spec}(\mathbf{k}[z', \xi']) \subset U' \times \mathbb{P}^1$$

where on $V_0 \cap V_0' = \mathrm{Spec}(\mathbf{k}[z, z^{-1}, \xi]) = \mathrm{Spec}(\mathbf{k}[z', z'^{-1}, \xi'])$ we have:

$$z' = z^{-1}, \quad \xi' = z^m \xi$$

Therefore we have:

[A.1.7]
$$\begin{aligned} \frac{\partial}{\partial z'} &= -z^2 \frac{\partial}{\partial z} + mz\xi \frac{\partial}{\partial \xi} \\ \frac{\partial}{\partial \xi'} &= z^{-m} \frac{\partial}{\partial \xi} \end{aligned}$$

We will describe a typical element $\theta \in H^0(T_{F_m})$ by describing its restriction to the open sets V_0 and V_0'. We have, by example (ii) above:

$$\theta_{|V_0} = g(z)\frac{\partial}{\partial z} + (a(z) + b(z)\xi + c(z)\xi^2)\frac{\partial}{\partial \xi}$$

with $g(z), a(z), b(z), c(z) \in \mathbf{k}[z]$ and similarly

$$\theta_{|V_0'} = \rho(z')\frac{\partial}{\partial z'} + (\alpha(z') + \beta(z')\xi' + \gamma(z')\xi'^2)\frac{\partial}{\partial \xi'}$$

with $\rho(z'), \alpha(z'), \beta(z'), \gamma(z') \in \mathbf{k}[z']$. Imposing their equality on $V_0 \cap V_0'$ and using [A.1.7] we obtain the following conditions:

[A.1.8]
$$\begin{aligned} g(z) &= -\rho(z^{-1})z^2 \\ a(z) &= \alpha(z^{-1})z^{-m} \\ b(z) &= \beta(z^{-1}) + \rho(z^{-1})mz \\ c(z) &= \gamma(z^{-1})z^m \end{aligned}$$

We distinguish the cases $m = 0$ and $m > 0$. If $m = 0$ [A.1.8] give:

$$\begin{aligned} g(z) &= g_0 + g_1 z + g_2 z^2 \\ a(z) &= a \\ b(z) &= b \\ c(z) &= c \end{aligned} \qquad g_0, g_1, g_2, a, b, c \in \mathbf{k}$$

In case $m > 0$ we have:

$$\begin{aligned} g(z) &= g_0 + g_1 z + g_2 z^2 \\ a(z) &= 0 \\ b(z) &= b - mz(g_1 + g_2 z) \\ c(z) &= c_0 + c_1 z + \cdots + c_m z^m \end{aligned} \qquad g_0, g_1, g_2, b, c_0, \ldots, c_m \in \mathbf{k}$$

Since the restriction $H^0(T_{F_m}) \to H^0(T_{V_0})$ is injective and we have described its image, we can conclude:

$$\begin{aligned} H^0(T_{F_0}) &\cong \mathbf{k}^6 \\ H^0(T_{F_m}) &\cong \mathbf{k}^{m+5} \end{aligned}$$

In particular F_m *and* F_n *are not isomorphic if* $m \neq n$. (note that $F_0 \cong I\!P^1 \times I\!P^1$ is not isomorphic to $F_1 \cong Bl_{(1,0,0)} I\!P^2$).

Since, by the calculations of the previous example (ii)

[*A*.1.9] $$h^i(T_{U \times I\!P^1}) = h^i(T_{U' \times I\!P^1}) = h^i(T_{(U \cap U') \times I\!P^1}) = 0, \quad i \geq 1$$

we deduce that:

$$H^1(T_{F_m}) = H^0(T_{(U \cap U') \times I\!P^1}) / H^0(T_{U \times I\!P^1}) + H^0(T_{U' \times I\!P^1})$$

An easy computation based on [A.1.8] shows that, for $m \geq 1$, $H^1(T_{F_m})$ consists of the classes, modulo $H^0(T_{U \times I\!P^1}) + H^0(T_{U' \times I\!P^1})$, of the vector fields

$$(b_1 z + \cdots + b_{m-1} z^{m-1}) \frac{\partial}{\partial \xi}$$

In particular

$$H^1(T_{F_m}) \cong \mathbf{k}^{m-1}$$

It also follows from [A.1.9] that

$$H^2(T_{F_m}) = (0)$$

NOTES

1. In $I\!P^1 \times I\!P^2$ with bihomogeneous coordinates $(x, y; u, v, w)$ consider the hypersurface Σ_m, $m \geq 0$, defined by the equation:

$$x^m v - y^m u = 0$$

Prove that $\Sigma_m \cong F_m$, and that the structure morphism $\pi : F_m \to I\!P^1$ is induced by the projection $I\!P^1 \times I\!P^2 \to I\!P^1$. (Solution: see Andreotti(1957)).

2. Let $X \to Y$ be a morphism of algebraic schemes. Prove that there is an exact sequence

[*A*.1.10] $$0 \to \Omega^1_{X/Y} \to \mathcal{P}^1_{X/Y} \to \mathcal{O}_X \to 0$$

which globalizes [A.1.1]. $\mathcal{P}^1_{X/Y}$ is called the *sheaf of principal parts of* X *over* Y, denoted by $\mathcal{P}^1_X$ if $Y = \mathrm{Spec}(\mathbf{k})$.

Let $X = I\!P(V)$ for a finite dimensional $\mathbf{k}$-vector space V. Then the exact sequence [A.1.10] is the dual of the Euler sequence; in particular

$$\mathcal{P}^1_{I\!P(V)} \cong \mathcal{O}_{I\!P(V)}(-1) \otimes V^\vee$$

Therefore [A.1.10] is a generalization of the Euler sequence to any $X \to Y$.

3. Consider $I\!P = I\!P(V)$ for a finite dimensional $\mathbf{k}$-vector space V and the *incidence relation*:

$$[A.1.11] \qquad \mathbf{I} = \{(x, H) : x \in H\} \subset I\!P \times I\!P^\vee$$

Consider the twisted and dualized Euler sequence:

$$0 \to \Omega^1_{I\!P(V)}(1) \to \mathcal{O}_{I\!P(V)} \otimes V^\vee \to \mathcal{O}_{I\!P(V)}(1) \to 0$$

From its definition it follows that $\mathbf{I} = I\!P(\Omega^1_{I\!P(V)}(1))$ and $I\!P \times I\!P^\vee = I\!P(\mathcal{O}_{I\!P(V)} \otimes V^\vee)$ and the inclusion in [A.1.11] is induced by the first homomorphism in the above sequence.

A.2. FLATNESS

The algebraic notion of flatness, first introduced in Serre(1955-56), is the basic technical tool for the study of families of algebraic varieties and schemes. In this section we will overview the main algebraic results needed.

A module M over a ring A is A-*flat* (or *flat over* A, or simply *flat*) if the functor $N \mapsto M \otimes_A N$ from the category of A-modules into itself is exact. Since this functor is always right exact, the flatness means that it takes monomorphisms into monomorphisms. An A-*algebra* B is *flat over* A if B is flat as an A-module.

The A-module M is said to be *faithfully flat* if for every sequence of A-modules $N' \to N \to N''$ the sequence

$$M \otimes_A N' \to M \otimes_A N \to M \otimes_A N''$$

is exact if and only if the original sequence is exact. Obviously, if M is faithfully flat then it is flat. In a similar way we give the notion of faithfully flat A-algebra. It is straightforward to check that if $A \to B$ is a local homomorphism of local rings, then a B-module of finite type is faithfully A-flat if and only if it is flat and nonzero.

Recall that the flatness of an A-module M is equivalent to any of the following conditions:

(1) $\mathrm{Tor}_i(M, N) = (0)$ for all $i > 0$ and for every A-module N.

(2) $\mathrm{Tor}_1(M, N) = (0)$ for every A-module N.

(3) $\mathrm{Tor}_1(M, N) = (0)$ for every finitely generated A-module N.

(4) $\mathrm{Tor}_1(M, A/I) = (0)$ for every ideal $I \subset A$.

