ERRATA & ADDENDA
to
Deformations of Algebraic Schemes
Last update sept. 15, 2020

Obstructions

Lemma B.10, p. 285, is false as stated. The following counterexample has been kindly provided by M. Manetti.

Let k be a field of characteristic zero, x, y indeterminates, and let $f(x,y) \in k[[x,y]]$ be such that

$$f \notin \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right), \quad \sqrt{\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, f\right)} = (x,y)$$

and consider the ideal

$$J = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, xf, yf\right)$$

Then the following is a small extension:

$$0 \longrightarrow k \xrightarrow{f} k[[x,y]]/J \longrightarrow k[[x,y]]/\left(\tfrac{\partial f}{\partial x}, \tfrac{\partial f}{\partial y}, f\right) \longrightarrow 0$$

which contradicts Lemma B.10. An example of an f as above is:

$$f(x,y) = x^7 + x^4y^4 + y^7$$

as shown by the following

Lemma 0.1 $f(x,y) := x^7 + x^4y^4 + y^7 \notin \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, xf, yf\right)$

Proof. We have

$$f_x = \frac{\partial f}{\partial x} = 7x^6 + 4x^3y^4, \quad f_y = \frac{\partial f}{\partial y} = 4x^4y^3 + 7y^6$$

Assume that $f - cxf - dyf \in (f_x, f_y)$. Then, since $1 - cx - dy$ is invertible, we have that $f \in (f_x, f_y)$, say $f = af_x + bf_y$. Looking at the homogeneous components of degree ≤ 7 we deduce

$$a \in \frac{x}{7} + (x,y)^2, \quad b \in \frac{y}{7} + (x,y)^2$$

and therefore

$$f - \frac{x}{7}f_x - \frac{y}{7}f_y = \frac{1}{7}x^4y^4 \in (x,y)^2(f_x, f_y)$$

which is impossible. □

Note that when $\mathrm{char}(k) = 0$ the conclusion of Lemma B.10 is valid for *curvilinear extensions* i.e. extensions of the form

$$0 \longrightarrow \frac{t^{n-1}}{t^n} \longrightarrow k[t]/(t^n) \longrightarrow k[t]/(t^{n-1}) \longrightarrow 0$$

This is Example B.9(v), p. 284.

Lemma B.10 is used in the proof of Proposition 2.4.8, p. 70, and this proof is therefore incomplete as it stands. Nevertheless the proof of of Proposition 2.4.8 can be adapted to prove the following weaker statement:

Proposition 0.2 *Let X be a reduced l.c.i. algebraic scheme, and assume char$(\mathbf{k}) = 0$. If $\mathrm{Ext}^2_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_X) = 0$ then X is unobstructed, i.e. Def_X is smooth.*

Proof. (outline) The steps are the following:

1) In $\mathrm{char}(\mathbf{k}) = 0$ in order to prove unobstructedness it is sufficient to check that deformations over $\mathbf{k}[t]/(t^n)$ lift to $\mathbf{k}[t]/(t^{n+1})$. This is proved in Fantechi - Manetti, J. Alg. 202 (1998), Lemma 5.6 p. 561.

2) The following Lemma is true:

Lemma 0.3 *Let X be a reduced l.c.i. algebraic scheme, and assume char$(\mathbf{k}) = 0$. Let $\xi_n = (\mathcal{X} \to \mathrm{Spec}(\mathbf{k}[t]/(t^n)))$ be a deformation of X. Then the obstruction to lift ξ_n over $\mathbf{k}[t]/(t^{n+1})$ lies in $\mathrm{Ext}^2_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_X)$.*

Proof of the Lemma (outline):

– The conormal sequence of the curvilinear extension $\mathbf{k}[t]/(t^{n+1}) -> \mathbf{k}[t]/(t^n)$ is exact: this is Example B.9(v) at the bottom of p. 284 of the book.

– The proof of Proposition 2.4.8 can be repeated for deformations ξ of X over $\mathbf{k}[t]/(t^n)$ and for elements of $\mathrm{Ex}_{\mathbf{k}}(\mathbf{k}[t]/(t^n), \mathbf{k})$ represented by curvilinear extensions. It shows that if $\mathrm{Ext}^2_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_X) = 0$ then every such deformation ξ can be lifted to a deformation over $\mathbf{k}[t]/(t^{n+1})$. □

In view of 1) above this proves the proposition. □

A detailed proof of the Lemma can be found in

T. Sano: On Deformations of $\mathbb{Q}$-Fano Threefolds, arXiv:1203.6323 (Proposition 2.6).

Correction to Prop. 2.6.2

The proof has to be changed by replacing the last 9 lines on p. 91 (starting from "in particular, we obtain etc.") by the following text.

