# The work of Beniamino Segre on curves and their moduli

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## 1 The context

The purpose of this article is to overview some of B. Segre's early work on algebraic curves and to outline its relevance to today's research on the subject, hoping that this can be of some interest both to historians and to researchers. The papers we will consider represent only a very small part of the impressive scientific production of B. Segre (see [36] for the list of its publications). In particular I will concentrate on few papers published at the beginning of his career, in the years 1928/30, when he served as an assistant professor of Severi in Roma. His scientific production of those few years opened the way to his appointment on the chair of *Geometria Superiore* in Bologna in 1931.

Recall that Beniamino Segre studied with Corrado Segre in Torino, where he was born in 1903. After spending one year in Paris with E. Cartan, he arrived in Roma in 1927. There he found the three greatest italian algebraic geometers, Castelnuovo, Enriques and Severi, and his research interests immediately focused on some of the hottest problems of contemporary algebraic geometry. He was only 24 years old.

## 2 Severi's Vorlesungen

In order to put Segre's work in the appropriate perspective it is necessary to review the advancement of curve theory at that time. Severi's *Vorlesungen* [46] had appeared in 1921. Many authors have already criticized and commented this well known work, so we don't need to discuss it in detail. We only want to highlight some parts relevant to Segre's work. This book contained some incomplete proofs and even some false statement, but nevertheless it indicated the state of the art on algebraic curves and their moduli. His main asserted achievement, supported by confused and unconvincing arguments, was the positive solution of *Zeuthen's Problem* asking whether, for each $r \geq 3$, every irreducible nonsingular and non-degenerate curve in $\mathbb{P}^r$ can be flatly degenerated to a *polygon* (also called a *stick figure*), i.e. to a curve which is a reduced nodal union of lines.

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This statement had been proved previously in some special cases [5] but we now know it to be false in general: a counterexample was found by Hartshorne [26]. The problem was motivated by the need of having some discrete invariants, finer than just degree and genus, that could distinguish among distinct irreducible components of the Hilbert scheme of curves. As of today such invariants have not been discovered yet. It is worth observing that the flat degeneration to a stick figure is correctly described by Severi in the case of linearly normal curves of degree $n = g + r$ in $\mathbb{P}^r$, and that there are no counterexamples known in the case of curves of degree $n$ and genus $g$ in $\mathbb{P}^r$ such that the inequality $\rho(g, r, n) \geq 0$ holds, where

$$\rho(g, r, n) := g - (r + 1)(g - n + r)$$

is the *Brill-Noether number*.

In a previous series of two notes [45] Severi had sketched his results and indicated new directions of research. Here he outlined several ideas and made remarkable conjectures. Among the topics he considered we find:

**(a)** *Riemann's existence theorem (RET) and algebraic geometry.*

He outlined ([45], n. 4) an idea, which was expanded in [46], Anhang F n. 8, for a purely algebro-geometrical statement and proof of RET. He later gave a modified argument in [47]. The lack of such a proof has been remarked by Mumford ([33], p. 15) and apparently the idea of Severi, based on a degeneration argument, still awaits for a modern critical reconsideration.

**(b)** *Rationality and unirationality properties of moduli spaces of curves.*

He formulated ([45], n. 2) the famous conjecture about the unirationality of the moduli space $M_g$ of curves of any genus $g$, and sketched a proof for genus $g \leq 10$ using families of plane curves. We will come back on his proof below, when discussing the content of [39]. The problem of unirationality of $M_g$ is mentioned as a very important and difficult one in [33], p. 37. The conjecture has been confirmed up to genus 14 [44, 13, 50] and disproved in general [25, 17, 19]. It is still unsettled for $15 \leq g \leq 21$ (even though we know that the moduli space of *stable* curves of genus 15 is rationally connected [8] and that $M_{16}$ has negative Kodaira dimension [14]).

**(c)** *Families of plane curves,* especially of nodal curves.

In [46], Anhang F, he considered the family $\mathcal{V}_{n,g}$ of plane nodal irreducible curves of given degree $n$ and geometric genus $g$ and sketched a proof of its irreducibility. His argument is incomplete (see [21] for a discussion), and the irreducibility of $\mathcal{V}_{n,g}$ has remained unsettled until 1986, when Harris proved it [24].

