PLANE QUARTICS

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Contents

1. Introduction

These notes are an exposition of some results about plane quartic curves that have been treated during the workshop in Catania. We explain three classical ways to costruct plane quartics, namely by means of the Geiser involutions associated to nets of plane cubics, by nets of quadrics and by projecting cubic surfaces. We only included the projective geometric aspects, while the algebraic ones have been deliberately omitted. A much more detailed treatment can be found in Dolgachev's book [16]. The bibliography includes more than needed, and it can be useful for further reading.

2. Principal parts

We work over $\mathbb{C}$. Let X be a projective nonsingular variety and $\Delta \subset X\times X$ the diagonal. Denote by $p_i : X \times X \to X$ the i-th projection. For a given $k \geq 1$ let $k\Delta \subset X \times X$ be the subscheme defined by the ideal sheaf $\mathcal{I}_\Delta^k \subset \mathcal{O}_{X\times X}$.

If $L \in \mathrm{Pic}(X)$ we define the *sheaf of* $(k-1)$*-st principal parts* of L as:

$$\mathcal{P}^{k-1}(L) := p_{1*}(p_2^*L \otimes \mathcal{O}_{k\Delta})$$

The sequence on $X \times X$:

$$0 \longrightarrow \mathcal{I}_\Delta^k \longrightarrow \mathcal{O}_{X\times X} \longrightarrow \mathcal{O}_{k\Delta} \longrightarrow 0$$

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induces an exact sequence on X:

$$0 \longrightarrow M_k(L) \longrightarrow H^0(L)\otimes\mathcal{O}_X \longrightarrow \mathcal{P}^{k-1}(L) \tag{1}$$

where we have denoted

$$M_k(L) := p_{1*}(p_2^*L\otimes\mathcal{I}_\Delta^k)$$

In this section we will be interested in the cases $k=1,2$.

We have $\mathcal{P}^0(L)=L$ and if L is globally generated with $h^0(L)=r+1$ then

$$M_1(L)=\varphi_L^*\Omega^1_{\mathbb{P}^r}\otimes L$$

The above sequence is:

$$0 \longrightarrow \varphi_L^*\Omega^1_{\mathbb{P}^r}\otimes L \longrightarrow H^0(L)\otimes\mathcal{O}_X \longrightarrow L \longrightarrow 0 \tag{2}$$

and coincides with the pullback of the Euler sequence under $\varphi_L: X\to\mathbb{P}^r$.

If $k=2$ and L is very ample then the sequence (1) is also exact on the right:

$$0 \longrightarrow M_2(L) \longrightarrow H^0(L)\otimes\mathcal{O}_X \longrightarrow \mathcal{P}^1(L) \longrightarrow 0$$

Note that we have a surjective restriction $\mathcal{P}^k(L)\to\mathcal{P}^{k-1}(L)$ for all k. In particular for $k=1$ we obtain the following exact sequence:

$$0 \longrightarrow \Omega^1_X\otimes L \longrightarrow \mathcal{P}^1(L) \longrightarrow L \longrightarrow 0 \tag{3}$$

Assuming that L is globally generated and combining (2) and (3) we obtain the following exact and commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & \varphi_L^*\Omega^1_{\mathbb{P}^r}\otimes L \arrow[r] \arrow[d, "\delta"'] & H^0(L)\otimes\mathcal{O}_X \arrow[r] \arrow[d, "e"] & L \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \Omega^1_X\otimes L \arrow[r] & \mathcal{P}^1(L) \arrow[r] & L \arrow[r] & 0
\end{tikzcd}$$

where e is the map associating to a section $\sigma\in H^0(L)$ its principal part and δ is the codifferential $d\varphi_L^\vee:\varphi_L^*\Omega^1_{\mathbb{P}^r}\to\Omega^1_X$ tensored by 1_L. We can now observe that:

- $\ker(\delta)=\ker(e)=M_2(L)$. If φ is an embedding then $M_2(L)=\mathcal{N}^\vee\otimes L$ where $\mathcal{N}^\vee$ is the conormal bundle of $\varphi_L(X)\subset\mathbb{P}^r$.
- the degeneracy schemes of δ and of e are the same, and both coincide with the degeneracy scheme of $d\varphi_L^\vee$, called *critical scheme* or *ramification scheme* of φ_L. Let's denote by $R(\varphi_L)$, or simply by R, this closed subscheme of X.

The scheme R is locally defined by the maximal minors of a matrix defining δ or of a matrix defining e. These two ways of defining R have different meaning a priori. In fact δ degenerates where the differential $d\varphi_L$ does not have maximal rank. On the other hand the degeneration of e is related with the singularities of the members of the linear system $|L|$. In order to clarify this point we discuss some special cases.

Example 2.1. Assume that $\dim(X) = n$ and that L is globally generated and $h^0(L) = n+1$. Then $\varphi_L : X \to \mathbb{P}^n$ and R is an effective divisor in the linear system $|\omega_X L^{n+1}|$. In the special case when X is a curve of genus g and L a globally generated invertible sheaf of degree d then $\varphi_L : X \to \mathbb{P}^1$ and R is defined by a section of $\omega_X L^2$. Therefore it has degree

$$\deg(R) = 2(d + g - 1)$$

This is *Hurwitz formula.*

Example 2.2. Let X be a surface and L globally generated and such that $h^0(L) = 2$. Then $\varphi_L : X \to \mathbb{P}^1$. If all fibres of φ are reduced then R is 0-dimensional and, since $L^2 = 0$:

$$\begin{aligned}\deg(R) = c_2[\Omega^1_X \otimes \varphi^*\omega^{-1}_{\mathbb{P}^1}] &= c_2(\Omega^1_X(2L)) \\ &= c_2(X) + 2(L \cdot \omega_X) \\ &= c_2(X) + 4(g-1)\end{aligned}$$

where g is the genus of the general fibre of φ_L. Interpreted via the map e this is the total number of singularities of the fibres of φ_L, counted with multiplicities.

A special important case is given by a K3 surface X with a pencil $|E|$ of elliptic curves. In this case we obtain $\deg(R) = c_2(X) = 24$.

More generally, consider an invertible sheaf L on the surface X and a pencil $|V| \subset |L|$ defined by a 2-dimensional vector space $V \subset H^0(L)$. Assume that all curves in $|V|$ are reduced. We can count the total number $s(L)$ of singularities in the members of $|V|$ as follows. We blow-up the base locus of $|V|$ and we obtain $\sigma : S \to X$ with exceptional divisor E. The pencil $|V|$ corresponds on S to the base-point free pencil defined by the invertible sheaf $M := \sigma^*L(-E)$ and all we have to do is to apply the previous formula to φ_M.

We have $\omega_S = \sigma^*\omega_X(E)$, $E^2 = -L^2$ and $c_2(S) = c_2(X) + L^2$ by Noether's formula; therefore:

$$\begin{aligned}s(L) = \deg(R(\varphi_M)) &= c_2(S) + 2(M \cdot \omega_S) \\ &= c_2(X) + 2(L \cdot \omega_X) + 3L^2\end{aligned} \tag{4}$$

For example let $X = \mathbb{P}^2$ and $L = \mathcal{O}(d)$. Then the above formula counts the number $s(d)$ of nodal curves in a general pencil of plane curves of degree d giving, since $c_2(\mathbb{P}^2) = 3$:

$$s(d) = 3(d-1)^2 \tag{5}$$

This is the degree of the *discriminant hypersurface* in $\mathbb{P}H^0\mathcal{O}(d)$, i.e. of the locus of singular curves of degree d. The formula is of course classical, see [16].