(5) $I \otimes_A M \to M$ is injective for every ideal $I \subset A$.

(6) $I \otimes_A M \to IM$ is an isomorphism for every ideal $I \subset A$.

(A.2.1) EXAMPLE. Let k be a ring, u, v indeterminates and $f : k[u, uv] \to k[u, v]$ the inclusion. Then

$$\frac{k[u, uv]}{(uv)} = k[u] \xrightarrow{u} k[u] = \frac{k[u, uv]}{(uv)}$$

is injective. Tensoring by $\otimes_{k[u,uv]} k[u, v]$ we obtain:

$$\frac{k[u, v]}{(uv)} \xrightarrow{u} \frac{k[u, v]}{(uv)}$$

which is not injective. Therefore f is not flat.

We list without proof a few *basic properties of flat modules*:

(I) M is A-flat if and only if M_p is A_p-flat for every prime ideal p.

(II) Every projective module is flat.

(III) Assume M is finitely generated. Then M is flat if and only if it is projective; if A is local then M is flat if and only if it is free.

(IV) If $S \subset A$ is a multiplicative subset then A_S is A-flat.

(V) A direct sum $M = \oplus_{i\in I} M_i$ is flat if and only if all M_i's are flat.

(VI) Let

$$0 \to M' \to M \to M'' \to 0$$

be an exact sequence of A-modules with M'' flat. Then M is flat if and only if M' is flat.

(VII) *Base change*: if M is A-flat and $f : A \to B$ is a ring homomorphism, then $M \otimes_A B$ is B-flat.

(VIII) *Transitivity*: if B is a flat A-algebra and N is a flat B-module, then N is A-flat.

(IX) If A is a noetherian ring and I is an ideal, the I-adic completion $\hat{A}$ is a flat A-algebra. If I is contained in the Jacobson radical of A then $\hat{A}$ is a faithfully flat A-algebra.

(X) If B is an A-algebra and if there exists a B-module M which is faithfully flat, then the morphism $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is surjective.

The following result is frequently used:

(A.2.2) PROPOSITION *If A is an artinian local ring with residue field k the following are equivalent for an A-module M:*
(i) M *is free*
(ii) M *is flat*
(iii) $\mathrm{Tor}_1^A(M,k) = (0)$

Proof
$(i) \Rightarrow (ii) \Rightarrow (iii)$ are clear.
$(iii) \Rightarrow (ii)$. Let N be a finitely generated A-module and let

$$N = N_0 \supset \cdots \supset N_n = (0)$$

be a composition series for N such that

$$N_i/N_{i+1} \cong k$$

for $i = 0, \dots, n-1$. Using the Tor exact sequences from the hypothesis (iii) we deduce that $\mathrm{Tor}_1(M, N) = (0)$ and the flatness of M follows from (3).

Let's now prove $(ii) \Rightarrow (i)$. Let $\{e_j\}_{j \in J}$ be a system of elements of M which induces a basis of $M \otimes_A k$ over k. The system $\{e_j\}$ defines a homomorphism $f : A^J \to M$ which induces an isomorphism $k^J \to M \otimes_A k$. From the following Lemma it follows that f is an isomorphism, and therefore M is free. *q.e.d.*

(A.2.3) LEMMA *Let R be a ring, I an ideal and $f : F \to G$ a homomorphism of R-modules with G flat. Assume that one of the following conditions is satisfied:*

(a) I is nilpotent.

(b) R is noetherian, I is contained in the Jacobson radical of R and F and G are finitely generated.

If the induced homomorphism $F/IF \to G/IG$ is an isomorphism, then f is an isomorphism.

Proof

Let $K = \mathrm{coker}(f)$. Tensoring the exact sequence

$$F \to G \to K \to 0$$

with R/I we get $K/IK = 0$: from Nakayama's lemma (which holds in either of the hypothesis (a) and (b)) it follows that $K = 0$, and therefore F is surjective. Letting $H = \ker(f)$ we deduce an exact sequence

$$0 \to H/IH \to F/IF \to G/IG \to 0$$

using the flatness of G. By Nakayama again we deduce $H = 0$ and the conclusion follows. *q.e.d.*

The following is a basic criterion of flatness.

(A.2.4) LOCAL CRITERION OF FLATNESS *Suppose that $\varphi : A \to B$ is a local homomorphism of local noetherian rings, and let $k = A/m_A$ be the residue field of A. If M is a finitely generated B-module, then the following conditions are equivalent:*

(i) M is A-flat

(ii) $\mathrm{Tor}_1^A(M, k) = 0$.

(iii) $M \otimes_A (A/m_A^n)$ is flat over A/m_A^n for every integer $n \geq 1$.

(iv) $M \otimes_A (A/m_A^n)$ is free over A/m_A^n for every integer $n \geq 1$.

Proof

$(i) \Rightarrow (ii)$ is obvious.

$(ii) \Rightarrow (i)$ see Eisenbud(1995), Th. 6.8, p. 167.

$(i) \Rightarrow (iii)$ is obvious.

$(iii) \Rightarrow (i)$ It suffices to show that for every inclusion $N' \to N$ of A-modules of finite type we have an inclusion $M \otimes_A N' \to M \otimes_A N$. For this purpose it suffices to show that the kernel of this last map is contained in

$$K_n := \ker[M \otimes_A N' \to M \otimes_A (N'/N' \cap m_A^n N)]$$

for all n, because $\bigcap_n K_n = (0)$. We have a commutative diagram with exact rows:

$$\begin{array}{ccccccc} 0 \to & K_n & \to & M \otimes_A N' & \to & M \otimes_A (N'/N' \cap m_A^n N) & \to 0 \\ & & & \downarrow & & \downarrow & \\ & & & M \otimes_A N & \to & M \otimes_A (N/m_A^n N) & \to 0 \end{array}$$

The last vertical arrow coincides with the map obtained from the injection

$$N'/N' \cap m_A^n N \to N/m_A^n N$$

after tensoring over A/m_A^n with the A/m_A^n-flat module $M \otimes_A (A/m_A^n)$, and therefore it is injective. The conclusion follows from the above diagram.

(iii) $\Leftrightarrow$ *(iv)* follows from Proposition (A.2.2) because A/m_A^n is artinian. *q.e.d.*

For a more general version of the local criterion we refer to [SGA1], exp. IV.

(A.2.5) COROLLARY *Suppose that $\varphi : A \to B$ is a local homomorphism of local noetherian rings, let $k = A/m_A$ be the residue field of A, M, N two finitely generated B-modules, and suppose that N is A-flat. Let $u : M \to N$ be a B-homomorphism. Then the following are equivalent:*

(i) u is injective and coker(u) is A-flat.

(ii) $u \otimes 1 : M \otimes k \to N \otimes k$ is injective.

Proof

(i) $\Rightarrow$ (ii). Let $G = \mathrm{Coker}(u)$. Tensoring by k the exact sequence

$$0 \to M \xrightarrow{u} N \to G \to 0$$

by k we obtain the exact sequence:

$$\mathrm{Tor}_1^A(G,k) \to M \otimes_A k \xrightarrow{u \otimes 1} N \otimes_A k \to G \otimes_A k \to 0$$

Since G is A-flat we have $\mathrm{Tor}_1^A(G,k) = 0$, and it follows that $u \otimes 1$ is injective.

