Severi, Zappa, and the Characteristic System

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Abstract Two examples of obstructed curves on an algebraic surface, due to G. Zappa, are given in modern language, after a short description of the historical context of Zappa's work.

Keywords Vector bundle · Characteristic series · Obstructed curve · Hilbert scheme

1 Introduction

This is a partial report of joint work with G. Ottaviani. In Algebraic Geometry the name of Guido Zappa is associated with his discovery of an important example, namely, of a positively dimensional family of smooth curves on an algebraic surface all of whose members are singular points of the Hilbert scheme, i.e., they are *obstructed curves*. In classical language, this is expressed by saying that all members of the family have "incomplete characteristic linear system." This example, published in [24], has some peculiar, not widely known, historical features. Its publication was preceded by the paper [23], where another interesting similar example appeared. In this note, I will briefly describe the historical context of Zappa's work, focusing on the late attempts to give a geometric proof of the theorem of completeness of the characteristic linear system of a complete continuous system of curves on an algebraic surface. In the final part, I will describe both examples in detail.

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2 The Fundamental Problem

At the turn of the twentieth century, the main general problem, called "fundamental problem," in algebraic surface theory was to prove that the irregularity $h^{01}(S) := h^1(S, \mathcal{O}_S)$ of a (complex projective nonsingular) surface S is equal to $h^{10}(S) := h^0(S, \Omega^1_S)$, the dimension of the Picard variety. This theorem, called fundamental theorem, was proved by Poincaré [14] in 1910 using transcendental methods. There still remained the problem of giving a purely algebro-geometric proof of it. Severi, in 1904 [18], introduced the notion of *characteristic linear series (or system)* on the curves of a continuous system of curves. He proved that the fundamental theorem is equivalent to proving the completeness of the characteristic linear system on the curves of a sufficiently good complete continuous system. The meaning of "sufficiently good" remained vague for a long time and was clarified only at the very end of the story (see below). The completeness statement is equivalent, in modern language, to the unobstructedness of the curves of the system, i.e., to the nonsingularity of the corresponding points of the Hilbert scheme. This equivalence is not difficult to prove and very clearly explained in [11], Lecture 2.

Enriques, B. Segre, and Severi tried to prove the fundamental theorem without success for long time. The most significant highlights of the entire story are [4, 15, 19], but I will skip them, being mostly spoiled by errors and sterile controversies.[1] In this volume, C. Ciliberto gives an accurate explanation of the fundamental problem as well as historical details [3]. I also recommend the papers [1, 12], for an extended historical discussion. So I will point directly to what happened at the very end, between 1941 and 1945.

In 1941, Severi published [20] where he criticized Enriques' work on the problem, and Enriques harshly responded in 1942 in [5], published on *Commentarii Mathematici Helvetici*. Severi was informed by the editors about this paper and was given the possibility of answering with another paper [21] which appeared on the same issue of the journal.

Here Zappa, who was assistant of Severi, comes on stage. In the last mentioned paper, Severi claimed to prove that the general curve of any positive dimensional complete continuous system is unobstructed. Zappa, following the suggestion of his master to look for examples on ruled surfaces, published a paper in the same year [23] with the purpose of showing the sharpness of Severi's criterion. In fact it contains an example of a positive dimensional system of curves on a ruled surface of genus 2 whose general curve is unobstructed but containing a special obstructed curve. Shortly after, Zappa discovered his second example [24], the important one, which showed that Severi's criterion was incorrect: it consisted of a positive dimensional, everywhere obstructed, system of curves on another ruled surface of genus 2. It is likely that Severi was inspired by this example and pushed by his self-esteem: he then published another paper [22] where he succeeded in

[1] Mumford in [11], p. 7, calls them "depressing."

giving the celebrated notion of *semiregularity* (called by him *emiregolarità*) which is a sufficient condition to guarantee the completeness of the characteristic system. This is the notion studied in modern language by Kodaira and Spencer [9] and subsequently by S. Bloch [2] in higher codimension. So, finally, Severi was able to give a substantial contribution to the proof of the fundamental theorem by giving a satisfactory definition of "sufficiently good" curve.

