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On the existence of certain families of curves

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Introduction

In this paper we study families of projective curves with good properties from the point of view of moduli. Our main goal is to investigate, for given r, n, g, the existence of a smooth irreducible open subset V of the Hilbert scheme of $\mathbb{P}^r$ parametrizing irreducible (and non-singular if $r \geqq 3$) curves of degree n and genus g having "the expected number of moduli". This means that the image of the natural functorial morphism

$$\pi: V \to \mathcal{M}_g$$

of V into the moduli space of curves of genus g has dimension (i.e. "number of moduli") equal to the expected dimension, which is

$$\min(3g-3, 3g-3+\rho(g,r,n))$$

where

$$\rho(g,r,n) = g-(r+1)(g-n+r)$$

is the Brill-Noether number. This expression for the expected number of moduli is the obvious postulation which comes from the well known interpretation, in terms of maps between vector bundles, of the existence of special divisors on a curve (cfr. [14] and [9]).

Of course when

$$\rho(g,r,n) \geqq 0$$

a family V has the expected number of moduli

$$3g-3=\dim \mathcal{M}_g$$

precisely when every sufficiently general curve of genus g belongs to it, i.e. when the family has *general moduli.*

When

$$\rho(g,r,n)<0$$

it follows from the results of [9] that every family V does not have general moduli (i.e. it has *special moduli*). In this case the number $-\rho(g,r,n)$ expresses the expected codimension of $\pi(V)$ in $\mathcal{M}_g$.

Our results are quite complete for plane curves. In §4 we prove in fact the following result (see Theorem (4.2) for a more precise statement).

Theorem. *For all n and g such that*

$$n-2 \leqq g \leqq \binom{n-1}{2}, \qquad n \geqq 5$$

there is an irreducible component of $\mathscr{V}_{n,g}$, the family of plane irreducible nodal curves of degree n and genus g, having the expected number of moduli.

When

$$\rho(g,2,n) \geqq 0,$$

i.e. in the range

$$n-2 \leqq g \leqq 3n/2-3$$

this result is well known (cfr. [9] and [1]).

It was also known classically, although expressed in a different way, for $g = \binom{n-1}{2}$, i.e. in the case of smooth plane curves (see §4 for more details on this).

Our result has been proved independently by M.R.M. Coppens in the range

$$2n-4 \leqq g \leqq \binom{n-1}{2}$$

(cfr. [3]). A priori for some n, g there might be another component of $\mathscr{V}_{n,g}$ not having the expected number of moduli, besides the one we prove to exist. The existence of such a component would of course imply that $\mathscr{V}_{n,g}$ is reducible: to decide whether this is so or not is a long standing open problem.

In §6 we prove the following existence theorem for families of curves in $\mathbb{P}^r$, $r \geqq 3$.

Theorem. *For all n, g such that*

$$(*) \qquad n-r \leqq g \leqq \frac{r(n-r)-1}{r-1}, \qquad n \geqq r+1$$

there is an open set V of an irreducible component of the Hilbert scheme of $\mathbb{P}^r$ which parametrizes smooth irreducible curves of degree n and genus g and has the expected number of moduli. Moreover if Γ is a curve parametrized by a point of V then Γ is embedded in $\mathbb{P}^r$ by a complete linear system $|D|$, its normal bundle N_Γ satisfies

$$H^1(\Gamma, N_\Gamma)=0$$

and the natural map

$$\mu_0(D)\colon H^0(\Gamma, \mathcal{O}(D))\otimes H^0(\Gamma, \mathcal{O}(K_\Gamma-D))\to H^0(\Gamma, \mathcal{O}(K_\Gamma))$$

where K_Γ is a canonical divisor, has maximal rank.

This result is improved in § 7 for curves in $\mathbb{P}^3$. We prove that for $r=3$ the same conclusions of the above theorem hold for all n, g such that

$$(**) \qquad n-3\leqq g\leqq 3n-18.$$

These results are not the best possible: inequalities (*) and (**) depend on the method of proof and probably they can be improved.