**(d)** *Brill-Noether theory.*

He proposed several approaches to the proof of the main statements of the so-called *Brill-Noether theory* concerning the existence and dimension of the varieties $W^r_n$ parametrizing special linear series of dimension $r$ and degree $n$ on a general curve. Such statements had been made without proof, because considered as evident by the authors, in the seminal paper [6]. Severi proposed at least two proofs of them by degeneration. The first one, less known, is sketched in [45], n. 3: it is by flatly degenerating a nonsingular curve of genus $g$ to an irreducible nodal curve of genus $g-1$, and then proceeding by induction. It has never been fixed or disproved. The second proof is outlined in [46], Anhang G, and it uses a degeneration of a nonsingular curve of genus $g$ to an irreducible $g$-nodal rational curve. These problems, and the second argument of Severi, have attracted much attention in post-Grothendieck times. The existence statement of Brill-Noether theory, which is far from trivial, has been proved independently and almost simultaneously by Kempf [28] and by Kleiman-Laksov [29]. The statement about the dimension of the varieties $W^r_n$ has remained unsettled for many years and finally proved by Griffiths-Harris [23] using a degeneration argument which follows quite closely Severi's idea.

**(e)** *Classification of maximal families of projective curves,* i.e. what today goes under the name of *Hilbert and Chow schemes.*

Severi claimed something equivalent to the statement that, whenever $n \geq g + r$ the Hilbert scheme of $\mathbb{P}^r$ has a unique irreducible component $\mathbf{I}_{g,r,n}$ whose general point parametrizes an irreducible nonsingular and non-degenerate curve of genus $g$ and degree $n$. The existence of $\mathbf{I}_{g,r,n}$ is a slightly stronger assertion that the existence statement in Brill-Noether theory (see (d) above), but it is trivial in this case. What we know today is that there is a unique component $\mathbf{I}_{g,r,n}$ of $\mathrm{Hilb}^r$ which parametrizes *general curves* (in the sense of moduli): this has been proved by Fulton and Lazarsfeld [22] using the irreducibility of $M_g$ for all $g, r, n$ such that $\rho(g, r, n) \geq 0$. The irreducibility statement of Severi has been proved in some special cases [16] but it is false in general. Harris found a series of counterexamples, i.e. of irreducible components different from $\mathbf{I}_{g,r,n}$, reported in [16]; more examples have been found by Mezzetti and Sacchiero [32]. The components found by Harris generically parametrize nonsingular trigonal curves of degree $n$ and genus $g$ which are *not linearly normal.* The same happens for the components described in [32]. No components different from $\mathbf{I}_{g,r,n}$ and generically consisting of linearly normal curves have been found yet for any value of $\rho(g, r, n) \geq 0$, so that Severi's irreducibility conjecture is still open for the extended set of values of $g, r, n$ such that $\rho(g, r, n) \geq 0$, if we interpret it as a statement about components of $\mathrm{Hilb}^r$ generically parametrizing *linearly normal* curves.

It is interesting to observe that Severi was constantly considering curves from the point of view of their moduli spaces. Brill and Noether had the same point of view when they spoke of *the general curve,* but with Severi all questions were clearly related to moduli, even though there was no definition of $M_g$ as an algebraic variety at that time. With the language of today we could say that

Severi had a *stacky* point of view, because he was looking at those properties of moduli spaces which could be detected by means of properties of families of curves, so that for him $M_g$ was just the target of the functorial morphisms induced by such families. Another notable fact is the systematic use of degeneration methods. Such methods derived from Schubert, and had already been applied by Castelnuovo [9] to enumerative problems on algebraic curves, but now Severi tried to use them to prove irreducibility statements, where very delicate monodromy-type of phenomena take place. We can say that, more than proving theorems, Severi raised questions and made conjectures, and they had a long lasting influence on curve theory.

# 3 On Riemann's existence theorem

This scenery must have been very stimulating for the young and talented Segre, and certainly Severi exherted a strong influence on him. As we will see next, Beniamino worked on moduli of curves, but he never ventured into the use of degeneration techniques: he rather relied on a solid knowledge of the geometry of plane curves and Cremona transformations.