(ii) $\Rightarrow$ (i). Factor $u \otimes 1$ as

$$M \otimes_A k \xrightarrow{\alpha} \mathrm{Im}(u) \otimes_A k \xrightarrow{\beta} N \otimes_A k$$

Then α is an isomorphism and β is injective. Tensoring by k the exact sequence

$$[A.2.1] \qquad 0 \to \mathrm{Im}(u) \to N \to G \to 0$$

we obtain the exact sequence:

$$\mathrm{Tor}_1^A(N,k) \to \mathrm{Tor}_1^A(G,k) \to \mathrm{Im}(u) \otimes_A k \xrightarrow{\beta} N \otimes_A k \to G \otimes_A k \to 0$$

Since N is A-flat we have $\mathrm{Tor}_1^A(N,k) = 0$; from the injectivity of β we deduce $\mathrm{Tor}_1^A(G,k) = 0$ and from (A.2.4) it follows that G is A-flat. Applying (VI) to the exact sequence [A.2.1] we deduce that $\mathrm{Im}(u)$ is A-flat as well. Consider the exact sequence:

$$0 \to \ker(u) \to M \to \mathrm{Im}(u) \to 0$$

and tensor by k. We obtain the exact sequence:

$$0 \to \ker(u) \otimes_A k \to M \otimes_A k \xrightarrow{\alpha} \mathrm{Im}(u) \otimes_A k \to 0$$

Since α is an isomorphism we deduce that $\ker(u) \otimes_A k = 0$, and therefore $\ker(u) = 0$ by Nakayama's lemma. *q.e.d.*

A related result is the following:

(A.2.6) LEMMA *Let B be a local ring with residue field K, and let $d : G \to F$ be a homomorphism of finitely generated B-modules, with F free. Then d is split injective if and only if $d \otimes_B K : G \otimes_B K \to F \otimes_B K$ is injective. In such a case also G is free.*

Proof
d is split injective if and only if $\mathrm{coker}(d)$ is free and d is injective. If this last condition is satisfied then clearly $d \otimes_B K$ is injective.

Conversely, assume that $d \otimes_B K$ is injective, and factor d as

$$G \to \mathrm{Im}(d) \to F$$

We see that

$$\begin{array}{cccl} G \otimes_B K & \to & \mathrm{Im}(d) \otimes_B K & \text{is bijective} \\ \mathrm{Im}(d) \otimes_B K & \to & F \otimes_B K & \text{is injective} \end{array}$$

From the exact sequence

$$0 \to \mathrm{Im}(d) \to F \to \mathrm{coker}(d) \to 0$$

we get

$$0 \to \mathrm{Tor}_1(\mathrm{coker}(d), K) \to \mathrm{Im}(d) \otimes_B K \to F \otimes_B K$$

so $\mathrm{Tor}_1(\mathrm{coker}(d), K) = (0)$ and this implies that $\mathrm{coker}(d)$ is free. From the above exact sequence we deduce that $\mathrm{Im}(d)$ is free as well, so that

$$0 \to \ker(d) \to G \to \mathrm{Im}(d) \to 0$$

is split exact. Recalling that $G \otimes_B K \cong \mathrm{Im}(d) \otimes_B K$ we deduce that $\ker(d) \otimes_B K = (0)$, hence $\ker(d) = (0)$ by Nakayama. *q.e.d.*

For the reader's convenience we include the proof of the following well known Lemma:

(A.2.7) LEMMA *Let (B, m) be a noetherian local integral domain, with residue field K and quotient field L. If M is a finitely generated B-module and if*

$$\dim_K(M \otimes_B K) = \dim_L(M \otimes_B L) = r$$

then M is free of rank r.

Proof

Let $m_1, \ldots, m_r \in M$ be such that their images in $M \otimes_B K = M/mM$ form a basis. Then they define a homomorphism $\varphi : B^r \to M$ and we have an exact sequence:

$$0 \to N \to B^r \xrightarrow{\varphi} M \to Q \to 0$$

where N and Q are kernel and cokernel of φ. Since tensoring with K we get

$$K^r \xrightarrow{\bar{\varphi}} M/mM \to Q/mQ \to 0$$

and $\bar{\varphi}$ is surjective, we get $Q/mQ = (0)$ and from Nakayama's lemma it follows that $Q = (0)$: hence φ is surjective. Now we tensor the above exact sequence with L, which is flat over B (by (IV)), and we obtain the exact sequence:

$$0 \to N \otimes_B L \to L^r \xrightarrow{\tilde{\varphi}} M \otimes_B L \to 0$$

Since $M \otimes_B L \cong L^r$ and $\tilde{\varphi}$ is surjective, it follows that $N \otimes_B L = \ker(\tilde{\varphi}) = (0)$. Therefore N is a torsion module. But $N \subset B^r$ and therefore $N = (0)$. *q.e.d.*

We have the following useful criterion:

(A.2.8) LEMMA *Let $A \to A'$ be a small extension in $\mathcal{A}$, and let $g : A \to R$ be a homomorphism of* **k***-algebras. Let $R_0 = R \otimes_A \mathbf{k}$. Then g is flat if and only if*

$$\ker(R \to R \otimes_A A') \cong R_0$$

and the homomorphism $g' : A' \to R \otimes_A A'$ induced by g is flat.

Proof

Assume that g is flat. Then since $R \otimes_A (\epsilon) \cong R \otimes_A \mathbf{k} = R_0$ and $\mathrm{Tor}_1^A(R, A') = 0$, from the exact sequence

$$[A.2.2] \qquad 0 \to \mathrm{Tor}_1^A(R, A') \to R \otimes_A (\epsilon) \to R \to R \otimes_A A' \to 0$$

we deduce that the first condition is satisfied. The flatness of g' is obvious.

Assume conversely that the conditions of the Lemma are satisfied. Then the sequence [A.2.2] implies that $\mathrm{Tor}_1^A(R, A') = 0$. If $A' = \mathbf{k}$ the conclusion follows from (A.2.2). If not, from the exact sequence

$$0 \to m_{A'} \to A' \to \mathbf{k} \to 0$$

one gets the exact sequence:

$$\begin{array}{ccccccccccc} \mathrm{Tor}_1^A(R, A') & \to & \mathrm{Tor}_1^A(R, \mathbf{k}) & \xrightarrow{\partial} & R \otimes_A m_{A'} & \to & A' & \to & R \otimes_A \mathbf{k} & \to & 0 \\ \| & & & & \| & & & & \| & & \\ 0 & & & & R' \otimes_{A'} m_{A'} & & & & R' \otimes_{A'} \mathbf{k} & & \end{array}$$

From the flatness of R' over A' we deduce that $\partial = 0$, hence $\mathrm{Tor}_1^A(R, \mathbf{k}) = 0$, and we conclude by (A.2.2). *q.e.d.*

* * * * * *

Flatness in terms of generators and relations.

Let P be a noetherian $\mathbf{k}$-algebra, $J \subset P$ an ideal. Let A be in $\mathcal{A}$, $P_A = P \otimes_{\mathbf{k}} A$, and $\mathbf{J} \subset P_A$ an ideal such that $(P_A/\mathbf{J}) \otimes_A \mathbf{k} \cong P/J$. We want to find the conditions $\mathbf{J}$ has to satisfy so that $P_A/\mathbf{J}$ is A-flat.

We have the following

(A.2.9) THEOREM *Let*

$$\Pi_0 : \quad P^n \to P^N \to P \to P/J \to 0$$

be a presentation of P/J as a P-module. Then the following conditions are equivalent for an ideal $\mathbf{J} \subset P_A$:
(i) $P_A/\mathbf{J}$ is A-flat and $(P_A/\mathbf{J}) \otimes_A \mathbf{k} \cong P/J$.
(ii) There is an exact sequence

$$\Pi : \quad P_A^n \to P_A^N \to P_A \to P_A/\mathbf{J} \to 0$$

such that $\Pi_0 = \Pi \otimes_A \mathbf{k} \quad (= \Pi/m_A\Pi)$.
(iii) There is a complex

$$\Pi : \quad P_A^n \xrightarrow{\varphi} P_A^N \to P_A \to P_A/\mathbf{J} \to 0$$

which is exact except possibly at P_A^N, such that $\quad \Pi_0 = \Pi \otimes_A \mathbf{k}$.

Proof
$(ii) \Rightarrow (i)$. We have:

$$\mathrm{Tor}_1^A(P_A/\mathbf{J}, \mathbf{k}) = H_1(\Pi \otimes \mathbf{k}) = H_1(\Pi_0) = (0)$$

From (A.2.2) it follows that $P_A/\mathbf{J}$ is A-flat. Moreover (ii) implies that $(P_A/\mathbf{J}) \otimes_A \mathbf{k} \cong P/J$.

