LECTURES ON THE BIRATIONAL GEOMETRY OF $\overline{M}_g$*

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1 Unirationality vs. uniruledness

My lectures will be devoted to the birational geometry of $\overline{M}_g$, the moduli space of stable curves of genus g, which is a projective variety obtained as a compactification of M_g, the quasi-projective moduli variety, or moduli space, of complex projective nonsingular curves of genus g. Stable curves are certain types of singular curves of arithmetic genus g (we don't need to define them) which are added as new elements to obtain $\overline{M}_g$.
Recall that $\overline{M}_g$ is a normal irreducible projective variety of dimension $3g-3$ if $g \geq 2$. Moreover it has a *universal property:* for each family $f : \mathcal{C} \to S$ of stable curves of genus g the map $\psi_f : S \to \overline{M}_g$, sending a point $s \in S$ to the isomorphism class $[f^{-1}(s)]$ of its fibre, is a morphism. ψ_f is the *functorial morphism* defined by f.
If a family $f : \mathcal{C} \to S$ is such that the functorial morphism $\psi_f : S \to \overline{M}_g$ is dominant, then we say that f is a family *with general moduli*, and that the family contains the general curve of genus g, or *a curve with general moduli.*
We know the following to be true:

$$\overline{M}_g \text{ is} \begin{cases} \text{rational} & \text{for } 0 \leq g \leq 6 \text{ ([11],[12],[13],[22],[23])} \\ \text{unirational} & \text{for } 7 \leq g \leq 14 \text{ (Severi,[18],[3],[24])} \\ \text{rationally connected} & \text{for } g = 15 \text{ ([2])} \\ \text{of } k\text{-dim} = -\infty & \text{if } g = 16 \text{ ([4])} \\ \text{of } k\text{-dim} \geq 2 & \text{if } g = 23 \text{ ([6])} \\ \text{of general type} & \text{if } g \geq 22 \text{ ([10],[9],[5],[7])} \end{cases}$$

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Nothing is known for $17 \leq g \leq 21$.
When $g \geq 22$ we have precise information about the Kodaira dimension. This is a fundamental invariant in the birational classification of algebraic varieties, and its computation in the case of $\overline{M}_g$ is a very deep result. Unfortunately it does not seem to be easy to translate this information into a geometrical condition concerning families of curves of genus g.
The properties of rationality and unirationality of $\overline{M}_g$ have a geometrical meaning in terms of families of curves. The unirationality of $\overline{M}_g$ is equivalent to the existence of a family of nonsingular curves of genus g with general moduli parametrized by a nonsingular and *rational* variety (which can be assumed to have dimension $3g-3$). Rationality means that such a family can be found which parametrizes general curves of genus g in a 1-1 way, i.e. such that ψ_f is birational (in particular $\dim(S) = 3g-3$ if $g \geq 2$).

Example 1.1 Every non-hyperelliptic curve of genus 3 can be realized as a plane nonsingular quartic. All such curves are parametrized by an open set S of the projective space $\mathbb{P}(H^0(\mathcal{O}_{\mathbb{P}^2}(4)) \cong \mathbb{P}^{14}$ of plane quartics. Since S is rational and since non-hyperelliptic curves of genus 3 correspond to an open dense subset of $\overline{M}_3$, this gives an easy proof of the fact that $\overline{M}_3$ is unirational. The rationality of $\overline{M}_3$ is much more difficult to detect (see [13]).
Plane irreducible quintics with two nodes have genus 4 and are parametrized by a locally closed irreducible and nonsingular subvariety S of $\mathbb{P}(H^0(\mathcal{O}_{\mathbb{P}^2}(5))$ of codimension two (a special example of a *Severi variety*). Since the Brill-Noether number $\rho(4,2,5) = 1$ is nonnegative it follows from Brill-Noether theory that S parametrizes the general curve of genus 4. Moreover S is rational because it is an open subset of the total space of a projective bundle over a dense open subset of the second symmetric product $(\mathbb{P}^2)^{(2)}$ of $\mathbb{P}^2$: the fibre over $p_1 + p_2$ is the linear system of quintics singular at p_1 and p_2. Therefore $\overline{M}_4$ is unirational.
If we try to repeat the same argument for $g = 5$ using plane quintics with one node we still find a rationally parametrized family, but the curves it parametrizes are special because $\rho(5,2,5) = -1$. If we want a family with general moduli in genus 5 we have to consider plane sextics with 5 nodes. In this case $\rho(5,2,6) = 2$ so that we obtain general curves of genus 5 in this family, again by Brill-Noether theory. Moreover for every general 5-tuple of points $\{p_1, \ldots, p_5\}$ the linear system of plane quintics singular at $\{p_1, \ldots, p_5\}$ is non-empty. It follows again that S is rational, because fibered over an open subset of $(\mathbb{P}^2)^{(5)}$ with fibres linear systems, and $\overline{M}_5$ is unirational.