The first paper I will discuss is [37]. Here Segre considered $d$-gonal curves of genus $g$ with $d < \frac{1}{2}g+1$ and showed that the general such curve has only finitely many linear series of degree $d$ and dimension 1 (i.e. $g^1_d$'s). This is equivalent to showing that the locus $M^1_{g,n} \subset M_g$ of such curves has the expected dimension $2g + 2d - 5$. The result was obtained by means of the construction of plane models of $d$-gonal curves. The question of counting the dimension of $M^1_{g,n}$ had been already considered in [47], but the estimate given there, even though correct, overlooked the finiteness question considered in [37]. In [3] this paper has been analyzed, commented and generalized. Arbarello and Cornalba improved Segre's result by proving that a general $d$-gonal curve as above has a *unique* $g^1_d$. Moreover, using the plane models constructed in [37], they obtained a new proof of the unirationality of $M_g$ for $g \leq 10$. It is interesting to note that Severi expressed the codimension of $M^1_{g,n}$ in $M_g$ at a point $C$ as $\dim[H^1(L^2)]$, where $L$ is the line bundle defining the (complete) $g^1_n$. Today, using cohomological methods, we understand the meaning of this estimate because we know that $H^1(L^2)$ is the conormal space of $M^1_{g,n}$ in $M_g$ at the point $C$. On the other hand the point of view of B. Segre is related with an interesting and still unsolved problem concerning the so-called *Petri loci* in $M_g$. For given $g, n$ such that $n \geq \frac{1}{2}g + 1$ one defines the Petri locus $P_{g,n} \subset M_g$ as the set of curves having a line bundle $L$ of degree $n$ such that the so-called *Petri map*

$$\mu_0(L) : \ker[H^0(L) \otimes H^0(K - L) \longrightarrow H^0(K)]$$

is not injective. This locus is not well understood yet, and the first question one can ask is whether it has pure codimension one. This would be true if a general curve $C$ in any component of $P_{g,n}$ had only finitely many $L$'s such that $\mu_0(L)$ is not injective. This is exactly what Segre's theorem says in the case $n < \frac{1}{2}g + 1$; unfortunately the problem is of a completely different nature in the case $n \geq \frac{1}{2}g + 1$, and Segre's proof does not extend. For partial results in this direction we refer to [49, 19, 20, 12, 7]. These examples show that the problems considered by Severi and Segre are still meaningful and interesting today.

# 4 On moduli of plane curves

Another interesting, but almost forgotten, paper is [39]. Here Segre consider a problem that has been suggested to him by Severi, as he says in the introduction, namely to try to extend Severi's proof of the unirationality of $M_g$ for $g \leq 10$, to higher values of the genus. Before describing Segre's paper we briefly recall the proof of Severi.

The degree $n$ of a plane model of a *general* curve of genus $g$ must satisfy the inequality

$$n \geq \frac{2}{3}g + 2$$

Taking $n$ minimal with this property and assuming that such a model $C$ has only nodes as singularities, one easily computes that when $g \leq 10$ the number $\delta$ of nodes of $C$ satisfies:

$$3\delta \leq \frac{1}{2}n(n+3) \tag{1}$$

The right hand side of this inequality is the dimension of the linear system of plane curves of degree $n$, while the left hand side is an upper bound for the number of linear conditions imposed to such curves if we want them to have nodes at given distinct points $P_1, \ldots, P_\delta$. Therefore (1) implies that there exist nodal curves of degree $n$ and genus $g$ with nodes at any *general* set of $\delta$ distinct points $P_1, \ldots, P_\delta$. Therefore, Severi concludes, the family $W$ of plane nodal curves of degree $n$ and genus $g$ is a union of linear systems parametrized by an open set of $(\mathbb{P}^2)^{(\delta)}$, the $\delta$-th symmetric product of $\mathbb{P}^2$, and therefore it is rational because it is a union of projective spaces parametrized by a rational variety: it follows that $M_g$ is unirational because it is dominated by $W$. This argument, as it stands, is not complete because some of its steps need to be justified: for example it might be a priori possible that the family of irreducible curves of degree $n$ and genus $g$ has two components, and that the one having its singular points in general position does not consist of general curves. This and other minor objections to this argument can be fixed easily, so that the proof can be made to satisfy modern standards (see [4] for a discussion).