It is interesting to observe that of the two Zappa's papers, only [24] has been quoted and acknowledged in the modern literature, but none of the examples has been explained in modern language in any published paper or book, as far as I know. More precisely:

(i) In [8], Kodaira quotes Zappa by saying that [24] contains an example of a positive dimensional complete continuous system whose general member is obstructed. But he does not give the example.
(ii) On p. 271 of [6], Grothendieck quotes [24] along the same lines as Kodaira does. Again, the example is not given explicitly.
(iii) In [11], p. 155, Mumford gives an example of an obstructed isolated curve inside an elliptic ruled surface, and he credits Severi and Zappa for it. Actually the example he gives is not given by Zappa neither in [23] nor in [24], even though his construction does not differ much from Zappa's. It is interesting that this example can be traced back to C. Segre [16] §15, where of course no mention is made of the characteristic system nor of the fundamental theorem. The same example is reproduced in my book [17] and attributed to Zappa.

3 The Examples

In this section, I will construct two classes of ruled surfaces X of genus 2 endowed with a section $C_0 \subset X$. The properties of the Hilbert scheme of X around $\{C_0\}$ in each class correspond to those of the examples appearing in [23] and in [24].

I work over $\mathbb{C}$. Let C be a projective nonsingular connected curve of genus 2, and consider a non-split exact sequence of the following form on C:

$$0 \longrightarrow \mathcal{O}_C \longrightarrow E \longrightarrow \omega_C \longrightarrow 0. \qquad (1)$$

It corresponds to a non-zero element e of

$$\mathrm{Ext}^1(\omega_C, \mathcal{O}_C) \cong H^1(C, \omega_C^{-1}) = H^0(C, \omega_C^2)^\vee.$$

Therefore, e defines a point of $\mathbb{P}H^0(C, \omega_C^2)^\vee \cong \mathbb{P}^2$, the target space of the bicanonical map of C. Note that $\varphi_{2K}(C) \subset \mathbb{P}^2$ is a conic q.

Lemma 3.1 $h^0(C, E) = 2$, *resp.* $= 1$, *according to whether* $e \in q$ *or* $e \notin q$.

Proof The multiplication map $S^2H^0(C,\omega_C) \longrightarrow H^0(C,\omega_C^2)$ is easily seen to be surjective. Therefore, the coboundary map ∂_e in (1) cannot be zero. Moreover, by [10], Lemma 2, ∂_e has rank one precisely when $e \in q$. □

We let $X = \mathbb{P}(E)$ and we denote by $\pi : X \longrightarrow C$ the projection.
Let $C_0 \subset X$ be the image of the section of π corresponding to the quotient $E \longrightarrow \omega_C$ in (1). We want to study the Hilbert scheme of X nearby the point $\{C_0\}$. We have $\mathcal{O}_{C_0}(C_0) = \omega_{C_0}$ by Sernesi [17, Cor. 4.6.3]. Thus, $C_0^2 = 2$ and

$$T_{\{C_0\}}\mathrm{Hilb}_X = H^0(C_0,\omega_{C_0})$$

has dimension 2. Moreover, by Hartshorne [7, Prop. V.2.6 p. 371], we have $\mathcal{O}_X(C_0) \cong \mathcal{O}_X(1)$ and therefore

$$\dim(|C_0|) = h^0(X,\mathcal{O}_X(C_0)) - 1 = h^0(C,E) - 1. \qquad (2)$$

A final ingredient in this analysis is the remark that all deformations of C_0 inside X are still sections of self-intersection 2 and therefore correspond to quotients of E of the form

$$0 \longrightarrow \zeta^{-1} \longrightarrow E \longrightarrow \omega_C(\zeta) \longrightarrow 0 \qquad (3)$$

for some $\zeta \in \mathrm{Pic}^0(C)$. Let's consider the two possibilities in the statement of Lemma 3.1.

First Possibility $e \in q$, i.e., $h^0(C,E) = 2$. The identity (2) implies that the linear system $|C_0|$ on X is a pencil. If $\zeta \neq \mathcal{O}_C$, then the exact sequence (3) implies that $h^0(C,E) \le 1$, contradicting our assumptions. Therefore, all quotients of E are of the form $E \longrightarrow \omega_C$, and therefore, C_0 can only deform inside $|C_0|$.