$(i) \Rightarrow (ii)$. Choose a P_A-homomorphism $p : P_A^N \to \mathbf{J}$ which makes the following diagram commute:

$$\begin{array}{rccc} p: & P_A^N & \to & \mathbf{J} \\ & \downarrow & & \downarrow \\ p_0: & P^N & \to & J \end{array}$$

where p_0 is the surjective homomorphism defined by the presentation Π_0. From the flatness of $P_A/\mathbf{J}$ it follows that $\mathrm{Tor}_1^A(P_A/\mathbf{J}, \mathbf{k}) = (0)$; hence the exact sequence

$$\begin{array}{cccccccccc} 0 \to & \mathrm{Tor}_1^A(P_A/\mathbf{J}, \mathbf{k}) & \to & \mathbf{J} \otimes \mathbf{k} & \to & P_A \otimes \mathbf{k} & \to & (P_A/\mathbf{J}) \otimes_A \mathbf{k} & \to 0 \\ & & & & & \| & & \| & \\ & & & & & P & & P/J & \end{array}$$

implies that $\mathbf{J} \otimes \mathbf{k} = J$. It follows that $p \otimes_A \mathbf{k} = p_0$ and therefore

$$\mathrm{coker}(p) \otimes_A \mathbf{k} = \mathrm{coker}(p_0) = (0)$$

so that $\mathrm{coker}(p) = (0)$ by Nakayama's lemma. Hence p is surjective.

Now consider the exact sequence

$$0 \to \ker(p) \to P_A^N \to \mathbf{J} \to 0$$

and the associated Tor sequence:

$$[A.2.3] \qquad \mathrm{Tor}_1^A(\mathbf{J}, \mathbf{k}) \to \ker(p)/m_A\ker(p) \to P^N \to J \to 0$$

From the flatness of $P_A/\mathbf{J}$ and from the exact sequence

$$0 \to \mathbf{J} \to P_A \to P_A/\mathbf{J} \to 0$$

we have $\mathrm{Tor}_1^A(\mathbf{J}, \mathbf{k}) = \mathrm{Tor}_2^A(P_A/\mathbf{J}, \mathbf{k}) = (0)$. Therefore from [A.2.3] we see that

$$\ker(p)/m_A\ker(p) \cong \ker(p_0)$$

Arguing as before we can find a surjective homomorphism $q : P_A^n \to \ker(p)$ which makes the following diagram commutative:

$$\begin{array}{ccc} P_A^n & \xrightarrow{q} & \ker(p) \\ \downarrow & & \downarrow \\ P^n & \to & \ker(p_0) \end{array}$$

$(ii) \Rightarrow (iii)$ is obvious.

$(iii) \Rightarrow (i)$ If Π is not exact at P_A^N then we can add finitely many generators of the kernel of $P_A^N \to P_A$ to obtain an exact sequence

$$\Pi' : \qquad P_A^{n'} \xrightarrow{\varphi'} P_A^N \to P_A \to P_A/\mathbf{J} \to 0$$

Then $\Pi' \otimes_A \mathbf{k}$ has the form:

$$P^{n'} \xrightarrow{\varphi' \otimes \mathbf{k}} P^N \to P \to P/J \to 0$$

Since

$$\mathrm{Im}(\varphi \otimes \mathbf{k}) \subset \mathrm{Im}(\varphi' \otimes \mathbf{k}) \subset \ker[P^N \to P]$$

we see that $\mathrm{Im}(\varphi' \otimes \mathbf{k}) = \ker[P^N \to P]$ and therefore $\Pi' \otimes_A \mathbf{k}$ is exact. Now (i) follows from (A.2.2). *q.e.d.*

(A.2.10) COROLLARY *Assume that* $J = (f_1, \ldots, f_N) \subset P$ *and that*

$$\mathbf{J} = (F_1, \ldots, F_N) \subset P_A$$

with $f_j = F_j \pmod{m_A P_A}$, $j = 1, \ldots, N$. *Then every relation among* $f_1, \ldots, f_N$ *lifts to a relation among* $F_1, \ldots, F_N$ *if and only if* $P_A/\mathbf{J}$ *is A-flat and* $(P_A/\mathbf{J}) \otimes_A \mathbf{k} \cong P/J$.

Proof

The condition that the F_j's reduce to the f_j's modulo $m_A P_A$ implies that the exact sequence

$$P_A^N \xrightarrow{\mathbf{F}} P_A \to P_A/\mathbf{J} \to 0$$

reduces to

$$[A.2.4] \qquad P^N \xrightarrow{\mathbf{f}} P \to P/J \to 0$$

when tensored by $\otimes_A \mathbf{k}$. Complete [A.2.4] to a presentation Π_0 of P/J. The condition that every relation among $f_1, \ldots, f_N$ lifts to a relation among $F_1, \ldots, F_N$ is a restatement of condition (iii) of (A.2.9). Therefore the conclusion follows from theorem (A.2.9). *q.e.d.*

(A.2.11) EXAMPLE

Let A be in $\mathcal{A}$. Suppose that $f_1, \ldots, f_N \in P$ form a regular sequence, and let $F_1, \ldots, F_N \in P_A$ be any liftings of $f_1, \ldots, f_N$, i.e. such that $f_j = F_j \pmod{mP_A}$, $j = 1, \ldots, N$. Then $\mathbf{J} = (F_1, \ldots, F_N) \subset P_A$ defines a flat family of deformations of $X = \mathrm{Spec}(P/J)$, where $J = (f_1, \ldots, f_N)$.

Infact every relation among $f_1, \ldots, f_N$ is a linear combination of the trivial ones

$$r_{ij} = (0, \ldots, f_j, \ldots, -f_i, \ldots, 0) \qquad 1 \le i < j \le N$$

and these can be lifted to the corresponding trivial relations

$$R_{ij} = (0, \ldots, F_j, \ldots, -F_i, \ldots, 0)$$

among $F_1, \ldots, F_N$. Applying Corollary (A.2.5) it is easy to show that $F_1, \ldots, F_N$ form a regular sequence.

NOTES

1. In the proof of Theorem (A.2.9) the condition that A is artinian has only been used in the proof of $(i) \Rightarrow (ii)$ in order to apply Nakayama's Lemma. In particular the implications $(ii) \Rightarrow (i)$, $(iii) \Rightarrow (i)$ and $(ii) \Rightarrow (iii)$ hold for any $A \in \mathrm{ob}(\mathcal{A}^*)$. Using the local criterion of flatness it is easy to verify that the implication $(i) \Rightarrow (ii)$ (and therefore the equivalence of the three conditions) holds as well if A is in $\hat{\mathcal{A}}$.

A.3. RELATIVE COMPLETE INTERSECTION MORPHISMS

Regular embeddings

If $X \subset Y$ is a closed embedding of schemes and $\mathcal{I} = \mathcal{I}_{X/Y} \subset \mathcal{O}_Y$ is the ideal sheaf of X in Y, then $\mathcal{I}/\mathcal{I}^2$ is a sheaf of $\mathcal{O}_X$-modules in a natural way, called the *conormal sheaf* of X in Y. Its dual

$$N_{X/Y} := Hom_{\mathcal{O}_X}(\mathcal{I}/\mathcal{I}^2, \mathcal{O}_X) = Hom_{\mathcal{O}_Y}(\mathcal{I}, \mathcal{O}_X)$$

is called the *normal sheaf* of X in Y. $N_{X/Y}$ (resp. $\mathcal{I}/\mathcal{I}^2$) is called the *normal bundle* (resp. the *conormal bundle*) of X in Y if it is locally free.

An embedding of schemes $j : X \subset Y$ is a *regular embedding of codimension n at the point $x \in X$* if $j(x)$ has an affine open neighborhood U in Y such that $X \cap U \subset U$ is a regular embedding of codimension n. If this happens at every point of X we say that j is a *regular embedding of codimension* n. In this case $\mathcal{I}/\mathcal{I}^2$ and $N_{X/Y}$ are both locally free of rank n. If for example X and Y are both nonsingular then $X \subset Y$ is a regular embedding. An open embedding is regular of codimension 0. The set of points of X where an embedding $j : X \subset Y$ is embedding is open.