In [39] Segre began by showing that there exists a *linear system* of plane irreducible curves of genus $g$ having general moduli, i.e. containing the general curve of genus $g$, if and only if $g \leq 6$. The existence part of his argument is elementary; nevertheless it is worth reading his elegant argument in the case $g = 5$. The proof of non-existence for $g > 10$ is based on the observation that a linear system containing the general curve must have dimension at least $3g - 3$, and therefore strictly larger than $2g + 7$. A theorem of Castelnuovo [10] (see [11] for a modern discussion) implies that a linear system of plane curves of genus $g$ and dimension larger than $2g + 7$ consists of hyperelliptic curves and therefore it cannot have general moduli if $g > 10$. The remaining cases $7 \leq g \leq 10$ are treated with a special, and non-obvious, argument.

In the second part of [39] Segre investigated whether there exist irreducible families of plane curves of genus $g > 10$, not necessarily linear systems, having general moduli and whose singularities are general points in $\mathbb{P}^2$; in other words, whether the proof of Severi could be extended by allowing the singularities of the curves of the system to be arbitrary, and not just nodes. The conclusion of Segre is that such a family $W$ cannot exist, under the following assumption. The general element $C$ of the family $W$ will be of degree $n$ and will have singularities of multiplicities $\nu_1 \geq \nu_2 \geq \cdots \geq \nu_r \geq 2$ say, at general points $P_1, \dots, P_r$; then the assumption is that the linear system $\mathcal{L}(n; \nu_1 P_1, \dots, \nu_r P_r)$ of curves of degree $n$ and having multiplicity $\geq \nu_i$ at $P_i$ is regular. Assuming this fact Segre gives an elaborated argument to show that $\nu_1 + \nu_2 + \nu_3 > n$ and from this fact the conclusion follows immediately because then the degree of the curve $C$ can be lowered by means of the quadratic transformation centered at $P_1, P_2, P_3$: it follows that the family $W$ can be replaced by an analogous one consisting of curves of lower degree and this leads to a contradiction. This interesting proof has not been rewritten in modern language yet. Moreover it must be remarked that today the regularity of the linear system $\mathcal{L}(n; \nu_1 P_1, \dots, \nu_r P_r)$ with general base points $P_1, \dots, P_r$ is known to be true only in special cases, but not for any choice of $n, \nu_1, \dots, \nu_r$; the question of regularity of such linear systems, which originates from Segre [43], has generated a great deal of research in the last few years, and many partial results are known. We refer to [15] for an overview about these problems. It follows that the truth of Segre's result is conditioned by the validity of the regularity assumption made. On the other hand, from his argument it follows that if a plane curve of genus $g \geq 11$ has general moduli, has degree $n$ and has singularities of multiplicities $\nu_1 \geq \nu_2 \geq \cdots \geq \nu_r \geq 2$ say, at general points $P_1, \dots, P_r$, then the linear system $\mathcal{L}(n; \nu_1 P_1, \dots, \nu_r P_r)$ is superabundant.

In the last part of his paper Segre drops the regularity assumption on the linear systems $\mathcal{L}(n; \nu_1 P_1, \dots, \nu_r P_r)$ and considers a similar problem. He arrives at the conclusion that *every family of curves of genus $g > 36$ having singular points in general position cannot consist of curves with general moduli.* The proof of this result is long and elaborate and it has not been reconsidered in recent times. It would deserve a critical screening.

In the paper [2], assuming a technical lemma used in [39], Arbarello gives a generalization of the last result of Segre to rational surfaces. To my knowledge this is the only relatively recent paper taking up the earlier work [39].

Also in this case the work of Segre turns out to be strongly related to problems of contemporary research. It has to do with the problem of deciding about the unirationality or uniruledness of the moduli space $M_g$ for low values of $g$, which is still unsettled when $16 \leq g \leq 21$.

# 5 On plane curves with nodes and cusps

As we have seen, B. Segre had a deep knowledge of plane algebraic curves, a subject he certainly had learned from his master C. Segre. No surprise then if he also ventured into difficult problems on families of plane curves with nodes, cusps and higher singularities. This subject was already classical, after the work of Lefschetz [31], Albanese [1] and, of course, Enriques and Severi. Let's briefly recall the set-up in modern language.