It is instructive and easier to deduce formula (5) by computing the degeneration scheme of e instead. If the pencil is given by $\lambda F + \mu G = 0$ then

$e : \mathcal{O}^2 \to \mathcal{O}(d-1)^3$ is given by the transposed of the matrix:

$$\begin{pmatrix} \frac{\partial F}{\partial X_0} & \frac{\partial F}{\partial X_1} & \frac{\partial F}{\partial X_2} \\ \frac{\partial G}{\partial X_0} & \frac{\partial G}{\partial X_1} & \frac{\partial G}{\partial X_2} \end{pmatrix}$$

and the locus of 2×2 minors has degree $3(d-1)^2$.

Example 2.3. Consider now a base-point free net of divisors on a threefold X, given by a globally generated invertible sheaf L such that $h^0(L) = 3$. Let $\varphi = \varphi_L : X \to \mathbb{P}^2$ be the corresponding morphism. Then $R(\varphi)$ is the locus of singular points of elements of the net. The virtual class of $R(\varphi)$ is given by

$$[c(\Omega^1_X)(\varphi^* c(\Omega^1_{\mathbb{P}^2}))^{-1}]_2$$

(Porteous formula, see e.g. [29], p. 29). We have: $\varphi^* c(\Omega^1_{\mathbb{P}^2}) = 1 - 3L + 3L^2$ and therefore:

$$(\varphi^* c(\Omega^1_{\mathbb{P}^2}))^{-1} = 1 + 3L + 6L^2 + \cdots$$

and

$$\begin{aligned} c(\Omega^1_X)(\varphi^* c(\Omega^1_{\mathbb{P}^2}))^{-1} &= (1 - c_1(X) + c_2(X) + \cdots)(1 + 3L + 6L^2 + \cdots) \\ &= 1 + (3L - c_1(X)) + (c_2(X) - c_1(X) \cdot 3L + 6L^2) + \cdots \end{aligned}$$

Hence the class of $R(\varphi)$ is $c_2(X) - 3c_1(X) \cdot L + 6L^2$.

Suppose now that we have a net of divisors in X defined by a 3-dimensional vector subspace $V \subset H^0(X, L)$ whose base locus B is zero-dimensional and of degree L^3. Let $\varphi_V : X \dashrightarrow \mathbb{P}^3$ be the rational map defined by V. If we want to compute the class of the locus of singular points of elements of the net we have to resolve the rational map φ_V and then apply the previous formula. Let $\sigma : S \to X$ be the blow-up of B; denote by $E = \sigma^{-1}(B)$. Then we have ([21], Ex. 15.4.2(c), p. 301):

$$c(\Omega^1_S) = 1 - \sigma^* c_1(X) + 2E + \sigma^* c_2(X) + \cdots$$

Moreover the resolved morphism is $\varphi_M : S \to \mathbb{P}^2$ where $M = \sigma^* L(-E)$. Therefore

$$(\varphi_M^* c(\Omega^1_{\mathbb{P}^2}))^{-1} = 1 + 3\sigma^* L - 3E + 6\sigma^* L^2 + 6E^2$$

Hence the class of $R(\varphi_M)$ is $\sigma^*(c_2(X) - 3c_1(X) \cdot L + 6L^2)$.

Let's apply this formula to a general net of surfaces of degree d in $\mathbb{P}^3$. In this case $L = dH$, $c_2(\mathbb{P}^3) = 6H^2$, $c_1(\mathbb{P}^3) = 4H$, where $H = \mathcal{O}(1)$. Then the degree of R is

$$6 - 12d + 6d^2 = 6(d-1)^2$$

For $d = 2$ the curve R is called the *Steiner curve* of the net of quadrics (see §6), and the formula says that it has degree 6. Arguing as in the previous

Example 2.2, we can deduce that the class of R is given by the maximal minors of the matrix:

$$\begin{pmatrix} \frac{\partial F_0}{\partial X_0} & \frac{\partial F_0}{\partial X_1} & \frac{\partial F_0}{\partial X_2} & \frac{\partial F_0}{\partial X_3} \\ \frac{\partial F_1}{\partial X_0} & \frac{\partial F_1}{\partial X_1} & \frac{\partial F_1}{\partial X_2} & \frac{\partial F_1}{\partial X_3} \\ \frac{\partial F_2}{\partial X_0} & \frac{\partial F_2}{\partial X_1} & \frac{\partial F_2}{\partial X_2} & \frac{\partial F_2}{\partial X_3} \end{pmatrix}$$

where F_0, F_1, F_2 are surfaces of degree d generating the net.

3. Del Pezzo surfaces of degree two and plane quartics

References for this section are [4], vol. 6 p. 122, and [18].
Let V be a $\mathbb{C}$-vector space of dimension 3, and $\mathbb{P}V$ the projective plane of subspaces of V of dimension one. For simplicity we will replace $\mathbb{P}V$ by $\mathbb{P}^2$ assuming that a basis of V has been chosen. But whenever necessary we will come back to $\mathbb{P}V$. Let $Z = \{P_1, \ldots, P_7\} \subset \mathbb{P}^2$ be a subscheme consisting of seven distinct point no three of which are on a line, no six on a conic. Such a Z is called a *regular 7-tuple.*

Let $\sigma : S \longrightarrow \mathbb{P}^2$ be the blow-up centered at Z, $e_i = \sigma^{-1}(P_i)$, $i = 1, \ldots, 7$, $h = \sigma^*\mathcal{O}(1)$. Then $K_S = -3h + \sum e_i$ and therefore $K_S^2 = 2$ and $h^0(S, -K_S) = h^0(\mathbb{P}^2, \mathcal{I}_Z(3)) = 3$. Moreover $|-K_S|$ is base-point free and therefore $\varphi := \varphi_{-K_S}$ is a morphism of degree two:

$$\begin{tikzcd} S \arrow[r, "\varphi"] \arrow[d, "\sigma"'] & {|-K_S|^\vee \cong \mathbb{P}^2} \\ \mathbb{P}^2 & \end{tikzcd}$$

S is a *Del Pezzo surface of degree 2.* The class of the ramification divisor of φ is

$$R = R(\varphi) \sim K_S + 3(-K_S) = -2K_S = 6h - 2\sum e_i$$

Therefore $\Sigma := \sigma(R) \subset \mathbb{P}^2$ is a sextic singular at Z. The curve R is mapped by φ isomorphically to the branch curve $B := \varphi(R) \subset \mathbb{P}^2$. We have:

$$\deg(B) = -K_S \cdot R = 4$$

i.e. B is a quartic. It is nonsingular because S is nonsingular and therefore it has genus 3.

The composition

$$\gamma_Z := \varphi \cdot \sigma^{-1} : \mathbb{P}V \dashrightarrow |-K_S|^\vee$$

is a rational map of degree two from a plane to another plane, classically known under the name of *Geiser involution*, which is defined by the net of cubics $|H^0(\mathbb{P}^2, \mathcal{I}_Z(3))|$. If $P \in \mathbb{P}V \setminus Z$ then $\gamma_Z(P) = \gamma_Z(P')$, where P' is the 9-th base point of the pencil of cubics through $Z \cup \{P\}$.

Definition 3.1. The quartic curve B is called the *branch curve of the Geiser involution* γ_Z and is denoted by $B(Z)$ whenever we want to emphasize its dependence from Z.