A ring B is called a *complete intersection* if Spec(B) can be regularly embedded in Spec(R) where R is a regular ring.

A scheme X is a *local complete intersection* (l.c.i.) if every local ring $\mathcal{O}_{X,x}$ is a complete intersection ring.

A *nonsingular scheme* X, i.e. a scheme all of whose local rings are regular, is an example of a l.c.i. scheme. If $X \subset Y$ is a regular embedding and Y is a l.c.i. scheme, then X is a l.c.i. scheme.

If we have a flag of embeddings of schemes $X \subset Y \subset Z$ and $\mathcal{I}_Y \subset \mathcal{I}_X \subset \mathcal{O}_Z$ are the ideal sheaves of X and Y, we have the exact sequence

$$[A.3.1] \qquad 0 \to \mathcal{I}_Y \to \mathcal{I}_X \to \mathcal{I}_{X/Y} \to 0$$

where $\mathcal{I}_{X/Y} \subset \mathcal{O}_Y$ is the ideal sheaf of X in Y. After tensoring by $\otimes_{\mathcal{O}_Z}\mathcal{O}_X$ we obtain an exact sequence of coherent $\mathcal{O}_X$-modules:

$$[A.3.2] \qquad \frac{\mathcal{I}_Y}{\mathcal{I}_Y^2} \otimes \mathcal{O}_X \xrightarrow{\alpha} \frac{\mathcal{I}_X}{\mathcal{I}_X^2} \to \frac{\mathcal{I}_{X/Y}}{\mathcal{I}_{X/Y}^2} \to 0$$

Its dual is the sequence:

$$[A.3.3] \qquad 0 \to N_{X/Y} \to N_{X/Z} \to N_{Y/Z} \otimes \mathcal{O}_X$$

(A.3.1) LEMMA
(i) If $f : X \subset Y$ and $g : Y \subset Z$ are regular embeddings of codimensions m and n rspectively, then $gf : X \to Z$ is a regular embedding of codimension $m+n$.
(ii) If the embeddings f and g are both regular then we have exact sequences of locally free sheaves on X:

$$[A.3.4] \qquad 0 \to \frac{\mathcal{I}_Y}{\mathcal{I}_Y^2} \otimes \mathcal{O}_X \xrightarrow{\alpha} \frac{\mathcal{I}_X}{\mathcal{I}_X^2} \to \frac{\mathcal{I}_{X/Y}}{\mathcal{I}_{X/Y}^2} \to 0$$

$$[A.3.5] \qquad 0 \to N_{X/Y} \to N_{X/Z} \to N_{Y/Z} \otimes \mathcal{O}_X \to 0$$

Proof
(i) left to the reader (see Grothendieck(1967), Prop. 19.1.5(iii)).
(ii) All sheaves in [A.3.4] are locally free because they are conormal bundles of regular embeddings. Since $\mathrm{Im}(\alpha)$ is a torsion free sheaf of the same rank of $(\mathcal{I}_Y/\mathcal{I}_Y^2) \otimes \mathcal{O}_X$, it follows that α must be injective. The sequence [A.3.5] is exact because $Ext^1_{\mathcal{O}_X}(\mathcal{I}_{X/Y}/\mathcal{I}_{X/Y}^2, \mathcal{O}_X) = 0$. *q.e.d.*

(A.3.2) PROPOSITION *Let $j : X \subset Y$ be an embedding of algebraic schemes, with X reduced and Y nonsingular. Consider the conormal sequence*

$$[A.3.6] \qquad \mathcal{I}/\mathcal{I}^2 \xrightarrow{\delta} \Omega^1_{Y|X} \to \Omega^1_X \to 0$$

(where $\mathcal{I} \subset \mathcal{O}_Y$ is the ideal sheaf of X) Then:
(i) The homomorphism δ is injective on the open set where j is a regular embedding.
(ii) If X and Y are nonsingular then the dual sequence

$$[A.3.7] \qquad 0 \to T_X \to T_{Y|X} \to N_{X/Y} \to 0$$

is exact.

Proof
(i) It suffices to show that δ is injective under the assumption that j is a regular embedding. Since X is regularly embedded in Y the conormal sheaf $\mathcal{I}/\mathcal{I}^2$ is locally free of rank equal to the codimension of X. At every nonsingular point $x \in X$ we have that $\Omega^1_{X,x}$ is free of rank equal to $\dim(X)$, so that

$$\dim(Y) \leq \mathrm{rk}(\Omega^1_{Y|X,x}) = \mathrm{rk}(\mathcal{I}_x/\mathcal{I}_x^2) + \mathrm{rk}(\Omega^1_{X,x}) = \dim(Y)$$

Therefore the sequence [A.3.6] is exact at every nonsingular point $x \in X$. Since X is reduced this happens on a dense open subset so that $\ker(\delta)$ is a torsion subsheaf of $\mathcal{I}/\mathcal{I}^2$; it follows that $\ker(\delta) = 0$ because $\mathcal{I}/\mathcal{I}^2$ is locally free.

(ii) Under the stated hypothesis j is a regular embedding and Ω^1_X is locally free, so we have $Ext^1(\Omega^1_X, \mathcal{O}_X) = 0$ and the exactness of [A.3.7] follows. *q.e.d.*

* * * * * *

Relative complete intersection morphisms

We now introduce a natural class of morphisms which generalize smooth morphisms and behave well with respect to differentials and base change.

(A.3.3) DEFINITION *A flat morphism of finite type $f : X \to S$ is called a* relative complete intersection (r.c.i.) morphism at the point $x \in X$ *if there is an open neighborhood U of x such that the restriction of f to U can be obtained as a composition*

$$U \xrightarrow{j} V \xrightarrow{g} S$$

where j is a regular embedding and g is smooth. If f is a r.c.i. morphism at every point we call it a r.c.i. morphism, *and we call X a* complete intersection over S.

This Definition is equivalent to Def. 19.3.6 of Ch. IV of [EGA]; the equivalence is proved in Berthelot(1971), prop. 1.4. Note that in case $S = \mathrm{Spec}(\mathbf{k})$ the morphism f is a r.c.i. if and only if X is a l.c.i. of finite type.

Before discussing the main properties of this notion we need two Lemmas:

(A.3.4) LEMMA *Let $A \to B$ be a ring homomorphism, M a B-module and $f_1, \dots, f_n$ an M-regular sequence of elements of B. Assume that for each $i = 1, \dots, n$ the module $M/(\sum_{j=1}^{i-1} f_j M)$ is A-flat. Then, for every ring homomorphism $A \to A'$, letting $B' = B \otimes_A A'$, $M' = M \otimes_A A'$, and $f'_i = f_i \otimes 1$ $(1 \le i \le n)$, the sequence $f'_1, \dots, f'_n$ of elements of B' is M'-regular and the modules $M'/(\sum_{j=1}^{i-1} f'_j M')$ are A'-flat.*

Proof

Consider the exact sequence:

$$0 \to M \xrightarrow{f_1} M \to M/f_1 M \to 0$$

Since M/f_1M is A-flat, the sequence:

$$0 \to M \otimes_A A' \xrightarrow{f_1 \otimes 1} M \otimes_A A' \to (M/f_1M) \otimes_A A' \to 0$$

is exact, and therefore f'_1 is not a zero-divisor for M'. Let $M_i = M/(\sum_{j=1}^{i} f_j M)$, $M'_i = M'/(\sum_{j=1}^{i} f'_j M')$; then we have $M'_i = M_i \otimes_A A'$, $M_{i+1} = M_i/f_{i+1}M_i$, $M'_{i+1} = M'_i/f'_{i+1}M'_i$. Replacing M and f_1 by M_i and f_{i+1} in the above argument, one deduces that f'_{i+1} is not a zero-divisor for M'_i, thereby proving the first assertion by induction. The last assertion follows from I.1.(VII). *q.e.d.*

(A.3.5) LEMMA *Let $A \to B$ be a local homomorphism of noetherian local rings, M a B-module of finite type, flat over A, and $f_1, \dots, f_n \in m_B$. For $1 \le i \le n$ let g_i be the image of f_i in $B \otimes_A k$, where $k = A/m_A$ is the residue field of A. Then the following conditions are equivalent:*

(i) $f_1, \dots, f_n$ is an M-regular sequence, and $M_i = M/(\sum_{j=1}^{i} f_j M)$ is A-flat for all

$1 \leq i \leq n$.
(ii) $g_1, \ldots, g_n$ *is an* $(M \otimes_A k)$*-regular sequence.*

Proof
(i) $\Rightarrow$ (ii) follows from (A.3.4) applied to $A' = k$.