Let $W_n^r(C)$ denote the closed subscheme of the jacobian of the curve C which parametrizes invertible sheaves of degree n with at least $r+1$ linearly independent sections.

It is known that if C is a general curve of genus g, $W_n^r(C)$ is reduced and irreducible of dimension $\rho(g,r,n)$ when

$$\rho(g,r,n)>0$$

and $W_n^r(C)$ consists of finitely many points when

$$\rho(g,r,n)=0$$

(cfr. [9] and [5]). An immediate consequence of our previous theorem is the following

Corollary. *Let $r\geqq 3$. If*

$$\rho(g,r,n)>0$$

and C is a general curve of genus g, the set of $L\in W_n^r(C)$ which are very ample is open and dense in $W_n^r(C)$. If

$$\rho(g,r,n)=0$$

and C is a general curve of genus g, there is at least one very ample $L\in W_n^r(C)$.

This result has already been proved by Eisenbud and Harris in [4] in a more general form using completely different techniques.

To describe our methods of proof let's suppose $r\geqq 3$ and let's fix n, g satisfying (*). The existence of the family V claimed in the above theorem follows, by standard deformation theory and semicontinuity theorems, if we can construct just one curve $\Gamma\subset\mathbb{P}^r$ of degree n and genus g having the properties stated in the theorem.

This construction is done in two steps.

The first step consists in proving that if Γ exists, $H\subset\mathbb{P}^r$ is a general hyperplane and γ is a rational normal curve belonging to H such that $\gamma\cap\Gamma$ consists of $r+1$ points, then the reducible curve

$$\Gamma'=\Gamma\cup\gamma,$$

as a curve of degree $n+r-1$ and arithmetic genus $g+r$, also has the properties stated in the theorem. A delicate point here is to prove that the map μ_0 for Γ' has maximal rank (Prop. 2.3). This gives a sort of inductive construction provided we can prove that Γ' can be flatly smoothed in $\mathbb{P}^r$; in fact the properties that Γ' shares with Γ ($H^1(\Gamma', N_{\Gamma'})=0$, completeness of the embedding linear system $|D'|$ and $\mu_0(D')$ of maximal rank) are locally preserved under flat deformations.

To prove that Γ' can be flatly smoothed in $\mathbb{P}^r$ is the second ingredient of the proof. This is accomplished by

a) giving a smoothability criterion for nodal curves; this involves a naturally defined subbundle of the normal bundle (see Prop. (1.6));

b) checking that the criterion is satisfied by certain reducible curves having a rational component (see Theorem (5.2) for a precise statement), among them by Γ'.

The inequalities (*) and (**) depend on the starting step of the inductive construction.

The result on plane curves is proved using a similar method.

We work in the category of schemes over $\mathbb{C}$, the field of complex numbers. We will only consider Cartier divisors; if Δ is a divisor on a scheme Y, $|\Delta|$ will denote the complete linear system associated to Δ. If $\mathscr{F}$ is an algebraic sheaf on Y we will often write

$$H^i(\mathscr{F}),\ h^i(\mathscr{F}),\ H^i(\Delta),\ h^i(\Delta)$$

instead of $H^i(Y,\mathscr{F})$, $\dim H^i(Y,\mathscr{F})$, $H^i(Y,\mathcal{O}(\Delta))$, $\dim H^i(Y,\mathcal{O}(\Delta))$ respectively. A set of $n\geqq r+1$ distinct points in $\mathbb{P}^r$ will be said to be *in general position* if no $r+1$ of them lie in a hyperplane. A nondegenerate curve in $\mathbb{P}^r$ is a curve which is not contained in a hyperplane.

I wish to thank C. Peskine for showing me how to start the induction in Theorem (7.1) and Ph. Ellia for pointing out some incorrect statements in the first version of this paper. I am grateful to the Mathematical Institute of the University of Warwick for its hospitality during the preparation of part of this work.

§ 1. Generalities on families of curves in $\mathbb{P}^r$

We shall very briefly recall Severi's theory of plane curves with nodes.