According to what we have seen in §2, if we choose a basis $\{F_0, F_1, F_2\}$ of the net the ramification curve Σ has equation:

$$\begin{vmatrix} \frac{\partial F_0}{\partial X_0} & \frac{\partial F_0}{\partial X_1} & \frac{\partial F_0}{\partial X_2} \\ \frac{\partial F_1}{\partial X_0} & \frac{\partial F_1}{\partial X_1} & \frac{\partial F_1}{\partial X_2} \\ \frac{\partial F_2}{\partial X_0} & \frac{\partial F_2}{\partial X_1} & \frac{\partial F_2}{\partial X_2} \end{vmatrix} = 0$$

Digression: The flexes of a general plane quartic C are the intersections of C with its hessian, which is a sextic. Therefore C has 24 flexes. The dual $C^\vee$ of C has degree 12, equal to the intersection number of C with the first polar from a general point. Therefore, since $C^\vee$ has arithmetic genus 55 and geometric genus 3, and it has 24 ordinary cusps coming from the flexes of C, it must also have 28 nodes, coming from the bitangents of C. Therefore *a general plane quartic C has 24 flexes and 28 bitangents.*

Proposition 3.2. *Every (-1)-curve E on S is mapped by φ to a line bitangent to B. Conversely every line $\ell \subset |-K_S|^\vee$ bitangent to B satisfies $\varphi^{-1}(\ell) = E_1 \cup E_2$ where $E_1, E_2 \subset S$ are (-1)-curves such that $E_1 \cdot E_2 = 2$.*

Proof. If $E \subset S$ is a (-1)-curve then also $E' = -K_S - E$ is a (-1)-curve and $\varphi(E) = \varphi(E')$. Moreover $\ell := \varphi(E) = \varphi(E') = \varphi(E + E')$ is a line which is bitangent to B because $E \cdot E' = 2$.

Conversely, if $\ell \subset |-K_S|^\vee$ is bitangent to B then $\varphi^{-1}(\ell) = E + E'$ and

$$2 = \varphi^{-1}(\ell)^2 = E^2 + 2E \cdot E' + E'^2 = E^2 + 4 + E'^2$$

and this implies $E^2 = E'^2 = -1$. □

Corollary 3.3. *S contains exactly 56 (-1)-curves. They are:*

$$\begin{aligned} &e_1, \dots, e_7, -K_S + e_1, \cdots, -K_S + e_7 \\ &h - e_i - e_j, \quad 1 \le i < j \le 7 \\ &2h - \sum_{k \ne i,j} e_k, \quad 1 \le i < j \le 7 \end{aligned}$$

□

The Geiser involution can be also treated involving the cubic surface defined by the linear system of cubics through 6 points in the plane. We pospone this point of view to a forthcoming section.

Remark 3.4. Seven general points of $\mathbb{P}^2$ can be also obtained as the zero locus of a general section of $T_{\mathbb{P}^2}(1)$. We have $c_2(T_{\mathbb{P}^2}(1)) = 7$ and $h^0(T_{\mathbb{P}^2}(1)) = 15$. Therefore an open subset of $\mathbb{P}(H^0(T_{\mathbb{P}^2}(1)))$ can be taken as a parameter space of general 7-tuples of points in $\mathbb{P}^2$. See [8] for a discussion from the point of view of representation theory.

The following are degenerate cases of Geiser involutions, for special configurations of 7-tuples Z.

Example 3.5. B has a double point when 3 points of Z are on a line or 6 are on a conic. B consists of two conics when 6 of the seven points are the vertices of a complete quadrilateral.

Example 3.6. Consider the Geiser involution determined by the point $(1,1,1)$, and by the coordinate points $(1,0,0),(0,1,0),(0,0,1)$ each fat of length 2 along the line joining it with $(1,1,1)$. Then the net of cubics through this set Z is generated by

$$(x-y)z^2, (x-z)y^2, (y-z)x^2$$

where x,y,z stand for homogeneous coordinates, and the jacobian sextic Σ is :

$$\begin{vmatrix} z^2 & y^2 & 2x(y-z) \\ -z^2 & 2y(x-z) & x^2 \\ 2z(x-y) & -y^2 & -x^2 \end{vmatrix} = -6xyz(x-y)(x-z)(y-z)$$

It is reducible into six lines. The branch quartic B is reducible in four lines. (Computed using WolframAlpha).

Example 3.7. Consider a set of distinct points $Z = \{O, P_1, \ldots, P_6\}$ with $P_1, \ldots, P_6$ on a conic θ and $O \notin \theta$. Then the linear system $\Lambda_Z := |H^0(\mathbb{P}^2, \mathcal{I}_Z(3))|$ has the property that for each line $\ell \ni O$ the cubic $\ell \cup \theta$ is in Λ_Z. Therefore Λ_Z induces a g^1_2 on each ℓ. This implies that there is a curve F consisting of the pairs of tangency points of the cubics of Λ_Z with the lines ℓ containing O. This curve is a quartic with a double point at O and, conversely, every quartic with a double point can be so constructed in a unique way. This is explained in [5], p. 279. It would be interesting to relate the curve F with the quartic branch curve B of the Geiser involution determined by Λ_Z, which is a nodal quartic as well.

4. Theta characteristics

If C be a nonsingular projective curve of genus g a *theta characteristic* on C is a $\theta \in \mathrm{Pic}^{g-1}(C)$ such that $2\theta = K_C$. The theta characteristic is called *even* (resp. *odd*) if $h^0(C,\theta)$ is even (resp. odd). C has exactly 2^{2g} theta characteristics, as many as the 2-torsion points of $\mathrm{Pic}^0(C)$, of which $2^{g-1}(2^g-1)$ are odd and $2^{g-1}(2^g+1)$ are even.

If $g=3$ then C has 28 odd theta characteristics and 36 even ones. If C is non-hyperelliptic then the odd thetas are the line bundles of the form $\mathcal{O}(p+q)$ where $2p+2q = \ell \cdot C$ as ℓ varies among the bitangents of the quartic plane canonical model of C. The 36 even thetas are more difficult to describe, as we will see, because they satisfy $h^0(C,\theta)=0$ and therefore they are not effective.

Suppose now that C is a nonsingular plane quartic.

Definition 4.1. An unordered set $\{\theta_1, \ldots, \theta_7\}$ of seven distinct bitangents to C is called an *Aronhold system* if the six points of contact with C of any three among them is not contained in a conic. In such a case the seven odd theta characteristics corresponding to $\theta_1, \dots, \theta_7$ are also called an Aronhold system.

Observe that the condition of the definition means that $|2K - \theta_i - \theta_j - \theta_k| = \emptyset$ for all $1 \le i < j < k \le 7$. Replacing $2K_C = 2\theta_i + 2\theta_j$ we can rephrase the condition of the definition by saying that $\theta_i + \theta_j - \theta_k$ is an even theta characteristic for all $1 \le i < j < k \le 7$. Therefore we have a second equivalent formulation of the definition.

Definition 4.2. A 7-tuple of distinct odd theta characteristics $\{\theta_1, \dots, \theta_7\}$ is an *Aronhold system* if $\theta_i + \theta_j - \theta_k$ is an even theta characteristic for all $1 \le i < j < k \le 7$.

We shall prove later that there are precisely 288 Aronhold systems on C. Among them 72 contain a given odd theta characteristic, and 16 contain two given odd theta characteristics.

5. Nets of quadrics

Let W be $\mathbb{C}$-vector space of dimension 4, and consider the projective space $\mathbb{P}W \cong \mathbb{P}^3$ of 1-dimensional subspaces of W. Let $\Lambda = \langle Q_1, Q_2, Q_3 \rangle$ be a net of quadrics such that its base locus consists of 8 distinct points:

$$X := \{x_1, \dots, x_8\} = Q_1 \cap Q_2 \cap Q_3$$

An octad of points of this type is called a *Cayley octad*. We will denote by Λ_X the net of quadrics whose base locus is the octad X.

Definition 5.1. A Cayley octad X is called *regular*, and the corresponding net Λ_X is also called regular, if Λ_X does not contain quadrics of rank ≤ 2.

Since the locus of quadrics of rank ≤ 2 in $\mathbb{P}H^0(\mathbb{P}^3, \mathcal{O}(2)) \cong \mathbb{P}^9$ has dimension 6 the general net of quadrics is regular.

Proposition 5.2. *A Cayley octad X is regular if and only if its points are 3 by 3 not on a line, 4 by 4 not on a plane, and none of them is vertex of a singular quadric of Λ_X.*

Proof. If x_i, x_j, x_k are on a line ℓ then ℓ is contained in the base locus of Λ_X, a contradiction.