(ii) $\Rightarrow$ (i) Applying Corollary (I.1.6), from the injectivity of $g_1 : M \otimes_A k \to M \otimes_A k$ we deduce that $f_1 : M \to M$ is injective and that $M_1 = M/f_1M$ is A-flat. Proceeding by induction on i, assume M_i flat over A. Since $g_{i+1} : M_i \otimes_A k \to M_i \otimes_A k$ is injective from (I.1.6) again we deduce that $f_{i+1} : M_i \to M_i$ is injective and that M_{i+1} is A-flat. *q.e.d.*

In the next Proposition some general properties of r.c.i. morphisms are proved.

(A.3.6) PROPOSITION
(i) An open embedding is a r.c.i. morphism. A smooth morphism of finite type is a r.c.i. morphism.
(ii) If $f : X \to S$ *is a r.c.i. morphism and* $h : S' \to S$ *is a morphism, then the morphism* $f' : X \times_S S' \to S'$ *induced by* f *after base change is a r.c.i. morphism.*

Proof
(i) is an immediate consequence of the definition and (ii) follows easily from Lemma (A.3.4). *q.e.d.*

From (A.3.6)(ii) it follows in particular that if $f : X \to S$ is a r.c.i. morphism then X_s is a l.c.i. for every $\mathbf{k}$-rational point $s \in S$.

The next result gives a useful characterization of r.c.i. morphisms.

(A.3.7) PROPOSITION *Let*

$$[A.3.8] \qquad \begin{array}{ccc} X & \xrightarrow{j} & Y \\ & \searrow f \quad \swarrow g & \\ & S & \end{array}$$

be a commutative diagram of morphisms of algebraic schemes, where f *is flat,* g *is smooth and* j *is an embedding. Then the following conditions are equivalent for a* $\mathbf{k}$*-rational point* $x \in X$*:*

(i) f *is a r.c.i. morphism at* x.
(ii) Letting $s = f(x)$*, the fibre* X_s *is a l.c.i. at* x.
(iii) j *is a regular embedding at* x.

Proof
(i) $\Rightarrow$ (ii) follows from (A.3.6)(ii) and (iii) $\Rightarrow$ (i) is obvious.

(ii) $\Rightarrow$ (iii) From (ii) it follows that the embedding $j_s : X_s \subset Y_s$ is regular at x. Let $\mathcal{I} \subset \mathcal{O}_Y$ be the ideal sheaf of X. Tensoring the exact sequence

$$0 \to \mathcal{I} \to O_Y \to \mathcal{O}_X \to 0$$

by $- \otimes_{\mathcal{O}_S} \mathbf{k}$ we obtain the sequence

$$0 \to \mathcal{I} \otimes_{\mathcal{O}_S} \mathbf{k} \to \mathcal{O}_{Y_s} \to \mathcal{O}_{X_s} \to 0$$

which is exact because f is flat. Therefore $\mathcal{I} \otimes_{\mathcal{O}_S} \mathbf{k}$ is the ideal sheaf of $j(X_s)$ in Y_s. Consider a sequence $f_1, \ldots, f_n$ of sections of $\mathcal{I}$ in an open neighborhood of $j(x)$ which induce a basis of $\mathcal{I}_{j(x)}/(m_s\mathcal{I}_{j(x)} + \mathcal{I}^2_{j(x)})$ as a $\mathcal{O}_{Y,j(x)}/(m_s\mathcal{O}_{Y,j(x)} + \mathcal{I}_{j(x)})$-module. Then the images $f_1 \otimes 1 = g_1, \ldots, f_n \otimes 1 = g_n$ are generating sections of $\mathcal{I} \otimes_{\mathcal{O}_S} \mathbf{k}$ in an open neighborhood of $j(x)$ in Y_s which form a regular sequence in $j(x)$. From Nakayama's Lemma it follows that $f_1, \ldots, f_n$ generate $\mathcal{I}$ in an open neighborhood of $j(x)$ in Y. From Lemma (A.3.5) it follows that $f_1, \ldots, f_n$ form a regular sequence in $j(x)$ and therefore (iii) holds.

(iii) $\Rightarrow$ (i) is true by definition. *q.e.d.*

(A.3.8) COROLLARY *Under the hypothesis of Proposition (A.3.7), the locus of points $x \in X$ such that f is a r.c.i. at x is open. If f is proper then the locus of points $s \in S$ such that X_s is a l.c.i. is open.*

Proof

The last assertion follows from the first because a proper map is closed. The first assertion can be proved using characterization (A.3.7)(iii) of r.c.i. morphism and the fact that the locus where an embedding is regular is open. *q.e.d.*

(A.3.9) THEOREM *Let*

$$\begin{array}{ccc} X & \xrightarrow{j} & Y \\ & \searrow f & \downarrow g \\ & & S \end{array}$$

be a commutative diagram of morphisms of schemes, with f a r.c.i., j an immersion and g smooth. Let $\mathcal{J} \subset \mathcal{O}_Y$ be the ideal sheaf of $j(X)$. Then if X is reduced the relative conormal sequence

$$0 \to \mathcal{J}/\mathcal{J}^2 \to j^*\Omega^1_{Y/S} \to \Omega^1_{X/S} \to 0$$

is exact and $\mathcal{J}/\mathcal{J}^2$ is locally free.

Proof

From the equivalence $(i) \Leftrightarrow (iii)$ in Proposition (A.3.7) it follows that j is a regular embedding and therefore $\mathcal{J}/\mathcal{J}^2$ is locally free. The proof of exactness follows the same lines of the proof of Proposition (A.3.2) and is left to the reader. *q.e.d.*

NOTES

1. An algebraic scheme can have different embeddings in $\mathbb{P}^r$, i.e. by means of non-isomorphic invertible sheaves, but with same normal sheaf. An example is given by a projective nonsingular curve C of genus 1, and by the embeddings in $\mathbb{P}^3$ given by two non isomorphic invertible sheaves $L_1$and L_2 of degree 4 such that $L_1^2 = L_2^2$. Then C is embedded as a nonsingular complete intersection of two quadrics by both sheaves, and the normal bundles are $L_1^2 \oplus L_1^2 = L_2^2 \oplus L_2^2$.

2. Let S be a scheme, and X, Y smooth over S. Prove that every closed S-embedding $X \subset Y$ is regular. In particular every section of a smooth morphism $f : Y \to S$ is a regular embedding of codimension equal to the relative dimension of f.

3. Let $f : \mathcal{X} \to S$ be a morphism of finite type and $s \in S$ a **k**-rational point. Let $m_s \subset \mathcal{O}_{S,s}$ be the maximal ideal and $\mathcal{I} = \mathcal{I}_{\mathcal{X}(s)}$ the ideal sheaf of the fibre $\mathcal{X}(s)$ of f over s. Prove that we have a surjective homomorphism

$$\frac{m_s}{m_s^2} \otimes_{\mathbf{k}} \mathcal{O}_{\mathcal{X}(s)} \to \mathcal{I}/\mathcal{I}^2$$

and an injection:

$$N_{\mathcal{X}(s)/\mathcal{X}} \subset T_{S,s} \otimes_{\mathbf{k}} \mathcal{O}_{\mathcal{X}(s)}$$

If f is flat then they are isomorphisms; in particular, if f is flat then $N_{\mathcal{X}(s)/\mathcal{X}}$ is free.