Inside the projective space $\mathbb{P}(n)$ of dimension $n(n+3)/2$ parametrizing all plane curves of degree $n\geqq 3$ there is, for each $0\leqq g\leqq\binom{n-1}{2}$, a locally closed functorially defined subscheme $\mathscr{V}_{n,g}$; it parametrizes all irreducible curves of degree n having

$$\delta := \binom{n-1}{2} - g$$

nodes (i.e. ordinary double points) and no other singularities.

If X is such a curve, consider the functor of Artin rings

$H_X(A) = \{$divisors $X_A \subset \mathbb{P}^2_A$ flat over A and inducing $X \subset \mathbb{P}^2$ on the closed fibre$\}$ for each Artin local $\mathbb{C}$-algebra A, and the subfunctor

$$H'_X(A) = \{X_A \subset \mathbb{P}^2_A \text{ locally trivial over } A\}.$$

These functors are easily seen to be prorepresented by the complete local rings $\hat{\mathcal{O}}_{\mathbb{P}(n),p(X)}$ and $\hat{\mathcal{O}}_{\mathscr{V}_{n,g},p(X)}$ respectively, where $p(X) \in \mathbb{P}(n)$ is the closed point parametrizing X.

Denoting by T^1_X the first cotangent sheaf of X (cfr. [16]) there is a natural surjective morphism

$$\mathcal{O}_X(n) \to T^1_X$$

whose kernel we denote by N'_X.

The tangent and obstruction spaces of H'_X are respectively the 0-th and first cohomology groups of the sheaf N'_X. One easily computes that

$$h^0(N'_X) = 3n + g - 1, \qquad h^1(N'_X) = 0.$$

It follows that $\mathscr{V}_{n,g}$ is smooth of dimension $3n+g-1$ at the point parametrizing X.

$\mathscr{V}_{n,g}$ is called *the family of plane irreducible nodal curves of degree n and genus g.*

It should be noted that

$$3n + g - 1 = n(n+3)/2 - \delta;$$

in other words $\mathscr{V}_{n,g}$ has codimension δ in $\mathbb{P}(n)$.

Suppose that X' is a possibly reducible nodal curve of degree n with $\delta' > \delta$ nodes. Let I be a set of δ nodes of X' and consider the subfunctor of $H_{X'}$:

$$H^I_{X'}(A) = \{X'_A \subset \mathbb{P}^2_A \mid X'_A \text{ is locally trivial at all } P \in I\}.$$

This is the functor of infinitesimal deformations of X' in $\mathbb{P}^2$ which preserve the selected δ nodes; it defines locally a closed subscheme of $\mathbb{P}(n)$ which is again smooth and of dimension $3n+g-1$ and parametrizes a family of nodal curves.

For dimension reasons a general element X'' of this family has precisely δ nodes, and no other singularities, which specialize to the chosen δ nodes of X'; it also follows easily that X'' is irreducible if and only if $X' \backslash I$ is connected. In this case the set I defines a local branch of the closure $\bar{\mathscr{V}}_{n,g}$ at the point parametrizing X'. It follows that all the branches defined in this way meet transversely there.

An easy consequence of this theory is that there exist irreducible nodal curves of degree n and genus g for all

$$0 \leqq g \leqq \binom{n-1}{2}$$

i.e. $\mathscr{V}_{n,g} \neq \phi$ for all such n and g.

In the following we will freely use the notations and the results we have just introduced; for more details the reader is referred to [22] and [23].

Unfortunately for curves in a projective space of higher dimension there is not a theory analogous to Severi's theory of plane nodal curves. We will prove some properties of deformations of nodal curves in $\mathbb{P}^r$, $r \geqq 3$, which will be needed in the sequel.

Let $X \subset \mathbb{P}^r$, $r \geqq 3$, be a connected reduced curve of degree n with only nodes as singularities, let $\operatorname{Sing}(X)$ be its set of double points, and

$$\varphi: C \to X$$

be the desingularization of X.

We denote by

$$g(X) = 1 - \chi(\mathcal{O}_X)$$

and

$$g(C) = 1 - \chi(\mathcal{O}_C)$$

the arithmetic genus of X and C respectively; it is

$$g(X) = g(C) + \delta$$

where δ is the number of double points of X.