If we fix $n, \delta, \kappa$, then there is a universal, possibly empty, family (called a "maximal" family in Segre's language):

$$\begin{array}{ccc} \mathcal{C} & \hookrightarrow & V_{n,\delta,\kappa} \times \mathbb{P}^2 \\ \downarrow & & \\ V_{n,\delta,\kappa} & & \end{array}$$

parametrizing all plane curves of degree $n$ with $\delta$ nodes, $\kappa$ cusps and no other singularities. Here $V_{n,\delta,\kappa} \subset \mathbb{P}^{n(n+3)/2}$ is a locally closed subscheme. The existence of $V_{n,\delta,\kappa}$ has been proved in modern standards by Wahl [51]. One of the main tools available at that time, and still now, in the study of $V_{n,\delta,\kappa}$ is the so-called "characteristic linear series" which can be introduced as follows. Given an irreducible $[C] \in V_{n,\delta,\kappa}$ one considers the linear system of curves of degree $n$ which is the projective tangent space to $V_{n,\delta,\kappa}$ at $[C]$: then the induced linear series on the normalization $\widetilde{C}$ of $C$ is the *characteristic linear series* of $V_{n,\delta,\kappa}$ at $[C]$. With a suggestive terminology the classical geometers said that the characteristic linear series is cut on $\widetilde{C}$ by the curves of the family $V_{n,\delta,\kappa}$ which are "infinitely near" $C$. This series turns out to be defined by sections of the sheaf $\mathcal{O}_{\widetilde{C}}(K + 3H - p_1 - \cdots - p_\kappa)$, where $p_1, \ldots, p_\kappa \in \widetilde{C}$ are the inverse images of the cusps and $H$ is the pullback of a line section under the normalization morphism $\widetilde{C} \longrightarrow C$; therefore the Zariski tangent space $T_{[C]}V_{n,\delta,\kappa}$ is naturally a subspace of $H^0(\widetilde{C}, \mathcal{O}_{\widetilde{C}}(K + 3H - p_1 - \cdots - p_\kappa))$. On the other hand a standard count of constants shows that $V_{n,\delta,\kappa}$ has dimension at least

$$3n+g-1-\kappa = h^0(\widetilde{C}, \mathcal{O}_{\widetilde{C}}(K+3H-p_1-\cdots-p_\kappa)) - h^1(\widetilde{C}, \mathcal{O}_{\widetilde{C}}(K+3H-p_1-\cdots-p_\kappa))$$

at $[C]$, where $h^i(-)$ means $\dim[H^i(-)]$ and

$$g = \frac{(n-1)(n-2)}{2} - \delta - \kappa$$

is the geometric genus of $C$, i.e. the genus of $\widetilde{C}$. The lower bound $3n + g - 1 - \kappa$ is called the *virtual dimension* of $V_{n,\delta,\kappa}$. The above facts imply:

(i) If
$$H^1(\widetilde{C}, \mathcal{O}_{\widetilde{C}}(K + 3H - p_1 - \cdots - p_\kappa)) = 0 \tag{2}$$
then $V_{n,\delta,\kappa}$ is nonsingular at $[C]$ of dimension equal to the virtual dimension. We then say that $V_{n,\delta,\kappa}$ is *regular at* $[C]$. If this happens at all $[C]$

in an irreducible component $W \subset V_{n,\delta,\kappa}$ (resp. in $V_{n,\delta,\kappa}$) we say that $W$ (resp. $V_{n,\delta,\kappa}$) is *regular*.

(ii) If $\kappa < 3n$ then (2) holds and therefore $V_{n,\delta,\kappa}$ is regular.

The main problems concerning the varieties $V_{n,\delta,\kappa}$ are:

(a) Establish for which values of $n, \delta, \kappa$ we have $V_{n,\delta,\kappa} \neq \emptyset$.

(b) prove or disprove irreducibility. In the negative case prove or disprove equidimensionality.

(c) Prove or disprove the existence of non-regular components.

(d) Investigate the existence of singular points of $V_{n,\delta,\kappa}$, and give examples.

All these problems (except irreducibility, as we saw before) had been completely solved by Severi [46] in the case $\kappa = 0$ (no cusps), but they were widely open in the presence of cusps.