If $\langle x_i, x_j, x_k, x_h \rangle =: \Pi$ is a plane then the restriction map:

$$H^0(\mathbb{P}^3, \mathcal{I}_X(2)) \longrightarrow H^0(\Pi, \mathcal{I}_{\{x_i, x_j, x_k, x_h\}/\Pi}(2))$$

has rank two. Therefore there is a quadric $Q \in \Lambda_X$ containing Π, and Q has rank ≤ 2, a contradiction.

If say x_1 is vertex of a cone $Q \in \Lambda_X$ then $\Lambda_X = \langle Q, Q_2, Q_3 \rangle$ for some $Q_2, Q_3 \in \Lambda_X$. But then X is non-reduced at x_1, and this contradicts the hypothesis that $x_1, \dots, x_8$ are distinct. □

Denote by $\Delta \subset \mathbb{P}H^0(\mathbb{P}^3, \mathcal{O}(2))$ the *discriminant hypersurface*, namely the locus of singular quadrics; it has degree four. Consider a regular net of quadrics $\Lambda_X \cong \mathbb{P}^2$, where $X = \{x_1, \dots, x_8\}$. The curve $C := \Lambda_X \cap \Delta$ is a quartic which is called the *Hesse quartic* of the net Λ_X, or of X.

Lemma 5.3. *C is nonsingular.*

Proof. Assume that $Q \in C$ is singular; then Q has rank three. Let $v(Q) \in \mathbb{P}^3$ be its vertex. Since Q is a nonsingular point of Δ, the fact that C is singular at Q means that $\Lambda_X \subset T_Q\Delta = |H^0(\mathcal{I}_{v(Q)}(2))|$, and therefore $v(Q) \in X$, contradicting Proposition 5.2. □

The lines of Λ_X are the pencils of quadrics contained in Λ_X. Which ones are the bitangents to C? Here is the answer.

Proposition 5.4. *Let X be a regular Cayley octad. For each $1 \leq i < j \leq 8$ the quadrics of Λ_X containing the line $L_{ij} := \langle x_i, x_j \rangle \subset \mathbb{P}^3$ form a pencil ℓ_{ij}. The 28 lines $\ell_{ij} \subset \Lambda_X$ are the bitangent of C.*

Proof. Let $\ell \subset \Lambda_X$ be bitangent to C at Q_1, Q_2. Then $\ell = \langle Q_1, Q_2 \rangle$ and every $Q \in \ell$ contains $v_1 = v(Q_1)$ and $v_2 = v(Q_2)$. But then $E := Q_1 \cap Q_2$ is singular at v_1 and v_2, and therefore it is reducible as $E = L \cup \Gamma$, where Γ is a twisted cubic and $L = \langle v_1, v_2 \rangle$. Moreover $X \cup \{v_1, v_2\} \subset E$. Since $|H^0(\mathcal{I}_\Gamma(2))|$ is a net of quadrics that cannot coincide with Λ_X it follows that $X \not\subset \Gamma$. More precisely at most 6 points of X are in Γ because any 7 of them impose independent conditions to quadrics. Therefore $L = L_{ij}$ for some i, j. □

Example 5.5. A remarkable class of Cayley octads is described by Bateman ([5], p. 286) as follows. Consider a quartic surface $S \subset \mathbb{P}^3$ having a double line ℓ. Such surface belongs to a class studied by Castelnuovo in [9] (see also [33], p. 156). There are 8 planes through ℓ that are tangent to S. Then the 8 points of contact are a Cayley octad. This is proved by Bateman by an elaborate argument involving the equations of the objects involved.

Recall that a quartic S with a double line is the image of the rational morphism $\varphi : \mathbb{P}^2 \dashrightarrow \Lambda^\vee \cong \mathbb{P}^3$ defined by the linear system $\Lambda := \mathcal{L}(4; 2p, p_1, \dots, p_8)$ of plane quartics with one double point p and 8 simple base points $p_1, \dots, p_8$ in general position. The double line ℓ of S is the image of the unique cubic through $p, p_1, \dots, p_8$. Denote by $e_1, \dots, e_8 \subset S$ the lines that are images of the base points $p_1, \dots, p_8$ respectively. The pencil $\mathcal{L}(1; p)$ of lines through p represents the residual sections, which are conics, of S with the pencil of planes through ℓ. Of these conics exactly 8 are singular, i.e. reducible, namely those corresponding to the lines $\langle p, p_i \rangle \in \mathcal{L}(1; p)$, $i = 1, \dots, 8$. These conics are of the form $\ell_i \cup e_i$, where $\ell_1, \dots, \ell_8$ are lines incident to ℓ, and the 8 points of contact $x_1, \dots, x_8$ are the double points of these conics, i.e. $x_i = e_i \cap \ell_i$, $i = 1, \dots, 8$.

Problem: count the parameters of these octads. Quartics with a double line depend on 25 parameters (the computation can be found for example in [14], p. 163), but Cayley octads depend on 21 parameters. Therefore every octad of this family is contained in infinitely many quartics with double line, and the problem is to compute in how many.

Exercises: 1) Prove that there are 8 planes through ℓ that are tangent to S using the methods of §2.

2) Prove that if the points $p, p_1, \dots, p_8 \in \mathbb{P}^2$ are base points of a pencil of irreducible cubics then the surface $S \subset \mathbb{P}^3$, image of the rational map defined by the linear system $\Lambda := \mathcal{L}(4; 2p, p_1, \dots, p_8)$, is a double quadric.

6. The Steiner curve

Consider a regular Cayley octad $X \subset \mathbb{P}^3$ and the associated net of quadrics Λ_X. Let $C \subset \Lambda_X$ be the Hesse curve of the net. The *Steiner curve* of $\Lambda = \Lambda_X$ is

$$\Gamma_X := \{v(Q) : Q \in C\} \subset \mathbb{P}W \cong \mathbb{P}^3$$

If $\Lambda_X = \langle Q_1, Q_2, Q_3 \rangle$ then Γ_X is defined schematically by the maximal minors of the matrix (see §2):

$$\begin{pmatrix} \frac{\partial Q_0}{\partial X_0} & \frac{\partial Q_0}{\partial X_1} & \frac{\partial Q_0}{\partial X_2} & \frac{\partial Q_0}{\partial X_3} \\ \frac{\partial Q_1}{\partial X_0} & \frac{\partial Q_1}{\partial X_1} & \frac{\partial Q_1}{\partial X_2} & \frac{\partial Q_1}{\partial X_3} \\ \frac{\partial Q_2}{\partial X_0} & \frac{\partial Q_2}{\partial X_1} & \frac{\partial Q_2}{\partial X_2} & \frac{\partial Q_2}{\partial X_3} \end{pmatrix}$$

and therefore it is a sextic of genus 3. We can be more precise.

