Deformations of plane curves with nodes

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1 Nodes

We consider only schemes defined over a fixed algebraically closed field $k$ of characteristic $\neq 2$.

Lemma 1.1. Let $Y \subset \mathbb{A}^2$ be a curve of equation $f(x, y) = 0$ and let $p = (\alpha, \beta) \in Y$. The following conditions are equivalent:

(i) $\left( f, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right) O_{\mathbb{A}^2, p} = (x - \alpha, y - \beta) O_{\mathbb{A}^2, p}$ (1)

(ii) $f(x + \alpha, y + \beta) = q(x, y) + \text{higher order terms}$

where $q(x, y) = a_{20}x^2 + 2a_{11}xy + a_{02}y^2$ factors as a product of distinct linear forms.

Proof. Exercise. \hfill \Box

A point $p \in Y$ satisfying the conditions of the Lemma is called a node, or an ordinary double point, or an $A_1$-singularity. A nodal plane curve is a plane curve having only nodes as singularities.

A family of affine plane curves parametrized by an affine scheme $S = \text{Spec}(R)$ (or over $S$) a morphism of the form:

$$\pi : \text{Spec} \left( R[x, y]/(f) \right) \longrightarrow S$$

for some non-constant $f \in R[x, y]$. The morphism $\pi$ is a family of curves with nodes (or a family of nodal curves) if all fibres $\mathcal{Y}(s)$ over $k$-rational points $s \in S$ are nodal curves and moreover for every node $p \in \mathcal{Y}(s)$ the morphism:

$$\text{Spec} \left[ O_{S, s}[x, y]/\left( f, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right) \right] \longrightarrow \text{Spec}(O_{S, s})$$
is étale at \( p \).

**Example 1.2.** Let \( 0 \neq c \in k \) and consider the family of plane curves \( xy + \epsilon c = 0 \) over \( D := \text{Spec}(k[\epsilon]) \), where \( k[\epsilon] = k[t]/(t^2) \). The fibre over the unique \( k \)-rational point \( (\epsilon) \in D \) is the nodal curve \( xy = 0 \). On the other hand:

\[
\left( xy + \epsilon c, \frac{\partial(xy + \epsilon c)}{\partial x}, \frac{\partial(xy + \epsilon c)}{\partial y} \right) = (xy + \epsilon c, y, x) = (\epsilon, x, y)
\]

and \( k[\epsilon, x, y]/(\epsilon, x, y) = k \) is not flat, hence not étale, over \( k[\epsilon] \) (Exercise: check this). Therefore this is not a family of nodal curves.

The constant family \( \text{Spec}(k[\epsilon, x, y]/(xy) \to D \) is a family of nodal curves because

\[
k[\epsilon, x, y]/(xy, y, x) = k[\epsilon]
\]

is étale over itself.

**Proposition 1.3.** Let \( f(\epsilon, x, y) = xy + \epsilon g(x, y) \in k[\epsilon, x, y] \). Then the following conditions are equivalent:

(a) \( f \) defines a family of nodal curves over \( D \).

(b) \( g(0, 0) = 0 \).

(c) \( f(\epsilon, x, y) = (x + \epsilon \alpha)(y + \epsilon \beta) \) for some \( \alpha, \beta \in k[x, y] \).

**Proof.**

(c) \( \Rightarrow \) (b) is obvious.

(b) \( \Rightarrow \) (c): write \( g(x, y) = \alpha y + \beta x \).

(c) \( \Rightarrow \) (a): define a \( k[\epsilon] \)-automorphism \( \phi: k[\epsilon, x, y] \to k[\epsilon, x, y] \) by

\[
\phi(x) = x - \epsilon \alpha, \quad \phi(y) = y - \epsilon \beta
\]

Then \( \phi(f) = xy \) and therefore \( \phi \) induces a \( D \)-isomorphism

\[
\text{Spec}(k[\epsilon, x, y]/(f) \cong \text{Spec}(k[\epsilon, x, y]/(xy)
\]

and (a) follows from Example 1.2.

(a) \( \Rightarrow \) (b): assume by contradiction that \( g(x, y) = c + \alpha y + \beta x \) with \( 0 \neq c \in k \). define a \( k[\epsilon] \)-automorphism \( \phi: k[\epsilon, x, y] \to k[\epsilon, x, y] \) by

\[
\phi(x) = x - \epsilon \alpha, \quad \phi(y) = y - \epsilon \beta
\]

Then \( \phi(f) = xy + \epsilon c \) and \( \phi \) induces a \( D \)-isomorphism

\[
\text{Spec}(k[\epsilon, x, y]/(f) \cong \text{Spec}(k[\epsilon, x, y]/(xy + \epsilon c)
\]

Therefore \( f \) does not define a family of nodal curves, by Example 1.2. \( \square \)
2 Families of projective plane nodal curves

Let $\Sigma_d \cong \mathbb{P}^N$, where $N = \frac{d(d+3)}{2} = \binom{d+2}{2} - 1$, be the projective space parametrizing all curves of degree $d$ in $\mathbb{P}^2$. The subset parametrizing curves having exactly $\delta$ nodes is denoted by $V_{d,\delta}$.