In [40], after introducing some terminology, Segre gave examples of non-regular, but everywhere nonsingular, components of $V_{n,\delta,\kappa}$ for infinitely many values of $n, \delta, \kappa$. He then went on by giving, in [41], examples of reducible $V_{n,\delta,\kappa}$ both in distinct regular components and in components of different dimensions. The simplest of these examples are sextics with 6 cusps: $V(6, 0, 6)$ consists of two components, both regular hence of dimension 15, distinguished by the condition that the cusps do/do not belong to a conic; in fact the sextics with 6 cusps on a conic appear as branch curves of general projections of cubic surfaces. With these examples Segre took care of problems (b) and (c). His examples are beautifully simple, and are essentially the only one known. They have been quoted by Zariski in [52], p. 220 and 223. For a modern treatment we refer to [48].

In the second part of [40] Segre suggested a procedure to construct new components starting from given ones, in the attempt to solve problem (a). He claimed, at the end, the non-emptyness of $V(2n, 0, n^2)$, for all $n \geq 3$ and of $V(2n+1, 0, n^2+n-1)$ for all $n \geq 1$. These claims are not clearly explained, the inductive procedure he proposes involves smoothing arguments of non-reduced curves, and the all construction needs a careful inspection. Certainly his conclusions, as they stand, are incorrect. In fact Zariski ([52], p. 222) showed the non-existence of curves of degree 7 with 11 cusps, even though they appear in the above list of Segre. Nevertheless Segre's procedure seems to be correct when he constructs new *regular* components by deforming cusps to nodes and smoothing nodes in regular components ([40], n. 7). This idea still awaits for a modern treatment. Note that the existence problem (a) is not completely solved yet. For recent partial results see [27, 30].

Recall that branch curves of multiple planes are curves with nodes and cusps, and not only nodal. This fact introduces a whole variety of cases which are partly responsible for the many possibilities that can occur in problems (a), (b), (c), as opposed to the case of families of curves with only nodes, that are always irreducible and regular. Segre took up this point of view in [42], where he characterized those plane curves which are branch curves of general projections of nonsingular surfaces of $\mathbb{P}^3$. His treatment of this problem is entirely algebro-geometrical and extremely elegant, and the characterization is given in terms of degrees of curves containing the singular points.

Problem (d) has remained out of reach for the classical geometers, essentially due to lack of technique. Only with the help of scheme theory the study of families has reached a level of sophistication sufficient to understand such questions. The first example of a singularity of a family $V(n, \delta, \kappa)$ or, in modern terminology, of an *obstructed* curve, has been given by Wahl [51]: it is a curve of degree 104 with 3636 nodes and 900 cusps.

Finally I would like to mention the interesting Note [38], which has been completely forgotten so far, where Segre showed that the characteristic linear series of a family of curve with an ordinary tacnode is always incomplete. He raised the question of giving a geometrical interpretation of the defect of completeness of this series. It is a question which deserves to be studied.

The all subject of families of plane curves with nodes and cusps is discussed at length in ch. VIII of [52], to which we refer the reader for other classical references.

# 6 Final considerations

From the above discussion it should hopefully emerge how much Segre's work was inspired, if not guided, by the scientific figure of Severi. At the same time, despite his young age and the presence near him of such an influential personality, his independent creativity emerged quite strongly. We have seen this in the way he went beyond Severi's [47] in computing the dimension of $M^1_{g,n} \subset M_g$ in [37], or in the way he explored moduli of plane curves using his technique that, we can guess, came to him directly from C. Segre. The papers on plane curves with nodes and cusps reflect perhaps another influence, coming from Castelnuovo and especially Enriques, even though I was not able to find direct confirmation about any scientific relation of B. Segre with them. We should also note that everything he did on algebraic curves at the time was restricted to plane curves, a very classical and safe point of view. He did not venture into the geometry of curves in higher projective spaces, and in this respect he did not take up the spirit of [46]. Nevertheless his work is far from being forgotten today, since the problems he considered were deep and difficult and, in fact, still unsolved or related with important open problems. In closing I want to observe that in the 1920's there has been another important development in curve theory, represented by the nowadays well known papers by Petri [34, 35], who was the last student of M. Noether. It is interesting to see how distant the points of view of Severi/Segre and of Petri were. The italians insisted in carrying on a purely synthetic point of view, no algebra being allowed to contaminate their world. The work of Petri was instead bringing the equations to the center of attention, mixing up projective geometry with elegant methods of an algebraic-homological nature, reflecting traditions and developments going back to the german and british schools of algebra and invariant theory. It seems sterile to speculate about the superiority of either point of view, but it cannot be left unnoticed the impression of isolation coming from the headquarters of italian algebraic geometry at that time.

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