Proposition 6.1. *There is an even theta characteristic θ on C such that $|\omega_C\theta|$ defines a morphism $f : C \longrightarrow \mathbb{P}W$ mapping C isomorphically onto Γ_X.*

Proof. Since every quadric on $\mathbb{P}W$ defines a self-dual linear map $W \to W^\vee$ we can associate to Λ_X an exact sequence:

$$0 \longrightarrow W \otimes \mathcal{O}_\Lambda(-2) \longrightarrow W^\vee \otimes \mathcal{O}_\Lambda(-1) \longrightarrow \theta \longrightarrow 0$$

where the first map is self dual and θ is an invertible sheaf on C of degree two because $h^0(\theta) = h^1(\theta) = 0$. Dualizing and twisting by $\omega_\Lambda = \mathcal{O}_\Lambda(-3)$ we obtain:

$$0 \longrightarrow W \otimes \mathcal{O}_\Lambda(-2) \longrightarrow W^\vee \otimes \mathcal{O}_\Lambda(-1) \longrightarrow Ext^1(\theta, \omega_\Lambda) \longrightarrow 0$$

and therefore

$$\theta \cong Ext^1(\theta, \omega_\Lambda) = Ext^1(\mathcal{O}_C, \omega_\Lambda)\theta^{-1} = \omega_C\theta^{-1}$$

Thus θ is an even theta characteristic. After tensoring the previous exact sequence by $\mathcal{O}_\Lambda(1)$ we obtain:

$$0 \longrightarrow W \otimes \mathcal{O}_\Lambda(-1) \longrightarrow W^\vee \otimes \mathcal{O}_\Lambda \longrightarrow \omega_C\theta \longrightarrow 0$$

Since θ is even $\omega_C\theta$ is very ample and $H^0(C, \omega_C\theta) = W^\vee$. The conclusion now is clear. $\square$

Corollary 6.2. *Keeping the same notations as in Prop. 5.4 we have:*

(i) *the 28 lines L_{ij} are chords of Γ_X such that $\Gamma_X \cap L_{ij} = p_{ij} + q_{ij}$ are the odd theta characteristics of Γ_X.*
(ii) *For each $1 \leq i \leq 8$ the seven odd theta characteristics*

$$\{\theta_{ij} = p_{ij} + q_{ij} : j \neq i\}$$

are an Aronhold system.

Proof. (i) Left to the reader.

(ii) Denote by θ_{rs} the odd theta characteristic defined by the line L_{rs}. For simplicity we give the proof for $i = 1$. Assume by contradiction that $\theta_{1i}+\theta_{1j}-\theta_{1k}$ is odd, equal say to θ_{rs} for some r, s. Then $\theta_{1i}+\theta_{1j}+\theta_{ij} \in |\omega_C\theta|$ because x_1, x_i, x_j are coplanar. Therefore $\theta_{rs} + \theta_{1k} + \theta_{ij} \in |\omega_C\theta|$, and this means that the lines L_{1k} and L_{ij} are coplanar, contradicting Proposition 5.2. $\square$

We also have the following:

Corollary 6.3. *Let $Z = \{P_1, \ldots, P_7\} \subset \mathbb{P}^2$ be a regular 7-tuple. With the notations of §3 let $\Sigma \subset \mathbb{P}^2$ be the ramification sextic and $B \subset |-K_S|^\vee$ the branch quartic curve of the Geiser involution defined by Z. Then the bitangents to B corresponding to the (-1)-curves $e_1, \ldots, e_7$ on S form an Aronhold system.*

Proof. From Proposition 5.2 it follows that $X \cap \Gamma_X = \emptyset$. If we project Γ_X in $\mathbb{P}^2$ from x_i we obtain a sextic Σ of genus 3 with 7 double points at the images of the points x_j, $j \neq i$. By construction Σ is the ramification curve of the Geiser involution defined by these 7 points and C is isomorphic to its branch curve B. In particular it follows from Corollary 6.2(ii) that the bitangents of B corresponding to the 7 base points of the net are an Aronhold system. $\square$

We can now prove easily that every even theta characteristic θ is associated to a net of quadrics as above.

Proposition 6.4. *For each even theta characteristic θ on C there is a net of quadrics whose Steiner curve is $\varphi_{\omega_C\theta}(C) \subset \mathbb{P}^3$.*

Proof. (sketch) Let $V := H^1(\mathcal{O}_C)$ and embed canonically $C \subset \mathbb{P}V \cong \mathbb{P}^2$. Then the multiplication maps:

$$H^0(\omega_C^k) \otimes H^0(C, \omega_C\theta) \longrightarrow H^0(\omega_C^{k+1}\theta)$$

are easily seen to be surjective for all $k \geq 1$. It follows that there is a resolution:

$$0 \longrightarrow W \otimes \mathcal{O}_{\mathbb{P}^2}(-1) \xrightarrow{\psi} H^0(C, \omega_C\theta) \otimes \mathcal{O}_{\mathbb{P}^2} \longrightarrow \omega_C\theta \longrightarrow 0$$

where $\dim(W) = h^0(C, \omega_C\theta) = 4$. Therefore the map ψ is defined by a square matrix A of linear forms. Dualizing and arguing as in the proof of Proposition 6.1 we deduce that $W = H^0(C, \omega_C\theta)^\vee$ and A is symmetric. Now we have a net of quadrics in $\mathbb{P}W$ parametrized by $\mathbb{P}V$ whose Steiner curve is precisely $\varphi_{\omega_C\theta}(C)$. $\square$

Remark 6.5. Now the picture is the following. To each even theta characteristic on a plane quartic C there is associated a net of quadrics in $\mathbb{P}^3$, defined up to PGL(4) action. To such a net we have associated 8 Aronhold systems on C, one for each base point of the net. We thus obtain $288 = 36 \times 8$ Aronhold systems. They are distinct and account for all (not yet proved here). In particular each Aronhold system $\{\theta_1, \dots, \theta_7\}$ on C is associated to a unique even theta characteristic, which is nothing but the one not among the 35 even thetas $\theta_i + \theta_j - \theta_k$ (not yet proved either).

In particular, each regular 7-tuple $Z \subset \mathbb{P}V$ defines uniquely a pair $(B(Z), \theta)$ consisting of the branch quartic $B(Z)$ of the Geiser involution γ_Z together with the even theta θ associated to the Aronhold system defined by the points of Z. This correspondence is originally due to Aronhold [3]. In modern language it is explained in [16].

Parameter count: the nets of quadrics depend on 21 parameters, the 7-tuples of points in $\mathbb{P}^2$ depend on 14 parameters, the pairs (C, θ) with θ even theta characteristic, depend on 6 parameters. Moreover:

$$21 - \dim[\mathrm{PGL}(4)] = 6 = 14 - \dim[\mathrm{PGL}(3)]$$

Remark 6.6. In the proof of Corollary 6.3 we considered a regular Cayley octad $X = \{x_1, \dots, x_8\}$ and, by projecting to a plane, we associated to each $x_i \in X$ a 7-tuple Z of points in the plane so that Λ_X projects to the net of plane cubics through Z.

This correspondence can be made more intrinsic by avoiding the projection in the following way. Consider Λ_X and $x_i \in X$. Then instead of projecting from x_i we can consider the plane $\mathbb{P}V$ where $V = T_{x_i}\mathbb{P}^3$. Then the points of $\mathbb{P}V$ are the lines of $\mathbb{P}^3$ through x_i. Now we may associate to each $Q \in \Lambda_X$ the lines of Q passing through x_i. This given a rational map $\gamma : \mathbb{P}V \dashrightarrow \Lambda_X$ of degree two, which ramifies at the lines contained in a singular quadric of Λ_X, i.e. at the lines joining x_i with a point of the Steiner curve. The branch curve is the Hesse quartic, and γ is not defined at the 7-tuple of lines $Z = \{\langle x_i, x_j \rangle : j \neq i\}$. It is clear that $\gamma = \gamma_Z$.

7. Geometry of a web of quadrics

Here we give still another description of Geiser involutions and further speculations. Our reference is [6], §5.

Consider six points $A = \{A_1, \dots, A_6\} \subset \mathbb{P}^3$ in general position. They define a 3-dimensional linear system (web) of quadrics $\Lambda_A = |H^0(\mathbb{P}^3, \mathcal{I}_A(2))|$ and consequently a rational map:

$$\eta : \mathbb{P}^3 \dashrightarrow \Lambda_A^\vee = \mathbb{P}^3$$

This map has degree two because, given a general $P \in \mathbb{P}^3$, the quadrics through $A \cup \{P\}$ contain another point P'. Moreover η contracts the 15 lines $\langle A_i, A_j \rangle$ and the (unique) twisted cubic Γ containing A. The ramification surface $\mathcal{W}$ is the jacobian determinantal quartic of Λ_A, which is singular at A; it is called the *Weddle surface.* The image of $\mathcal{W}$ is the branch surface $\mathcal{K}$

of Λ_A: it is a *Kummer surface*, singular at the 16 points images of the lines $\langle A_i, A_j\rangle$ and of Γ. For detais on this configuration see [18], p. 128.