This method works well until genus 10 and it is essentially how Severi proved the unirationality of $\overline{M}_g$ for all $g \leq 10$. We refer to [1] for a modern discussion of his proof.

There is another birational notion, namely *uniruledness*, which has an interpretation in terms of families. I will give a negative formulation of it:

Proposition 1.2 *The following conditions are equivalent:*

1. *$\overline{M}_g$ is not uniruled.*
2. *A general curve of genus g cannot occur in a non-trivial linear system in any non-ruled surface.*

Proof. Uniruledness means that given a general curve C of genus g, there is a non-constant morphism $\psi : \mathbb{P}^1 \longrightarrow \overline{M}_g$ whose image contains $[C]$. If such a ψ exists then we can pullback to an open subset of $\mathbb{P}^1$ the universal family of curves of genus g, which is defined over an open set of $\overline{M}_g$ containing $[C]$ because C has no automorphisms. We obtain a family of curves of genus g parametrized by an affine rational curve and then we can embed the total space in a projective surface X extending the family to one over $\mathbb{P}^1$. This gives a linear pencil of curves in X containing C among its members.

Conversely, if a general curve C of genus g moves in a surface X in a non-trivial linear system, then, after blowing-up the base points, we can assume that there is a morphism $f : X \to \mathbb{P}^1$ with C as a fibre. Let $\psi_f : \mathbb{P}^1 \longrightarrow \overline{M}_g$ be the induced map. If $\overline{M}_g$ is not uniruled then ψ_f is constant. In other words, the fibration f is *isotrivial*. By the structure theorem about such fibrations, there is a nonsingular curve Γ and a finite group G acting on both C and Γ such that there is a birational isomorphism

$$X \dashrightarrow (C \times \Gamma)/G$$

and a commutative diagram:

$$\begin{tikzcd} X \arrow[r, dashed] \arrow[d, "f"] & (C \times \Gamma)/G \arrow[d] \\ \mathbb{P}^1 \arrow[r, equal] & \Gamma/G \end{tikzcd}$$

where the right vertical arrow is the projection. But since C is general, it has no non-trivial automorphisms, and therefore G acts trivially on C: thus X is birational to $C \times (\Gamma/G)$. $\square$

As a corollary of what is known about k-dim$(\overline{M}_g)$ we obtain:

Corollary 1.3 *A general curve of genus $g \geq 22$ cannot occur in a non-trivial linear system in any non-ruled surface.*

This can be viewed as a result belonging to the theory of algebraic surfaces. It would be interesting to have a direct proof of it which does not use any information about k-dim$(\overline{M}_g)$ and which, hopefully, gives some new insight into the open cases $17 \leq g \leq 21$. In these lectures we will explore this point of view.
Note that the difference between unirationality and uniruledness is that in the first case one can parametrize almost all curves of genus g simultaneously by a rational variety, while in the second case almost all curves of genus g are fibres of a one-parameter family parametrized by $\mathbb{P}^1$, but all these families cannot in general be obtained as restrictions of a single larger rational family.

2 Fibrations

A rational curve containing a general point $[C] \in \overline{M}_g$ can be represented by a surface fibered over a curve having C among its fibres. For this reason we will study such fibrations. Let's introduce some terminology.
By a *fibration* we mean a surjective morphism

$$f : X \to S$$

with connected fibres and nonsingular general fibre from a projective nonsingular surface to a projective nonsingular connected curve. We will denote by

$g =$ the genus of the general fibre. *We will always assume* $g \geq 2$.

$b =$ the genus of S.

A fibration is called:

- *relatively minimal* if there are no (-1)-curves contained in any of its fibres.

- *semistable* if it is relatively minimal and every fibre has at most nodes as singularities.

- *isotrivial* if all its nonsingular fibres are mutually isomorphic; equivalently, if two general nonsingular fibres of f are mutually isomorphic (the equivalence of the two formulations follows from the separatedness of $\overline{M}_g$).