**Theorem 2.1** (Severi). $V_{d,\delta}$ is a locally closed subset of $\Sigma_d$ smooth of pure dimension $N - \delta$.

$V_{d,\delta}$ is called the *Severi variety* of plane curves of degree $d$ with $\delta$ nodes. The proof of this theorem consists in introducing a sub-functor $\mathbb{V}_{d,\delta}$ of the Hilbert functor of plane curves of degree $d$ and proving that this sub-functor is represented by a nonsingular locally closed subscheme $V_{d,\delta} \subset \Sigma_d$. The definition of $\mathbb{V}_{d,\delta}$ is based on the following:

**Definition 2.2.** A family of projective plane curves of degree $d$ with $\delta$ nodes is a family of projective plane curves $\mathcal{Y} \rightarrow S \times \mathbb{P}^2$ such that all fibres are curves of degree $d$ having exactly $\delta$ distinct nodes and $\pi$ is locally a family of nodal curves as defined in §1.

It is easy to show that this notion is functorial, so that we have a well defined functor:

$\mathbb{V}_{d,\delta} : \text{(schemes/} k\text{)} \rightarrow \text{(sets)}$

by setting:

$\mathbb{V}_{d,\delta}(S) = \{\text{families of } \delta\text{-nodal curves of degree } d \text{ over } S\}$

The proof that $\mathbb{V}_{d,\delta}$ is representable can be found in [1], Theorem 4.7.3 p. 257. We will admit this theorem, and we will concentrate on the local properties of $V_{d,\delta}$.
3 Local properties of $V_{d, \delta}$

Let $C \subset \mathbb{P}^2$ be a $\delta$-nodal plane curve of degree $d$ and denote by $[C] \in V_{d, \delta}$ the corresponding point. Let $\Delta = \{p_1, \ldots, p_\delta\}$ be the set of nodes of $C$. We can identify the tangent space $T_{[C]}V_{d, \delta}$ with the set of families of $\delta$-nodal curves parametrized by $\text{Spec}(k[\epsilon])$. Such families belong to $T_{[C]}\Sigma_d = H^0(C, \mathcal{O}_C(C))$. Let $F(X_0, X_1, X_2) = 0$ be an equation of $C$. An element $\overline{G} \in T_{[C]}\Sigma_d$ consists of a family of the form:

$$F + \epsilon G = 0$$

where $G = G(X_0, X_1, X_2)$ is a homogeneous polynomial of degree $d$ representing $\overline{G} \in H^0(C, \mathcal{O}_C(C))$. From Proposition 1.3 it follows that this family defines an element of $T_{[C]}V_{d, \delta}$ if and only if $G \in H^0(\mathbb{P}^2, \mathcal{I}_\Delta(d))$, or equivalently $\overline{G} \in H^0(C, \mathcal{I}_\Delta \otimes \mathcal{O}_C(d))$. Therefore:

$$T_{[C]}V_{d, \delta} = H^0(C, \mathcal{I}_\Delta \otimes \mathcal{O}_C(d))$$

A simple local calculation shows that the obstructions to the smoothness of $V_{d, \delta}$ at $[C]$ lie in $H^1(C, \mathcal{I}_\Delta \otimes \mathcal{O}_C(d))$ (Exercise).

Since $\mathcal{I}_\Delta \otimes \mathcal{O}_C(d)$ is not an invertible sheaf at the singular points we proceed as follows.

Consider the normalization $\nu : Y \to C$, and assume for simplicity that $C$ is irreducible, so that $Y$ is nonsingular connected of genus $g = (\frac{d-1}{2}) - \delta$. Let $\nu^{-1}(p_i) = x_i + y_i$, $i = 1, \ldots, \delta$. Then

$$\mathcal{I}_\Delta(d) \otimes \mathcal{O}_C = \nu_* F$$

where $F := \nu^* \mathcal{O}_C(d)(-\sum_i(x_i + y_i))$. Note that $F$ is an invertible sheaf on $Y$ of degree

$$d^2 - 2\delta = 2g - 2 + 3d$$

Therefore:

$$H^1(C, \mathcal{I}_\Delta(d) \otimes \mathcal{O}_C) = H^1(C, \nu_* F) = H^1(Y, F) = 0$$

and

$$h^0(C, \mathcal{I}_\Delta(d) \otimes \mathcal{O}_C) = h^0(Y, F) = N - \delta$$

This concludes the proof of Theorem 2.1.

**Note:** The curves defined by sections of $I_\Delta(d)$ are the so-called *adjoints* to $C$ of degree $d$. 
References