Now consider a general $Q \in \Lambda_A$, restrict η to Q and call this restriction ζ. Then $\zeta : Q \dashrightarrow \mathbb{P}^2$ is defined by the net of elliptic quartics cut on Q by Λ_A, i.e. by the 2-dimensional linear system $|H^0(Q, \mathcal{I}_{A/Q}(2))|$. The ramification curve R of ζ is $Q \cap \mathcal{W}$, a curve of degree 8 with double points at A, hence of geometric genus 3. If we project $\pi_{A_1} : Q \dashrightarrow \mathbb{P}^2$ then the curves of $|H^0(Q, \mathcal{I}_{A/Q}(2))|$ are mapped to the linear system of cubics containing the projections $\overline{A}_2, \dots, \overline{A}_6$ of $A_2, \dots, A_6$ and the two points B_0, B_1 images of the two lines passing through A_1. Therefore the composition $\zeta \cdot \pi_{A_1}^{-1} : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ is the Geiser involution γ_Z where $Z = \{B_0, B_1, \overline{A}_2, \dots, \overline{A}_6\}$. The ramification curve is $\Sigma = \pi_{A_1}(R)$.

A variation on the same construction is the following. Identify $\mathbb{P}^2 = \Gamma_2$, the second symmetric product of Γ, with the set of chords of Γ. We can define a rational map:

$$\zeta : Q \dashrightarrow \Gamma_2$$

by setting $\zeta(P)$ to be the unique chord of Γ containing P. Then ζ has degree two and is well defined at $Q \setminus A$.

An equivalent but less intrinsic way of describing ζ is the following.
In $\mathbb{P}^3$ consider a pencil of quadrics $\Lambda = \langle Q_1, Q_2\rangle$ generated by two nonsingular quadrics Q_1, Q_2. Then:

(i) To each point $\xi \in \mathbb{P}^3$ there is associated a line $\ell(\xi)$, *the polar of* ξ *w.r. to* Λ, given by $\ell(\xi) = \Delta_\xi Q_1 \cap \Delta_\xi Q_2$. By reciprocity:

$$P \in \ell(\xi) \Leftrightarrow \xi \in \ell(P)$$

(ii) If ξ moves on a line ℓ the line $\ell(\xi)$ spans a quadric $Q(\ell)$.

(iii) To each plane $X \subset \mathbb{P}^3$ there is associated a twisted cubic $\Gamma(X)$, consisting of the poles of X with respect to the quadrics of Λ.

Now fix a plane X and a quadric Q in $\mathbb{P}^3$. Define a rational map

$$\zeta : Q \dashrightarrow X$$

by

$$\zeta(P) = X \cap \ell(P)$$

This map is defined at the points $P \in Q$ such that $\ell(P) \not\subset X$. By reciprocity

$$\zeta^{-1}(\xi) = \ell(\xi) \cap Q =: \{P_1(\xi), P_2(\xi)\}$$

Therefore ζ has degree two.

If $\ell \subset X$ is a line then $\zeta^{-1}(\ell) = Q(\ell) \cap Q =: E(\ell)$. If $\ell_1, \ell_2 \subset X$ and $\xi = \ell_1 \cap \ell_2$ then

$$\zeta^{-1}(\xi) \subset E(\ell_1) \cap E(\ell_2) = Q \cap Q(\ell_1) \cap Q(\ell_2)$$

Since $\zeta^{-1}(\xi)$ consists of two points while $Q \cap Q(\ell_1) \cap Q(\ell_2)$ consists of 8 points, 6 of these must be base points of the net spanned by $\zeta^{-1}(\ell) = E(\ell)$ as ℓ moves in X.

The 6 base points of the net $\{E(\ell)\}$ are precisely the points where ζ is not defined and coincide with $\Gamma(X) \cap Q$.
In fact let $P \in \Gamma(X)$ and let $Q_P \in \Lambda$ be the quadric such that $X = \Delta_P Q_P$. Then $\ell(P) = \Delta_P Q_P \cap \Delta_P \widetilde{Q} \subset X$ for some $Q_P \neq \widetilde{Q} \in \Lambda$ and therefore if $P \in \Gamma(X) \cap Q$ then ζ is not defined at P.
If $P \in Q$ is a base point of $\{E(\ell)\}$ then the line $\ell(P)$ must be contained in X by reciprocity because $\ell(P)$ contains infinitely many points ξ such that $P \in \ell(\xi)$.
Since there are 6 base points and $\Gamma(X) \cap Q$ consists if 6 points, this set coincides with the base points locus and the indeterminacy locus of ζ.

Denote by $\{A_1, \dots, A_6\} = \Gamma(X) \cap Q$. If we project Q from A_1 onto a plane Y then $\{E(\ell)\}$ projects into a net of cubics passing through 7 points, namely the projections $\overline{A}_2, \dots, \overline{A}_6$ of $A_2, \dots, A_6$ and the two points B_0, B_1 where the generators of Q through A_1 meet Y.

Denote by $\pi : Q \dashrightarrow Y$ the projection from A_1 and by $Z = \{B_0, B_1, \overline{A}_2, \dots, \overline{A}_6\}$. Then we have the following diagram:

$$\begin{tikzcd} S = Bl_Z Y \arrow[r, "\varphi"] \arrow[d] & X \\ Y \arrow[ur, dashed, "\zeta\pi^{-1}"'] & \end{tikzcd}$$

and $\zeta\pi^{-1} = \gamma_Z$ is the Geiser involution associated to Z.

Count of parameters. The construction of the Geiser involutions we have given depends on the choice of a pair (A, Q) with $Q \in \Lambda_A$. Therefore the number of parameters is 18+3=21.

8. Cayley octads and 7-tuples in $\mathbb{P}^2$

Denote by X_3^8 the set of regular Cayley octads. It is a locally closed subset of the symmetric product $(\mathbb{P}^3)^{(8)}$. It can be identified with the open set of the space of nets of quadrics consisting of regular nets, which in turn is identified with an open set of the grassmannian $G(3, S^2W^\vee)$. The identification is given by:

$$X_3^8 \ni X \longmapsto H^0(\mathbb{P}^3, \mathcal{I}_X(2)) \subset S^2W^\vee$$

Note that $X_3^8 \neq \emptyset$ because the locus of quadrics of rank ≤ 2 has codimension three in $|H^0(\mathbb{P}^3, \mathcal{O}_(2))|$, and therefore a general net $\Lambda \in G(3, S^2W^\vee)$ does not meet it. Note also that X_3^8 is irreducible,

$$\dim(X_3^8) = \dim[G(3, S^2W^\vee)] = 21$$

and that X_3^8 is rational. It is obvious that PGL(4) acts on X_3^8.

Let X_3^7 be the set of unordered 7-tuples $\{x_1, \dots, x_7\}$ of distinct points of $\mathbb{P}^3$ such that the net of quadrics they determine has as base locus a regular Cayley octad $\{x_1, \dots, x_7, x_8\}$. Elements of X_3^7 are called *regular 7-tuples* *of* $\mathbb{P}^3$. Again, we have an action of PGL(4) on X_3^7, and there is a natural PGL(4)-equivariant morphism

$$\widetilde{\zeta}: X_3^7 \longrightarrow X_3^8$$

which is finite of degree 8.

Let moreover Z_2^7 be the set of unordered 7-tuples of distinct points in $\mathbb{P}^2$ such that no three are on a line and no six are on a conic. Elements of Z_2^7 will be called *regular 7-tuples of* $\mathbb{P}^2$. It is obvious that Z_2^7 is mapped into itself by PGL(3).