We have an exact sequence

$$0 \to f^*\omega_S \to \Omega^1_X \to \Omega^1_{X/S} \to 0 \tag{1}$$

(which is exact on the left because the first homomorphism is injective on a dense open set and $f^*\omega_S$ is locally free). Note that $\Omega^1_{X/S}$ is torsion free rank-one but not locally free if f has singular fibres. In particular it is different from the relative dualizing sheaf

$$\omega_{X/S} = \omega_X \otimes f^*\omega_S^{-1}$$

which is invertible.

If we dualize the sequence (1) we obtain the exact sequence:

$$0 \to T_{X/S} \to T_X \to f^*T_S \to N \to 0 \tag{2}$$

where we have denoted

$$T_{X/S} := Hom_{\mathcal{O}_X}(\Omega^1_{X/S}, \mathcal{O}_X)$$

and

$$N := Ext^1_{\mathcal{O}_X}(\Omega^1_{X/S}, \mathcal{O}_X) \tag{3}$$

The sequence (2) shows, in particular, that N is the normal sheaf of f, and also the first relative cotangent sheaf $T^1_{X/S}$ of f. In particular, *N is supported on the set of singular points of the fibres of f.*

Moreover *$T_{X/S}$ is an invertible sheaf* because it is a second syzygy of the $\mathcal{O}_X$-module N.

We we will need the first part of the following classical result:

Theorem 2.1 (Arakelov, Serrano) *If f is a non-isotrivial fibration then*

$$h^0(X, T_{X/S}) = 0 = h^0(X, T_X) \tag{4}$$

If moreover f is relatively minimal then we also have:

$$h^1(X, T_{X/S}) = 0 \tag{5}$$

Proof. of (4). f is non-isotrivial if and only if $f_*T_X = 0$ (Serrano). Since

$$f_*T_{X/S} \subset f_*T_X$$

we also have $f_*T_{X/S} = 0$ if f is non-isotrivial. Thus (4) is a consequence of the Leray spectral sequence. □

Denoting by Ext^1_f the first derived functor of f_*Hom, we are interested in the sheaf $Ext^1_f(\Omega^1_{X/S}, \mathcal{O}_X)$ because its cohomology describes the deformation theory of f.

Lemma 2.2 *For any fibration $f : X \to S$ there is an exact sequence of sheaves on S:*

$$\begin{array}{ccccccc} 0 & \to R^1 f_*T_{X/S} & \xrightarrow{c_{10}} & Ext^1_f(\Omega^1_{X/S}, \mathcal{O}_X) & \xrightarrow{c_{01}} & f_*Ext^1_X(\Omega^1_{X/S}, \mathcal{O}_X) & \to 0 \\ & & & & & \| & \\ & & & & & f_*N & \end{array} \tag{6}$$

Proof. (6) is the sequence associated to the local-to-global spectral sequence for Ext_f. □

Proposition 2.3 *If the fibration f is non-isotrivial then we have:*

$$\chi(Ext^1_f(\Omega^1_{X/S}, \mathcal{O}_X)) = 11\chi(\mathcal{O}_X) - 2K^2_X + 2(b-1)(g-1) \tag{7}$$

In particular, if $S = I\!P^1$ then:

$$\chi(Ext^1_f(\Omega^1_{X/S}, \mathcal{O}_X)) = 11\chi(\mathcal{O}_X) - 2K^2_X - 2(g-1)$$

Proof. Since the fibres of f are 1-dimensional we have

$$R^2 f_*T_{X/S} = 0$$

Moreover $f_*Ext^2(\Omega^1_{X/S}, \mathcal{O}_X) = 0$ because $Ext^2(\Omega^1_{X/S}, \mathcal{O}_X) = 0$ by the exact sequence (1). Therefore, using the local-to-global spectral sequence for Ext_f we deduce that

$$R^1 f_*N = Ext^2_f(\Omega^1_{X/S}, \mathcal{O}_X) = 0$$

where the last equality is true because the fibres of f are 1-dimensional. This gives:

$$\chi(f_*N) = \chi(N)$$

Moreover, since f is non-isotrivial, from (4) and the Leray spectral sequence we get

$$\chi(R^1 f_* T_{X/S}) = -\chi(T_{X/S})$$

We now use the exact sequence (6) and we deduce that

$$\begin{aligned} \chi(Ext^1_f(\Omega^1_{X/S}, \mathcal{O}_X)) &= \chi(f_* N) + \chi(R^1 f_* T_{X/S}) \\ &= \chi(N) - \chi(T_{X/S}) \\ &= \chi(f^* T_S) - \chi(T_X) \qquad \text{(by (2))} \end{aligned}$$