A.4. FUNCTORIAL LANGUAGE

Let $\mathcal{C}$ be a category. A covariant (resp. contravariant) functor F from $\mathcal{C}$ to (sets) is said to be *representable* if there is an object X in $\mathcal{C}$ such that F is isomorphic to the functor

$$[A.4.1] \qquad Y \mapsto \mathrm{Hom}(X, Y)$$

(resp. $Y \mapsto \mathrm{Hom}(Y, X)$). We will denote by h_X a functor of the form [A.4.1]. The representable functors are a full subcategory, isomorphic to $\mathcal{C}^\circ$ (resp. to $\mathcal{C}$ in the contravariant case), of the category $Funct(\mathcal{C}, \text{(sets)})$ of covariant functors (resp. $Funct(\mathcal{C}^\circ, \text{(sets)})$ of contravariant functors) from $\mathcal{C}$ to (sets).

To fix ideas let's consider covariant functors. In order to investigate conditions for the representability of a given functor F it is convenient to study functorial morphisms $h_X \to F$. Such morphisms turn out to be easy to describe, thanks to the elementary:

(A.4.1) LEMMA(Yoneda) *Let $F : \mathcal{C} \to$ (sets) be a covariant functor. For each object X in $\mathcal{C}$ there is a canonical bijection:*

$$\begin{array}{ccc} \mathrm{Hom}(h_X, F) & \leftrightarrow & F(X) \\ \Phi & \mapsto & \Phi(X)(1_X) \end{array}$$

Let's mention, on passing, that functorial morphisms $F \to h_X$ are more interesting, but they are much harder to control. They are related to the notion of "coarse moduli space".

We may consider *couples* of the form (X, ξ), where X is an object of $\mathcal{C}$ and $\xi \in F(X)$. Yoneda's Lemma implies that to give such a couple is equivalent to giving a morphism of functors $h_X \to F$; if this morphism is an isomorphism then (X, ξ) is called a *universal couple*, and ξ a *universal element*, for F. The existence of a universal couple is equivalent to the representability of F.

The couples for F are the objects of a category in which a *morphism* $(X, \xi) \to (Y, \eta)$ between two couples is by definition a morphism $f : X \to Y$ in $\mathcal{C}$ such that $F(f)(\xi) = \eta$. We denote this category by I_F. A morphism $f : (X, \xi) \to (Y, \eta)$ in I_F corresponds to a commutative diagram of morphisms of functors:

$$\begin{array}{ccc} h_X & \xrightarrow{\xi} & F \\ \uparrow f & \nearrow \eta & \\ h_Y & & \end{array}$$

We have an obvious "forgetful functor"

$$I_F \to \mathcal{C}$$

The fibres of this functor are precisely the sets $F(X)$, which are embedded as subcategories of I_F by $\xi \mapsto (X, \xi)$.
(recall that, given a functor $G : \mathbf{C} \to \mathbf{D}$, the fibre $G^{-1}(D)$ of G over an object D of $\mathbf{D}$ is a subcategory of $\mathbf{C}$, consisting of all objects C such that $G(C) = D$ and of all morphisms f such that $G(f) = 1_D$. A set can be viewed as a category whose objects are its elements and the only morphisms are the identity morphisms).

(A.4.2) LEMMA *The functor F is representable if and only if the category I_F has an initial object (X, ξ). If this is the case, (X, ξ) is a universal couple for F.*

The proof is immediate. Note that, since an initial object is unique up to isomorphism, it follows that a representable functor has a unique universal couple, up to isomorphism.

* * * * * *

Let I and $\mathcal{D}$ be two categories. Given an object A of $\mathcal{D}$, the constant functor $c_A : I \to \mathcal{D}$ is defined as $c_A(i) = A$ for each object i of I and $c_A(f) = 1_A$ for each morphism f in I. Note that c_A is both covariant and contravariant. Every morphism $\alpha : A \to B$ in $\mathcal{D}$ induces an obvious morphism of functors $c_\alpha : c_A \to c_B$. Consider a covariant functor $\Phi : I \to \mathcal{D}$. An *inductive limit* of Φ is an object A of $\mathcal{D}$ and a functorial morphism $\lambda : \Phi \to c_A$ such that for every other morphism $\mu : \Phi \to c_B$ there is a morphism $\alpha : A \to B$ such that $\mu = c_\alpha \lambda$.

$$\begin{array}{ccc} \Phi & \xrightarrow{\lambda} & c_A \\ & \searrow \mu & \downarrow c_\alpha \\ & & c_B \end{array}$$

From the definition it follows that an inductive limit of Φ, if it exists, is unique up to unique isomorphism, and is denoted

$$\lim_{\rightarrow} \Phi$$

In practice an inductive limit is an object A of $\mathcal{D}$ such that there is a morphism $\Phi(i) \to A$ for each $i \in Ob(I)$ with the condition that the diagram

$$\begin{array}{ccc} \Phi(i) & \to & A \\ \downarrow \Phi(f) & \nearrow & \\ \Phi(j) & & \end{array}$$

is commutative for each morphism $f : i \to j$ in I; moreover these data must have a universal property.

Dually one has the notion of *projective limit* of a covariant functor $\Phi : I \to \mathcal{D}$: it is an object A of $\mathcal{D}$ and a morphism $\pi : c_A \to \Phi$ such that for every other morphism

$\rho : c_B \to \Phi$ there is a morphism $\beta : B \to A$ such that $\rho = \pi c_\beta$. The projective limit of Φ, if it exists, is denoted

$$\lim_{\leftarrow} \Phi$$

The above notions can be defined without changes replacing the covariant functor Φ by a contravariant one. We will write Φ_i for $\Phi(i)$, for each object i of I, and sometimes

$$\lim_{\rightarrow} \Phi_i \quad (\text{resp. } \lim_{\leftarrow} \Phi_i) \quad \text{instead of} \quad \lim_{\rightarrow} \Phi \quad (\text{resp. } \lim_{\leftarrow} \Phi)$$

(A.4.3) EXAMPLE Let J be a partially ordered set. We define a category $Ord(J)$ as follows. The objects of $Ord(J)$ are the elements of J; for any $i, j \in J$ the set $\mathrm{Hom}_{Ord(J)}(i, j)$ consists of one element if $i \leq j$ and is $\emptyset$ otherwise. A covariant (resp. contravariant) functor $\Phi : Ord(J) \to \mathcal{D}$ is called an *inductive system* (resp. a *projective system*) in $\mathcal{D}$ indexed by J; in case $\mathcal{D} =$(sets), we obtain the usual notions of inductive (projective) system and of inductive (projective) limit.

If I is a set and $\Phi : I \to \mathcal{D}$ is a functor, where $\mathcal{D}$ is a category with arbitrary coproducts, then

$$\lim_{\rightarrow} \Phi = \coprod_i \Phi_i$$

Similarly, if $\mathcal{D}$ has products then

$$\lim_{\leftarrow} \Phi = \prod_i \Phi_i$$

(A.4.4) PROPOSITION *The inductive limit and projective limit exist for every functor $\Phi : I \to$ (sets) from any category I.*

Proof
We take

$$\lim_{\rightarrow} \Phi = \coprod_i \Phi_i / R$$

where R is the equivalence relation generated by pair (x, y), $x \in \Phi_i$ and $y \in \Phi_j$, such that there exists $\varphi : i \to j$ with $\Phi(x) = y$. Similarly for the projective limit. *q.e.d.*

(A.4.5) EXAMPLE: Let $F : \mathcal{C} \to$ (sets) be a covariant functor, and let I_F be the category of couples for F. Then we have a contravariant functor

$$\Phi : I_F \to Funct(\mathcal{C}, (\text{sets}))$$

which sends a couple (X, ξ) to the functor $h_X : \mathcal{C} \to$ (sets), and a morphism $f : (X, \xi) \to (Y, \eta)$ to the functorial morphism $h_f : h_Y \to h_X$ induced by f. By construction there is a morphism $\Phi \to c_F$. This morphism makes F the inductive limit of the functor Φ (the proof is an easy exercise). We will write:

$$F = \lim_{\substack{\to \\ (X,\xi)}} h_X$$

(A.4.6) DEFINITION *A category I is* filtered *if*
(a) for every pair of objects i, j in I there exists an object k in I and morphisms:

$$\begin{array}{ccc} & & i \\ & & \downarrow \\ j & \to & k \end{array}$$

(b) each pair of morphisms $i \rightrightarrows j$ has a coequalizer $i \rightrightarrows j \to k$.
The category I is cofiltered *if the dual category I° is filtered.*

Assume from now on that $\mathcal{C}$ is a category with products and fibered products.