The tangent space to $|C_0|$ at $\{C_0\}$ is

$$T_{\{C_0\}}|C_0| = H^0(X,\mathcal{O}_X(C_0))/\langle C_0\rangle$$

(where $\langle - \rangle$ denotes linear span), and the characteristic map of this family

$$T_{\{C_0\}}|C_0| \longrightarrow T_{\{C_0\}}\mathrm{Hilb}_X = H^0(C_0,\omega_{C_0})$$

is induced by the restriction

$$H^0(X,\mathcal{O}_X(C_0)) \longrightarrow H^0(C_0,\omega_{C_0}).$$

Since

$$\dim[T_{\{C_0\}}|C_0|] = 1 < 2 = \dim[T_{\{C_0\}}\mathrm{Hilb}_X]$$

we see that the characteristic map has corank 1; in particular, it is not surjective. Therefore, $\{C_0\}$ is a singular point of Hilb_X, and the same clearly holds for every $\{C'\} \in |C_0|$. More precisely Hilb_X is isomorphic to a nonreduced scheme of dimension 1 supported on $|C_0| \cong \mathbb{P}^1$. So we have constructed an example having the same properties of the example constructed in [24].

Remark 3.2 A similar construction can also be made in genus $g = 1$; the output is the example given by Mumford in [11].

Second Possibility $e \notin q$, i.e., $h^0(C, E) = 1$. Arguing exactly as above, we find this time that

$$h^0(X, \mathcal{O}_X(C_0)) = h^0(C, E) = 1$$

and therefore the linear system $|C_0|$ is zero-dimensional. The exact sequences (3) are in 1–1 correspondence with the set

$$Z := \{\xi \in \mathrm{Pic}^0(C) : H^0(C, E \otimes \xi) \neq 0\}.$$

Assume for a moment that E is stable. Then, by [13], Theorem 2, Z is a curve,[2] and therefore Hilb_X is one-dimensional around $\{C_0\}$. Since $T_{\{C_0\}}\mathrm{Hilb}_X$ is two-dimensional, C_0 is obstructed. On the other hand, at a point $\xi \in Z$, the corresponding section $C_\xi \subset X$ satisfies

$$T_{\{C_\xi\}}\mathrm{Hilb}_X = H^0(C_\xi, \mathcal{O}_{C_\xi}(C_\xi)) \cong H^0(C, \omega_C \xi^2)$$

which has dimension 1 except at the finitely many points where $\xi^2 = \mathcal{O}_C$ and the dimension is 2. Therefore, Hilb_X is unobstructed at the general $\{C_\xi\}$. We then see that we have the same situation as that of the example in [23]. It remains to prove the following:

Lemma 3.3 *E is stable.*

Proof By contradiction, assume that there exists $\eta \subset E$, $\deg(\eta) \geq 1$, destabilizing E. Then we have a commutative diagram:

[2] In [13], it is assumed that $\det(E) = \mathcal{O}_C$, but one can easily reduce to the case $\det(E) = \omega_C$.

$$
\begin{tikzcd}
 & & 0 \arrow[d] & & \\
 & & \eta \arrow[d] \arrow[dr, "a"] & & \\
0 \arrow[r] & \mathcal{O}_C \arrow[r] & E \arrow[r] \arrow[d] & \omega_C \arrow[r] & 0 \\
 & & \omega_C\eta^{-1} \arrow[d] & & \\
 & & 0 & &
\end{tikzcd}
$$

where $a \neq 0$. If $\deg(\eta) = 2$, then a is an isomorphism and (1) splits, a contradiction. Therefore, $\deg(\eta) = 1$. Since $\chi(\eta) = 0$, it must be $h^0(C, \eta) = 1$, because otherwise the vertical exact sequence would imply that $h^0(C, E) = 0$, a contradiction. But since $h^0(C, E) = 1$, the above diagram shows that $\mathcal{O}_C \subset \eta \subset E$, which is clearly impossible because it implies that the torsion sheaf $\eta/\mathcal{O}_C$ is contained in ω_C. Therefore, E is stable. □

Remark 3.4 In the first case considered ($e \in q$), the subset $Z \subset \mathrm{Pic}^0(C)$ described above consists solely of the point $\{\mathcal{O}_C\}$. Remembering Theorem 2 of [13], one deduces that E is not stable. In fact, it is easy to show that E is strictly semistable and sits in an exact sequence of the form:

$$0 \longrightarrow \mathcal{O}_C(Q_1) \longrightarrow E \longrightarrow \mathcal{O}_C(Q_2) \longrightarrow 0$$

for some $Q_1 + Q_2 \in |\omega_C|$. Details will appear in a work in preparation in collaboration with G. Ottaviani, where generalizations to higher genera will also be discussed.

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