Using Riemann-Roch one computes that:

$$\begin{aligned} \chi(f^*(T_S)) &= \chi(\mathcal{O}_X) + 2(b-1)(g-1) \\ \chi(T_X) &= 2K_X^2 - 10\chi(\mathcal{O}_X) \end{aligned}$$

and by substitution one gets (7). □

3 Free fibrations

In order to check whether there exists a rational curve containing a general point of $\overline{M}_g$ one might naively try to construct directly a non-isotrivial rational fibration containing a *general* curve of genus g among its fibres. This turns out to be hopeless, simply because general curves are virtually impossible to produce in practise. This is related with the complexity of $\overline{M}_g$: in order to produce a general curve one should describe it in some way, for example by equations whose coefficients should vary in a suitable way, and this would give an explicit parametrization of $\overline{M}_g$, which is exactly what we are unable to do. For remarks and further discussion about this point we refer the reader to [17].
We will try another approach. Starting from any rational fibration, we will apply deformation theory techniques to check whether it can be deformed to another one which parametrizes a general curve of genus g. We give the following definition.

Definition 3.1 *Let $f : X \to \mathbb{P}^1$ be a non-isotrivial rational fibration. We say that* the fibres of f have general moduli *if there is a family of deformations* *of f:*

$$\begin{tikzcd} X \arrow[r, hook] \arrow[dd, "f"] & \mathcal{X} \arrow[dd, "F"] \arrow[dr, "\varphi"] & \\ & & V \\ I\!\!P^1 \times \{v\} \arrow[r, hook] & I\!\!P^1 \times V \arrow[ur] & \end{tikzcd}$$

parametrized by a nonsingular connected pointed algebraic scheme (V, v) such that F is a family of curves of genus g with surjective Kodaira-Spencer map at every point of a non-empty open subset of $I\!\!P^1 \times \{v\}$.

From the definition it follows that if the fibres of f have general moduli then, up to taking a general deformation of f, we may assume that a general fibre C of f is a general curve of genus g. We obtain in this way that the general curve C of genus g moves in a non-trivial linear system on the algebraic surface X, and therefore $\overline{M}_g$ is uniruled. Therefore, in order to have information about the uniruledness of $\overline{M}_g$ using this definition we need a criterion to check whether the fibres of a given rational fibration have general moduli.

Suppose given a rational fibration

$$f: \; X \longrightarrow I\!\!P^1$$

whose general fibre has genus g. We have:

$$Ext^1_f(\Omega^1_{X/I\!\!P^1}, \mathcal{O}_X) \cong \bigoplus_{i=1}^{3g-3} \mathcal{O}_{I\!\!P^1}(a_i)$$

for some integers a_i. We call f *free* if it is non-isotrivial and $a_i \geq 0$ for all i. The reason why are we considering free fibrations is because, using general deformation-theoretic techniques one can prove the following:

Theorem 3.2 *Assume that $f : X \to I\!\!P^1$ is a non-isotrivial rational fibration whose fibres have general moduli. Then f is free.*

The proof is straightforward and we omit it. We will try to apply this theorem in a negative way, looking for an upper bound on g for the existence of free rational fibrations with fibres of genus g. If we find such an upper bound g_0, then we have proved that $\overline{M}_g$ is not uniruled for $g > g_0$. In order to carry out this strategy we need numerical criteria for the freeness of a fibration.

Lemma 3.3 *Suppose given a rational fibration*

$$f: X \longrightarrow \mathbb{P}^1$$

whose general fibre has genus g. There is a homomorphism:

$$\kappa_f: T_{\mathbb{P}^1} \longrightarrow Ext^1_f(\Omega^1_{X/\mathbb{P}^1}, \mathcal{O}_X))$$

which is injective if and only if f is non-isotrivial.