To a regular 7-tuple $\bar{x} = \{x_1, \dots, x_7\}$ in $\mathbb{P}^3$ we can associate its projection from the eighth base point x_8 of the net of quadrics defined by $\bar{x}$, in the plane $\mathbb{P}(T_{x_8}\mathbb{P}^3)$. After choosing a system of coordinates in this plane, we obtain a 7-tuple of points of $\mathbb{P}^2$, which we denote by $\beta(\bar{x})$. From Proposition 5.2 it follows that $\beta(\bar{x})$ is regular. Note that $\beta(\bar{x})$ is defined only up to an element of PGL(3).

Conversely, given $\bar{z} = \{z_1, \dots, z_7\} \in Z_2^7$, we can consider two general cubics D_1, D_2 containing $\bar{z}$ and their residual intersections $y, t \in \mathbb{P}^2$. The linear system of conics $|H^0(\mathbb{P}^2, \mathcal{I}_{\{y,t\}}(2))|$ defines a rational map: $b : \mathbb{P}^2 \dashrightarrow \mathbb{P}^3$ whose image is a quadric Q. Let

$$\beta'(\bar{z}) := \{b(z_1), \dots, b(z_7)\} \subset Q$$

and $x_8 := b(y) = b(t)$. Then $X := \beta'(\bar{z}) \cup \{x_8\}$ consists of eight distinct points which are the base locus of a pencil of elliptic curves on Q, image of the pencil of plane cubics $\langle D_1, D_2 \rangle$. It follows that this pencil is the restriction to Q of a net of quadrics containing Q. By construction it follows immediately that X is a regular Cayley octad, and therefore $\beta'(\bar{z}) \in X_3^7$. Again this construction depends on several choices, which imply that $\beta'(\bar{z})$ is defined up to an element of PGL(4).

It is not difficult to show that the geometric quotients

$$U_3^7 := X_3^7/\mathrm{PGL}(4), \quad U_3^8 := X_3^8/\mathrm{PGL}(4)$$

and

$$U_2^7 := Z_2^7/\mathrm{PGL}(3)$$

exist and are nonsingular irreducible of dimension six: we will assume this fact, referring to [18] for the proof. The morphism $\widetilde{\zeta}$ induces a finite morphism of degree 8:

$$\overline{\zeta}: U_3^7 \longrightarrow U_3^8$$

Moreover the above constructions define isomorphisms:

$$\beta: U_3^7 \longrightarrow U_2^7$$

and

$$\beta': U_2^7 \longrightarrow U_3^7$$

which are inverses of each other. We will also consider the composition:

$$\zeta := \overline{\zeta} \cdot \beta' : U_2^7 \longrightarrow U_3^8$$

We can also consider *ordered regular Cayley octads*, i.e. ordered 8-tuples $(x_1, \dots, x_8) \in (\mathbb{P}^3)^8$ such that $\{x_1, \dots, x_8\} \in X_3^8$, and *ordered regular 7-tuples in* $\mathbb{P}^2$, i.e. ordered 7-tuples $(z_1, \dots, z_7) \in (\mathbb{P}^2)^7$ such that $\{z_1, \dots, z_7\} \in Z_2^7$. Again, the quotients of these sets by the respective projective linear groups exist [18], and will be denoted by O_3^8 and O_2^7 respectively. The above constructions induce *isomorphisms*

$$\begin{tikzcd} O_3^8 \arrow[r, "\beta"'] & O_2^7 \arrow[l, bend right, "\beta'"'] \end{tikzcd}$$

which are inverses of each other.

Proposition 8.1. *O_3^8 and O_2^7 are rational.*

Proof. Given a regular ordered 7-tuple $(z_1, \dots, z_7) \in (\mathbb{P}^2)^7$ there is a unique $G \in \mathrm{PGL}(3)$ sending $z_1, \dots, z_4$ to the reference points

$$(1,0,0), (0,1,0), (0,0,1), (1,1,1)$$

Therefore the orbits under PGL(3) are in 1–1 correspondence with the set of ordered triples $(G(z_5), G(z_6), G(z_7)) \in (\mathbb{P}^2)^3$, which is an open subset of $(\mathbb{P}^2)^3$. $\square$

Remark 8.2. Consider the closure $\overline{X_3^8} \subset (\mathbb{P}^3)^{(8)}$. Its element will be called *Cayley octads*. The structure of $\overline{X_3^8}$ does not seem to have been studied in detail. It is pretty clear that $\overline{X_3^8}$ is not isomorphic to $G(3, S^2W^\vee)$: the nets of quadrics defining rational normal cubic curves are in the closure of regular nets, but do not define any element of $\overline{X_3^8}$. On the other hand, octads $\{x_1, \dots, x_8\}$ contained in a rational normal cubic are likely to belong to $\overline{X_3^8}$.

Problem: given $\{x_1, \dots, x_8\}$ contained in a rational normal cubic Γ, find a line L in $G(3, S^2W^\vee)$ containing the net $|H^0(\mathbb{P}^3, \mathcal{I}_\Gamma(2))|$ such that the 1-parameter family of regular octads parametrized by the elements of L contains $\{x_1, \dots, x_8\}$ in its closure.

These remarks suggest that $\overline{X_3^8}$ should be a blow-up of $G(3, S^2W^\vee)$ along the 12-dimensional locus of nets defining rational normal cubics. The exceptional divisor should be the 20-dimensional locus of 8-tuples $\{x_1, \dots, x_8\}$ contained in some rational normal cubic.

9. Cubic surfaces and plane quartics

Let $Y = \{P_1, \dots, P_6\}$ consist of six distinct points of $\mathbb{P}^2$, not on a conic, no three on a line. Let $\sigma_Y : S \longrightarrow \mathbb{P}^2$ be the blow-up of $\mathbb{P}^2$ centered at Y. Then S is called a *Del Pezzo surface of degree three.* We have:

$$K_S \sim -3h + \sum_i e_i$$

where $e_i := \sigma_Y^{-1}(P_i)$, $i = 1, \dots, 6$, and $h = \sigma_Y^* \mathcal{O}(1)$. Therefore $|-K_S|$ is the pullback of the linear system $\Lambda_Y := |H^0(\mathbb{P}^2, \mathcal{I}_Y(3))|$ of cubics through Y. An elementary analysis shows that Λ_Y has dimension 3 and that $-K_S$ is very ample; since $K_S^2 = 3$, we see that the linear system $|-K_S|$ maps S isomorphically onto a nonsingular cubic surface of $|-K_S|^\vee \cong \mathbb{P}^3$:

(6)
$$\begin{tikzcd} S \arrow[r, "\varphi"] \arrow[d, "\sigma_Y"'] & {|-K_S|^\vee \cong \mathbb{P}^3} \\ \mathbb{P}^2 & \end{tikzcd}$$

Since this embedding is anticanonical, and therefore completely intrinsic, we will identify S with the cubic surface $\varphi(S)$.

Lines ((-1)-curves). It is immediate from the adjunction formula that a nonsingular rational curve $E \subset S$ can be a (-1)-curve if and only if $\deg(E) = (-K_S \cdot E) = 1$, i.e. if and only if it is a line. There are exactly 27 lines on the cubic surface, and they are the images of $e_1, \dots, e_6$, of the proper transforms $\widehat{C}_1, \dots, \widehat{C}_6$ of the six conics $C_1, \dots, C_6$ passing through five of the points of Y, and of the proper transforms of the 15 lines joining two points of Y. Each line meets exactly 10 other lines and therefore it is skew with 16 others (an easy verification).

