DE FRANCHIS' CONTRIBUTIONS TO THE THEORY OF ALGEBRAIC CURVES

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In my talk I will survey the scientific contributions of M. de Franchis to the theory of algebraic curves. In what follows by a *variety* I will always mean an irreducible nonsingular complex projective algebraic variety. A *curve* will be sinonimous of variety of dimension 1.

In 1893 Humbert and Castelnuovo, using a transcendental result of Painlevé, proved:

Theorem 1: *On a curve X does there not exist a continuous system of irrational involutions of genus $\pi \geq 2$.*

Recall that an *irrational involution of genus* π is a surjective morphism $X \to Y$ of X onto a curve Y of genus π. This a special case of a *correspondence* between two curves, a topic deeply studied in the second half of XIX century.

A special case of theorem 1 is the following:

Theorem 2 (Schwarz): *A curve X of genus $\pi \geq 2$ does not have a continuous group of automorphisms.*

From theorem 1 Humbert and Castelnuovo deduced the following:

Theorem 3: *On a curve X an algebraic system of divisors of dimension $r \geq 2$ and index 1 is linear unless it coincides with $X^{(r)}$, the r-th symmetric power of X.*

Both theorems 1 and 3 were considered of great importance at that time. Today we still attribute a central role to theorem 1, while theorem 3 has been practically forgotten.

The proofs given by Humbert and Castelnuovo are transcendental, making use of the theory of abelian integrals. Using today's language they can be entirely reformulated in geometrical terms (see below).

In 1903 de Franchis took up this subject from a completely different point of view. In studying the geometry of the surface $X \times Y$ product of two curves X and Y, he noticed that a correspondence between X and Y can be considered as a curve $\Gamma \subset X \times Y$ and, studied as such, the all subject of correspondences can be greatly simplified. This was an important conceptual progress, and actually it is the modern point of view.

Slightly later Severi also introduced this point of view, but the priority belongs to de Franchis, as Severi himself admitted ([S1], footnote (1)).

Using his idea de Franchis observed that an irrational involution $X \to Y$ of degree n can be viewed as a symmetric correspondence $(n-1, n-1)$ on X, thereby defining a curve Σ in $X \times X$. He studied this curve proving essentially that it has negative self intersection. This implies that it cannot move in a continuous family, and proves theorem 1. Then he easily deduced theorem 3 as well.

About theorem 3 it is worth recalling that in [T] Torelli gave a completely geometrical proof of it only using properties of the symmetric products $X^{(r)}$ and not as a consequence of theorem 1; this is the point of view from which this result is regarded today (see [Ma]).

In 1913 de Franchis came back to the subject of irrational involutions publishing a short note of one page where the following result is proved:

Theorem 4: *On a curve X there are only finitely many irrational involutions of genus $\pi \geq 2$.*

This is what today is called *the theorem of de Franchis*. This celebrated and important result and its extremely simple proof had escaped the attention of Castelnuovo, Severi and others. This is surprising because the proof given by de Franchis is in Severi's style: it consists in noting that, once X is suitably embedded, e.g. by $|3K|$, irrational involutions on X correspond to projections; since the centers of projection form finitely many algebraic families, the conclusion follows from theorem 1.

This proof is also reproduced on page 271 of [S2], and can be easily adapted to give a more general result. Given varieties X and Y, let's denote by $\mathcal{M}(X, Y)$ the set of surjective morphisms from X to Y. The generalization of the theorem of de Franchis we alluded to is the following:

Theorem 5: *For a fixed variety X, there are only finitely many pairs (Y, f), where Y is a curve of genus $\pi \geq 2$ and $f \in \mathcal{M}(X, Y)$.*

This extension of theorem 4 and the adaptation of its proof are explicitly given in [Sa] and attributed to Severi, referring infact to [S2], page 271.

In [HS] this result is refined further, by giving an upper bound on the number of pairs (Y, f) only in terms of the Chern numbers of X; in case X is a curve this upper bound depends only on the genus of X.

It is interesting to read the note on page 288 of [S2] where the author, while stating that theorem 4 is due to de Franchis, also claims that it is already contained in the work of Painlevé. It is likely that Severi referred to the fact that a proof of theorem 4 can be given in the spirit of the Castelnuovo-Humbert-Painlevé approach, in the following way. By Torelli's theorem and the functorial propertied of the jacobian variety of a curve, the existence of an irrational involution on X is equivalent to the existence of an abelian variety quotient $J(X) \to J'$ of the jacobian variety $J(X)$ of X; the conclusion follows from the fact that there are finitely many such quotients, by general properties of abelian varieties.

This proof uses Torelli's theorem, which was proved only in 1915, and therefore was beyond the possibilities of Painlevé and de Franchis at the time when they published their papers. Therefore the claim of Severi, made in 1926, has only the value of an *a posteriori* observation.

The theorem of de Franchis and its generalization 5 have undergone vast generalizations by several authors. One reason why such results raised so much interest is that they have to do with characteristic properties of what we call today "varieties of general type".

A *variety of general type* Y of dimension n is defined by the condition that the canonical line bundle K_Y has positive self intersection: $K_Y^n > 0$. A special case of this condition is that K_Y is ample. Curves of general type are simply curves of genus $\pi \geq 2$. Given varieties X and Y, we denote by $\mathcal{R}(X, Y)$ the set of dominant rational maps from X to Y. An important generalization of the theorem of de Franchis is the following result:

Theorem (Kobayashi-Ochai [KO]): *Given varieties X and Y, with Y of general type, $\mathcal{R}(X, Y)$ is finite.*

Actually in [KO] Kobayashi and Ochai prove a more general result, which applies to complex spaces rather than to varieties only.

The following is a generalization of theorem 5:

Theorem (Maehara [M]): *Given X, there are only finitely many varieties Y with ample canonical bundle such that $\mathcal{M}(X,Y) \neq \emptyset$.*

Other results of this type have been proved by Deschamps-Menegaux ([DM1], [DM2]).

Another reason why the theorem of de Franchis is interesting is because it is related with number theoretical finiteness results: this is well illustrated in the above mentioned article [Sa] of Samuel. Actually the entire classical topic of correspondences has been throughly studied in modern times from an arithmetical point of view. A vivid testimony of this fact is given by A. Weil on page 557 of volume 1 of [W], where it appears that for modern readers the main classical reference for this topic has been chapter 6 of [S2].

Notably de Franchis' theorem has been applied by Parshin in [P] to show that the Shafarevich conjecture for curves defined over number fields implies the Mordell conjecture; recall that the proof of the Mordell conjecture given by Faltings in 1983 consists in proving the Shafarevich conjecture and then using the result of Parshin.

For a more complete account of the consequences and generalizations of the theorem of de Franchis we refer to chapter 2 of [ZY].

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