Proof. The exact sequence (1) induces an exact sequence on $\mathbb{P}^1$:

$$0 \to f_* T_X \to T_{\mathbb{P}^1} \xrightarrow{\kappa_f} Ext^1_f(\Omega^1_{X/\mathbb{P}^1}, \mathcal{O}_X)$$

We have $f_* T_X = 0$ if and only if f is non-isotrivial, and this proves the assertion. $\square$

Proposition 3.4 *Suppose that $f : X \to \mathbb{P}^1$ is a rational fibration. If f is free then we have:*

$$11\chi(\mathcal{O}_X) - 2K_X^2 \geq 5(g-1) + 2 \tag{8}$$

Proof. By Lemma 3.3 we have $a_i \geq 2$ for some i. Therefore, since

$$H^1(Ext^1_f(\Omega^1_{X/\mathbb{P}^1}, \mathcal{O}_X)) = 0$$

we have

$$11\chi(\mathcal{O}_X) - 2K_X^2 - 2(g-1) = H^0(Ext^1_f(\Omega^1_{X/S}, \mathcal{O}_X)) \geq 3g - 1$$

$\square$

Let Y be a projective nonsingular surface, and let $C \subset Y$ be a projective nonsingular connected curve of genus g such that

$$\dim(|C|) \geq 1$$

Consider a linear pencil Λ contained in $|C|$ whose general member is nonsingular and let $\sigma : X \to Y$ be the blow-up at its base points (including the infinitely near ones). We obtain a rational fibration

$$f : X \to \mathbb{P}^1$$

obtained by composing σ with the rational map $Y \dashrightarrow \mathbb{P}^1$ defined by Λ. We will call f *the fibration defined by the pencil* Λ. If moreover the curve C is general then the fibres of the fibration f have general moduli and therefore f is free.
By expressing the invariants appearing in (8) in terms of Y and by some little extra work one can prove the following result:

Theorem 3.5 *Let Y be a projective nonsingular surface. Assume that $C \subset Y$ is a general projective nonsingular connected curve of genus g and that*

$$\dim(|C|) \geq 1$$

Then:

$$11\chi(\mathcal{O}_Y) - 2K_Y^2 + 2C^2 \geq 5(g-1) + h^0(\mathcal{O}_Y(C)) \tag{9}$$

If moreover $h^0(K_Y - C) = 0$ then

$$10\chi(\mathcal{O}_Y) - 2K_Y^2 \geq 4(g-1) - C^2 \tag{10}$$

Note that

$$10\chi(\mathcal{O}_Y) - 2K_Y^2 = -\chi(T_Y) = h^1(T_Y) - h^2(T_Y) - h^0(T_Y)$$

If $h^0(T_Y) = 0$ (which means that Y has no infinitesimal automorphisms) this is the *expected number of moduli* of Y. Therefore inequality (10) roughly says that if the surface Y has no infinitesimal automorphisms (e.g. it is of general type) and contains a curve of genus g with general moduli moving in a non-trivial linear system then it must have sufficiently many moduli.

Examples 3.6 (i) Let $Y = \mathbb{P}^2$. Then

$$10\chi(\mathcal{O}_{\mathbb{P}^2}) - 2K_{\mathbb{P}^2}^2 = -8$$

If $C \subset Y$ has degree d then $C^2 = d^2$, $4(g-1) = 2d(d-3)$ and (10) gives $d \leq 4$. This is the well known bound on d for a plane nonsingular curve of degree d to have general moduli.

(ii) Let Y be a K3-surface. Then

$$10\chi(\mathcal{O}_Y) - 2K_Y^2 = 20$$

If $C \subset Y$ has genus g then $C^2 = 2(g-1)$ and (10) implies $g \leq 11$. This bound is sharp because a general curve of genus 11 can be embedded in a K3-surface [15], but this is not possible in genus $g \geq 12$, as it can be shown by an elementary count of parameters [14]. In fact a curve of genus g moves in a g-dimensional linear system on a K3-surface Y; on the other hand Y depends on 19 moduli. Therefore the locus in M_g spanned by C when it varies in $|C|$ and Y varies cannot exceed $g + 19$. In order to have

$$g + 19 \geq 3g - 3$$

it must therefore be $g \leq 11$.