In order to prove that the natural map

$$\zeta : \mathrm{Aut}_{\hat{u}}(\widetilde{A}) \longrightarrow \mathrm{Aut}_{\hat{u}}(A') \times_{\mathrm{Aut}_{\hat{u}}(A)} \mathrm{Aut}_{\hat{u}}(A'')$$

is an isomorphism we use the isomorphism (2.27). Let's recall how ζ is defined. Let $\widetilde{\alpha} \in \mathrm{Aut}_{\hat{u}}(\widetilde{A})$. It can be identified with an automorphism

$$\widetilde{\alpha} : \mathcal{O}_{\mathcal{X}_{\widetilde{A}}} \longrightarrow \mathcal{O}_{\mathcal{X}_{\widetilde{A}}}$$

such that $\widetilde{\alpha} \otimes_{\widetilde{A}} A = \mathrm{id}_{\mathcal{O}_{\mathcal{X}_A}}$. Equivalently, by (2.27), $\widetilde{\alpha}$ can be identified with:

$$\widetilde{\alpha} : \mathcal{O}_{\mathcal{X}'_A} \times_{\mathcal{O}_{\mathcal{X}_A}} \mathcal{O}_{\mathcal{X}''_A} \longrightarrow \mathcal{O}_{\mathcal{X}'_A} \times_{\mathcal{O}_{\mathcal{X}_A}} \mathcal{O}_{\mathcal{X}''_A}$$

inducing the identity of $\mathcal{O}_{\mathcal{X}_A}$ after $\otimes_{\widetilde{A}} A$.

Then we define:

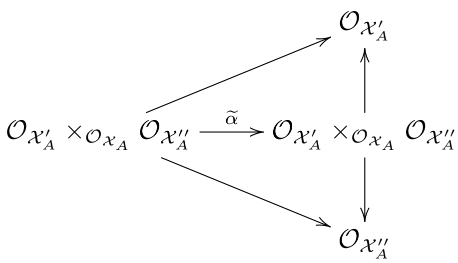
$$\zeta(\widetilde{\alpha}) = (\alpha', \alpha'') := (\widetilde{\alpha} \otimes_{\widetilde{A}} A', \widetilde{\alpha} \otimes_{\widetilde{A}} A'')$$

We need to check that α' and α'' are automorphisms. We have

$$\alpha'' \otimes_{A''} A : \widetilde{\alpha} \otimes_{\widetilde{A}} A = \mathrm{id}_{\mathcal{O}_{\mathcal{X}_A}}$$

Therefore, since $A'' \to A$ is surjective, α'' is an isomorphism (Lemma A.4). Similarly, since $\widetilde{A} \to A'$ is surjective, from $\widetilde{\alpha}$ isomorphism it follows that α' is an isomorphism as well. Therefore ζ *is well defined*.

It remains to be shown that ζ is a bijection. This follows from the universal property of the fibered product. In fact the pair (α', α'') is induced by $\widetilde{\alpha}$ composed with the pair of projections as in this diagram:



Note that we only used the surjectivity of $A'' \to A$, and not the full hypothesis that it is a small extension.

Other corrections/typos

In several occasions the symbol $T_{S,s}$ has to be replaced by T_sS: see p. 31, 159, 169, 179, 252.

p. 15, statement of Corollary 1.1.8: replace I/I^2 by J/J^2.

p. 132, l. 2: 'locally free' instead of locally trivial'.

p. 146, l. 4: "two-dimensional" instead of "finite dimensional".

p. 161, l. 5: remove -1.

p. 163, exact seq. (d): remove $0 \to$.

p. 170, l. 1 of Example 3.4.13(i): replace $\mathcal{F}$ by $\mathcal{F}_m$.

p. 171, last line: replace $\mathcal{O}_X$ by $\mathcal{O}_Y$.

p. 264, l. 1: remove $f_1(x_1, x_2)$.

p. 291, center of page: $b(z) = b - m(g_1 + g_2 z)$ instead of $b(z) = b - mz(g_1 + g_2 z)$.