Sixes and double sixes. A *six* of lines consists of a 6-tuple of lines on S which are pairwise skew. Examples of sixes are $\{e_1, \dots, e_6\}$ and $\{\widehat{C}_1, \dots, \widehat{C}_6\}$. There are exactly 72 sixes on S. The sixes are naturally associated in pairs, called *double sixes.* A double six consists of two 6-tuples of lines on S:

$$\{A_1, \dots, A_6; B_1, \dots, B_6\}$$

such that the lines of each 6-tuple are pairwise skew (i.e. each 6-tuple is a six), and each line A_i meets all the lines B_j, $j \neq i$, but not B_i. An example of a double six is $\{e_1, \dots, e_6, \widehat{C}_1, \dots, \widehat{C}_6\}$. There are exactly 36 double sixes on S.

The contraction of the lines of a six $\{A_1, \dots, A_6\}$ is isomorphic to $\mathbb{P}^2$, and the inverse of the rational map $\sigma_A : S \longrightarrow \mathbb{P}^2$ is defined by a linear system of cubics through 6 points of $\mathbb{P}^2$. Therefore there are exactly 72 distinct ways of obtaining S as in (6). Given a double six $\{A_1, \dots, A_6; B_1, \dots, B_6\}$, the composite birational map:

$$\sigma_B \sigma_A^{-1} : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$$

is defined by the linear system of quintics having multiplicity ≥ 2 at the points $\sigma_A(A_1), \dots, \sigma_A(A_6)$.

Schlaefi notation. Consider a double six

$$\{A_1, \dots, A_6; B_1, \dots, B_6\}$$

on S. For each $1 \leq i < j \leq 6$ consider the line

$$L_{ij} = \langle A_i, B_j \rangle \cap \langle A_j, B_i \rangle$$

Then the L_{ij}'s are the other 15 lines on S. Two lines L_{ij} and L_{kl} meet if and only if $\{i, j\} \cap \{k, l\} = \emptyset$. A line A_k (resp. B_k) meets L_{ij} if and only if $k \in i, j$.

Relation with Geiser involutions and plane quartics. Given

$$Y = \{P_1, \dots, P_6\} \subset \mathbb{P}^2$$

not on a conic and no three on a line, we can add a 7-th point P_7 so that $Z = \{P_1, \dots, P_6, P_7\}$ is a regular 7-tuple. Then the Geiser involution $\gamma_Z : \mathbb{P}^2 \dashrightarrow \Lambda_Z^\vee \cong \mathbb{P}^2$ is the composition:

$$\mathbb{P}^2 \overset{\sigma_Y^{-1}}{\dashrightarrow} S \overset{\pi_O}{\dashrightarrow} \mathbb{P}^2$$

where π_O is the projection of S from the point $O := \sigma_Y^{-1}(P_7)$. The assumption that Z is regular is equivalent to the fact that O is not on a line of S. Conversely, given a point $O \in S$ not on a line, the composition of σ_Y^{-1} with the projection π_O is a Geiser involution. The net Λ_Z corresponds, via σ_Y^{-1}, to the sections of S with the planes containing O. The pairs of the involution are transformed in the pairs of point of S aligned with O; the curve $\sigma_Y^{-1}(\Sigma)$ is a sextic $\Gamma \subset S$ having a node at O and nonsingular elsewhere, whose projection from O is the quartic C. Then Γ and C are the ramification curve and the branch curve respectively, of the double cover $\widehat{S} \longrightarrow \mathbb{P}^2$, where $\widehat{S}$ is the blow-up of S at O, and it is also the Del Pezzo surface of degree two defined by Z.

Various degenerate configurations associated to special choices of the points $P_1, \dots, P_7$ have special interest. For more on this point of view see [33], p. 141-142, 148-149, and [4], vol. VI p. 122-123).

The Sylvester pentahedron. The equation of a general cubic surface $S \subset \mathbb{P}^3$ can be put under the form:

$$a_1 X_1^3 + \cdots + a_5 X_5^3 = 0 \tag{7}$$

where $X_1, \dots, X_5$ are linear forms satisfying the linear relation:

$$\sum_i X_i = 0$$

and the a_i's are constants. The planes having equations $X_1 = 0, \dots, X_5 = 0$ are uniquely determined by S. They form the so-called *Sylvester pentahedron* of S. For a modern proof of this theorem we refer to [16]. It is easy to compute that the hessian of a cubic S whose equation is (7) is the quartic surface H given by:

$$\sum_i a_1 \cdots \widehat{a}_i \cdots a_5 X_1 \cdots \widehat{X}_i \cdot X_5 = 0$$

or, in equivalent rational form, by:

$$X_1X_2X_3X_4X_5 \left(\frac{1}{a_1X_1} + \cdots + \frac{1}{a_5X_5} \right) = 0$$

It is immediate to verify that H contains 10 lines and has 10 double points: they are the edges and the vertices of the Sylvester pentahedron. Each edge contains 3 vertices and each vertex is contained in 3 edges.

Cremona hexahedral equations. Another important description of nonsingular cubic surfaces by means of equations has been given by Cremona [15].

In $\mathbb{P}^5$ with coordinates $(Z_0, \dots, Z_5)$ consider the following equations:

$$\begin{cases} Z_0^3 + Z_1^3 + Z_2^3 + Z_3^3 + Z_4^3 + Z_5^3 = 0 \\ Z_0 + Z_1 + Z_2 + Z_3 + Z_4 + Z_5 = 0 \\ \beta_0 Z_0 + \beta_1 Z_1 + \beta_2 Z_2 + \beta_3 Z_3 + \beta_4 Z_4 + \beta_5 Z_5 = 0 \end{cases} \tag{8}$$

where the β_s's are constants. These equations define a cubic surface S in a $\mathbb{P}^3$ contained in $\mathbb{P}^5$. If S is nonsingular then (8) are called *Cremona hexahedral equations* of S. They have several remarkable properties, the most important being that these equations also determine a double-six of lines on the surface S.

Theorem 9.1. *Each system of Cremona hexahedral equations of a nonsingular cubic surface S defines a double-six of lines on S. Conversely, the choice of a double-six of lines on S defines a system of Cremona hexahedral equations (8) of S, which is uniquely determined up to replacing the coefficients $(\beta_0, \dots, \beta_5)$ by $(a + b\beta_0, \dots, a + b\beta_5)$ for some $a, b \in \mathbb{C}$, $b \neq 0$.*

We refer to [16], Theorem 9.4.6, for the proof. We need to point out the following:

Corollary 9.2. *To a pair (S, Δ) consisting of a nonsingular cubic surface $S \subset \mathbb{P}^3$ and a double-six of lines Δ on S, there is canonically associated a plane $\Xi \subset \mathbb{P}^3$ which is given by the equations*

$$\begin{cases} Z_0 + Z_1 + Z_2 + Z_3 + Z_4 + Z_5 = 0 \\ \beta_0 Z_0 + \beta_1 Z_1 + \beta_2 Z_2 + \beta_3 Z_3 + \beta_4 Z_4 + \beta_5 Z_5 = 0 \\ \beta_0^2 Z_0 + \beta_1^2 Z_1 + \beta_2^2 Z_2 + \beta_3^2 Z_3 + \beta_4^2 Z_4 + \beta_5^2 Z_5 = 0 \end{cases} \tag{9}$$

Proof. By replacing in the third equation β_i by $a + b\beta_i$, with $b \neq 0$, the plane Ξ remains the same. Therefore this plane depends only on the equations (8) which in turn depend only on (S, Δ). $\square$

Definition 9.3. The plane $\Xi \subset \mathbb{P}^3$ is called the *Cremona plane* associated to the pair (S, Δ).

From Theorem 9.1 and Corollary 9.2 it follows that every nonsingular cubic surface has 36 different systems of hexahedral equations and 36 Cremona planes. If the cubic surface $S \subset \mathbb{P}^3$ is given as the image of a linear system of plane cubic curves through six points $\{P_1, \dots, P_6\}$, then a double-six is implicitly selected by such a representation, and therefore also a system of equations (8) and a Cremona plane (9) are.

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