4 Nodal curves on surfaces of low Kodaira dimension

The examples 3.6 involve *minimal* surfaces. If we want to investigate the (non)-uniruledness of $\overline{M}_g$ we cannot restrict to this case. On the other hand if we allow Y to be just any surface then we loose control on the left hand side of (10), because each blow-up increases $10\chi(\mathcal{O}_Y) - 2K_Y^2$ by 2, so that (10) becomes less and less significant. A good compromise can be to consider *nodal curves* on minimal surfaces. They correspond to nonsingular curves on non-minimal surfaces where the number of blow-ups necessary to desingularize the curve can be kept under control. As an example we can prove the following result:

Theorem 4.1 *Let $C \subset \mathbb{P}^2$ be an irreducible curve of degree d having δ nodes $p_1, \ldots, p_\delta$ and no other singularities. Let*

$$g = \binom{d-1}{2} - \delta$$

be the geometric genus of C. Assume that there exists a linear pencil of curves of degree d singular at $p_1, \ldots, p_\delta$ and that the normalization D of C has general moduli. Then $g \leq 9$.

Proof. Let $\pi : Y \longrightarrow I\!P^2$ be the blow-up of $I\!P^2$ at $p_1, \ldots, p_\delta$. Then the proper transform $D \subset Y$ of C is its normalization and we have:

$$D^2 = d^2 - 4\delta, \qquad 10\chi(\mathcal{O}_Y) - 2K_Y^2 = -8 + 2\delta, \qquad 4(g-1) = 2d(d-3) - 4\delta$$

Therefore (10) gives:

$$-8 + 2\delta \geq 2d(d-3) - 4\delta - (d^2 - 4\delta)$$

which is equivalent to $2\delta \geq d^2 - 6d + 8$. Using the fact that $D^2 \geq 0$ we obtain:

$$0 \leq d^2 - 4\delta \leq -d^2 + 12d - 16$$

which implies $d \leq 10$. An elementary well known case-by-case analysis shows that there are no nodal irreducible curves of degree 9 or 10 having general moduli and moving in a pencil, and that in degree 8 the highest genus of such a curve is 9, corresponding to $\delta = 12$ (see [1] for details). □

Example 4.2 If $C \subset I\!P^2$ is an irreducible quintic with two nodes p_1, p_2 then $g = 4$ and we know (Example 1.1) that D has general moduli. Therefore a general pencil of quintic curves singular at p_1, p_2 defines a free fibration and inequality (10) must hold. In fact $D^2 = 25 - 8 = 17$, $10\chi(\mathcal{O}_Y) - 2K_Y^2 = -4$ and

$$-4 > -5 = 4(g-1) - D^2$$

If we consider a quintic $C \subset I\!P^2$ with one node ($g = 5$) then we obtain

$$10\chi(\mathcal{O}_Y) - 2K_Y^2 = -6 < -5 = 4(g-1) - D^2$$

i.e. (10) does not hold, compatibly with the fact that D is not a general curve (Example 1.1). But (10) holds for sextics with five nodes (of genus 5) as the reader can easily check.

Another result which follows from a similar analysis is the following:

Theorem 4.3 *Let Y be a minimal surface of geometric genus p_g and with k-dim$(Y) \geq 0$, and let $C \subset Y$ be an irreducible curve of geometric genus $g \geq 3$ having nodes $p_1, \ldots, p_\delta$ and no other singularities. Assume that:*

(i) C is contained in a linear pencil Λ whose general member is an irreducible curve with nodes at $p_1, \ldots, p_\delta$ and no other singularities.

(ii) The normalization D of C is a general nonsingular curve of genus g.

(iii) $h^0(K_Y - C) = 0$.

Then

$$g \leq 1 + 5\chi(\mathcal{O}_Y) - K_Y^2$$

In particular

$$g \leq \begin{cases} 6 & \textit{if } p_g = 0 \\ 11 & \textit{if } p_g = 1 \\ 16 & \textit{if } p_g = 2 \end{cases}$$

Proof. Let $Z \longrightarrow Y$ be the blow-up at $p_1, \ldots, p_\delta$; we may assume that the proper transform of Λ defines a free fibration. Since Y is not ruled we have $D^2 \leq 2g - 2 - 2\delta$. Therefore (10) gives:

$$2(g-1) \leq 4(g-1) - D^2 - 2\delta \leq 10\chi(\mathcal{O}_Z) - 2K_Z^2 - 2\delta = 10\chi(\mathcal{O}_Y) - 2K_Y^2$$

and we obtain the stated inequalities. □

Remarks 4.4 The bounds of Theorem 4.3 are not sharp if the surface Y is irregular. For example, in the case of an abelian surface Y, a more accurate computation would give the bound $g \leq 1$.
In [15] and [16] it is proved that a general nonsingular curve of genus g can be embedded in a K3 surface if and only if $g \leq 11$ and $g \neq 10$. In particular the bound of Theorem 4.3 is sharp for K3 surfaces. In [8] it is shown that even a general curve of genus 10 can be birationally embedded in a K3 surface provided one allows it to have one node (hence arithmetic genus 11).