(A.4.7) DEFINITION *A covariant functor $F : \mathcal{C} \to$ (sets) is called* left exact *if $F(B \times C) = F(B) \times F(C)$ and $F(B \times_A C) = F(B) \times_{F(A)} F(C)$ for each diagram*

$$\begin{array}{ccc} & & C \\ & & \downarrow \\ B & \to & A \end{array}$$

in $\mathcal{C}$ (i.e. F commutes with finite products and finite fibered products).

Every representable functor is left exact by definition of product and fibered product.

(A.4.8) LEMMA *Let I be a filtered category and $\Phi : I \to Funct(\mathcal{C}, \text{(sets)})$ a covariant functor. Then, for each diagram in $\mathcal{C}$:*

$$\begin{array}{ccc} & & C \\ & & \downarrow \\ B & \to & A \end{array}$$

there is a bijection:

$$\lim_{\to} \Phi_i(B) \times_{\lim \Phi_i(A)} \lim_{\to} \Phi_i(C) \cong \lim_{\to} [\Phi_i(B) \times_{\Phi_i(A)} \phi_i(C)]$$

The proof of this Lemma is straightforward and we omit it. The following result, which will be needed in §III.2, is a useful characterization of left exact functors.

(A.4.9) PROPOSITION *A covariant functor $F : \mathcal{C} \to$ (sets) is left exact if and only if the category I_F is cofiltered.*

Proof

Assume that I_F is cofiltered. Applying Lemma (A.4.8) to the functor Φ of Example (A.4.5), we see that the inductive limit $F = \lim_{(X,\xi)} h_X$ is left exact because each functor h_X is left exact.

Conversely assume that F is left exact. Let $(X,\xi),(Y,\eta) \in Ob(I_F)$; we must find

$$\begin{array}{ccc} (Z,\zeta) & \to & (X,\xi) \\ \downarrow & & \\ (Y,\eta) & & \end{array}$$

Take $(Z,\zeta) = (X \times Y,(\xi,\eta))$. Now consider $(X,\xi) \rightrightarrows (Y,\eta)$ coming from $\phi,\psi : X \to Y$. We have

$$F(\phi)(\xi) = F(\psi)(\xi) = \eta$$

Consider the diagram:

$$\begin{array}{ccc} X & \xrightarrow{\Gamma_\phi} & X \times Y \\ \uparrow & & \uparrow \Gamma_\psi \\ K & \to & X \end{array}$$

where $\Gamma_\phi = (1_X,\phi)$ and $\Gamma_\psi = (1_X,\psi)$ and $K = X \times_{X\times Y} X$. Since F is left exact

$$F(K) = F(X) \times_{F(X\times Y)} F(X)$$

and there is $\chi \in F(K)$ corresponding to (ξ,ξ):

$$\begin{array}{ccc} \xi & \stackrel{F(\Gamma_\phi)}{\longmapsto} & (\xi,\eta) \\ \uparrow & & \uparrow F(\Gamma_\psi) \\ \chi & \longmapsto & \xi \end{array}$$

Then (K,χ) is the equalizer of ϕ and ψ. Therefore I_F is cofiltered. *q.e.d.*

Let I be a category. A full subcategory J of I is *cofinal* if for each $i \in Ob(I)$ there is a morphism $f : i \to j$ for some $j \in Ob(J)$. It follows immediately from the definitions that if $\Phi : I \to \mathcal{D}$ is a covariant functor and $\Phi_J : J \to \mathcal{D}$ is its restriction, then

$$\lim_{\to} \Phi = \lim_{\to} \Phi_J$$

* * * * * *

Let Z be a scheme. In this subsection we will consider contravariant functors defined on (schemes/Z). All we will say holds, with obvious modifications, for functors defined on (algschemes/Z), the full subcategory of algebraic Z-schemes. A contravariant functor

$$F : (\text{schemes}/Z)^\circ \to (\text{sets})$$

defines on every Z-scheme S a presheaf of sets:

$$U \mapsto F(U)$$

for all open sets $U \subset S$. F is called a *sheaf* (more precisely a *sheaf in the Zariski topology*) if it defines a sheaf on every scheme; namely if for all Z-schemes S and for all open coverings $\{U_i\}$ of S the following is an exact sequence of sets:

$$F(S) \to \prod_i F(U_i) \mathrel{\substack{\to \\ \to}} \prod_{i,j} F(U_i \cap U_j)$$

The most important sheaves are the *representable functors*, i.e. functors isomorphic to one of the form:

$$S \mapsto \mathrm{Hom}_Z(S, X)$$

for some Z-scheme X.

If F is a sheaf then F is determined by its restriction to the category of affine schemes. Infact, if S is any Z-scheme we can consider an affine open cover $\{U_i\}$. For any i, j we take an affine open cover $\{V_{i,j,\alpha}\}$ of $U_i \cap U_j$; composing the map

$$F(U_i) \mathrel{\substack{\to \\ \to}} \prod_{i,j} F(U_i \cap U_j)$$

with the inclusions $F(U_i \cap U_j) \to \prod_\alpha F(V_{i,j,\alpha})$ we obtain the exact sequence:

$$F(S) \to \prod_i F(U_i) \mathrel{\substack{\to \\ \to}} \prod_{i,j,\alpha} F(V_{i,j,\alpha})$$

which shows that $F(S)$ is determined by its values on affine schemes.

It is very important to have conditions, easy to verify in practice, for a contravariant functor $F : (\text{schemes}/Z) \to (\text{sets})$ to be representable. Certainly a necessary condition is that F is a sheaf. Another necessary condition is the following.

Recall that a subfunctor G of F is said to be an *open* (resp. *closed*) *subfunctor* if for every scheme S and for every morphism of functors

$$\mathrm{Hom}(-, S) \to F$$

the fibered product $\mathrm{Hom}(-, S) \times_F G$, which is a subfunctor of $\mathrm{Hom}(-, S)$, is represented by an open (resp. closed) subscheme of S. A family of open subfunctors $\{G_i\}$ of F is a *covering* of F if for every Z-scheme S and for every morphism of functors $\mathrm{Hom}(-, S) \to F$ the family $\{\mathrm{Hom}(-, S) \times_F G_i\}$ of subschemes of S is an open covering of S.

An obvious example is obtained considering an open (resp. closed) subscheme X' of a Z-scheme X: correspondingly we obtain an open (resp. closed) subfunctor

Hom$(-, X')$ of Hom$(-, X)$. An open cover $\{X_i\}$ of X defines a cover of Hom$(-, X)$ by open subfunctors.

Therefore a second obvious necessary condition for a functor F to be representable is that it can be covered by representable open subfunctors. We will now show that these two necessary conditions are also sufficient.

(A.4.10) PROPOSITION *Let*

$$F : (\text{schemes}/Z)^\circ \to (\text{sets})$$

be a contravariant functor. Suppose that:

(i) F is a sheaf;

(ii) F admits a covering by representable open subfunctors F_i.

Then F is representable.

Proof

Letting $F_{ij} = F_i \times_F F_j$, by (ii) the projections $F_{ij} \to F_i$ correspond to open embedding of schemes $X_{ij} \to X_i$. Therefore the F_i's patch together to form a representable functor Hom$(-, X)$, where X is the scheme obtained by patching the X_i's together along the X_{ij}'s. By (i), F and Hom$(-, X)$ are isomorphic. *q.e.d.*

NOTES

1. For more on representable functors in algebraic geometry the reader may consult Murre(1965) and Vistoli(2003).

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