The following theorem covers the remaining cases of non-ruled elliptic surfaces:

Theorem 4.5 *Let Y be a non-ruled and non-rational minimal elliptic surface of geometric genus p_g, $\pi : Y \to B$ the elliptic fibration onto a nonsingular connected curve B. Let $C \subset Y$ be an irreducible curve of geometric genus $g \geq 3$ having nodes $p_1, \ldots, p_\delta$ and no other singularities. Assume that:*

(i) C is contained in a linear pencil Λ whose general member is an irreducible curve with nodes at $p_1, \ldots, p_\delta$ and no other singularities.

(ii) The normalization D of C is a general nonsingular curve of genus g.

Then $g \leq 16$.

For the proof we refer to [20]. It uses special properties of elliptic surfaces.

5 Curves on surfaces of general type

We now turn to the more difficult case of surfaces of general type. Here the situation is not as clear, and we do not have general results yet. Therefore we will only state the results which seem more remarkable to us. We will state them only for nonsingular curves on minimal surfaces, but they are valid more generally for nodal curves on minimal surfaces. Our main result is the following:

Theorem 5.1 *Let Y be a minimal surface of general type. Assume that $C \subset Y$ is a nonsingular irreducible curve of genus $g \geq 3$ with general moduli. Then*

(i) *If* $\dim(|C|) \geq 2$ *and* $K_Y^2 \geq 3\chi(\mathcal{O}_Y) - 10$ *then* $g \leq 19$.

(ii) *If* $\dim(|C|) \geq 2$ *and* $K_Y^2 \geq 4\chi(\mathcal{O}_Y) - 16$ *then* $g \leq 15$.

(iii) *If* $\dim(|C|) = 1$, $|K_Y - C| = \emptyset$ *and* $K_Y^2 \geq 4\chi(\mathcal{O}_Y) - 16$ *then* $g \leq 19$.

The proof of this theorem uses again the inequality (10) combined with the inequalities satisfied by $\chi(\mathcal{O}_Y)$ and K_Y^2. A new ingredient with respect to the cases analized before is the use of the Clifford index of the curve C. For the complete proof we refer to [20]. The bound $g \leq 19$ suggests that $\overline{M}_g$ could be non-uniruled for $g \geq 20$.
An immediate but noticeable special case is the following:

Corollary 5.2 *Let Y be a minimal surface of general type such that the canonical map is birational onto its image. Suppose that $C \subset Y$ is a nonsingular curve of genus $g \geq 3$ with general moduli and such that dim$(|C|) \geq 2$. Then $g \leq 19$.*

Proof. Y satisfies $K_Y^2 \geq 3\chi(\mathcal{O}_Y) - 10$ (Castelnuovo inequality). Then we are in case (i) of Theorem 5.1. □

One case not considered by these results is that of curves on minimal surfaces Y such that $K_Y^2 < 3\chi(\mathcal{O}_Y) - 10$. This case turns out to be easy to treat because the canonical map is 2:1 onto a rational surface and one can reduce to a previous case proving that $g \leq 9$ if $C \subset Y$ has genus g and general moduli.
The cases which seem to be more difficult to handle are *complete linear pencils*. The above results in this case are still unsatisfactory and perhaps

they hide the highest difficulties. Unfortunately, as we already remarked, it is difficult to produce explicit examples of curves with general moduli, and even more difficult to find them on surfaces of general type such that they move in a non-trivial linear system.
One of the few known examples is given in [2]. The authors show that a general curve C of genus 15 can be embedded as a non-degenerate nonsingular curve in $\mathbb{P}^6$, lying in a nonsingular canonical surface $Y \subset \mathbb{P}^6$ which is a complete intersection of 4 quadrics, and such that $\dim(|C|) = 2$ on Y. Therefore, by Theorem 3.2, a general pencil in $|C|$ defines a free rational fibration. The relevant numbers are in this case:

$$d = C^2 = 9, \qquad K_Y^2 = 16, \qquad \chi(\mathcal{O}_Y) = 8$$

We find:

$$10\chi(\mathcal{O}_Y) - 2K_Y^2 = 48 > 47 = 4(g-1) - d$$

as it must be, because of the existence of a free rational fibration of genus 15.

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