Denote by D the pullback on C of a general hyperplane section divisor on X and by K any canonical divisor on C. Let $\mathscr{I}_X \subset \mathcal{O}_{\mathbb{P}^r}$ be the ideal sheaf of X and

$$N_X := \operatorname{Hom}(\mathscr{I}_X, \mathcal{O}_X)$$

the *normal bundle of* X; it is a locally free sheaf of rank $r-1$ on X. Let moreover

$$T_X = \operatorname{Hom}(\Omega^1_{\mathbb{P}^r}, \mathcal{O}_X)$$

and

$$T_C = \varphi^* T_X.$$

On C we have the exact sequence

$$(1.1) \qquad 0 \to \mathcal{O}_C(-K) \to T_C \to N_\varphi \to 0$$

where N_φ is a locally free sheaf of rank $r-1$, *the normal bundle of the map* φ. We also have an exact sequence on X

$$(1.2) \qquad 0 \to \Theta_X \to T_X \to N_X \to T^1_X \to 0$$

where

$$\Theta_X := \mathrm{Hom}(\Omega^1_X, \mathcal{O}_X)$$

and where T^1_X is the *cotangent sheaf of* X. This is a torsion sheaf supported on $\mathrm{Sing}(X)$ with

$$T^1_{X,P} \cong \mathbb{C}$$

for each $P \in \mathrm{Sing}(X)$ (cfr. [16]).

For each non empty subset $I \subseteq \mathrm{Sing}(X)$ we will denote by $T^1_{X|I}$ the restriction of T^1_X to I extended by zero on X, and by

$$N^I_X := \ker(N_X \to T^1_{X|I}).$$

We will shortly denote $N^{\mathrm{Sing}(X)}_X$ by N'_X. For every inclusion

$$\phi \neq J \subseteq I \subseteq \mathrm{Sing}(X)$$

we have inclusions

$$N'_X \subseteq N^I_X \subseteq N^J_X \subseteq N_X$$

and with an abuse of notations we may view N_X as being N^{ϕ}_X.

Denote by $|I|$ the number of elements of I.

(1.3) **Lemma.** *For each* $I \subseteq \mathrm{Sing}(X)$ *we have*

$$\chi(N^I_X) = \chi(n, g(X)) - |I|$$

and

$$\chi(N_\varphi) = \chi(n, g(C))$$

where

$$\chi(n, g) := (r+1)n - (r-3)(1-g).$$

Proof. From the exact sequence (1.1) and from "Euler sequence"

$$0 \to \mathcal{O}_C \to \mathcal{O}(D)^{r+1} \to T_C \to 0$$

we deduce the assertion for $\chi(N_\varphi)$.

Since for each non empty $I \subseteq \mathrm{Sing}(X)$ we have an exact sequence

$$0 \to N^I_X \to N_X \to T^1_{X|I} \to 0$$

it suffices to prove the remaining assertion of the lemma for $I = \phi$.

By pulling (1.2) back on C we get a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{O}_C(-K) & \longrightarrow & T_C & \longrightarrow & N_\varphi & \longrightarrow & 0 & & \\
 & & \downarrow & & \| & & \downarrow & & & & \\
0 & \longrightarrow & \varphi^* \Theta_X & \longrightarrow & T_C & \longrightarrow & \varphi^* N_X & \longrightarrow & \varphi^* T^1_X & \longrightarrow & 0
\end{array}$$

and from this we deduce the exact sequence

(1.4) $$0 \to N_\varphi \to \varphi^* N_X \to \varphi^* T^1_X \to 0.$$

On X we have an exact sequence

$$0 \to \mathcal{O}_X \to \varphi_* \mathcal{O}_C \to \mathcal{T} \to 0$$

with $\mathcal{T}$ a torsion sheaf supported on $\mathrm{Sing}(X)$, such that

$$h^0(\mathcal{T}) = \delta;$$

we deduce the exact sequence

$$0 \to N_X \to \varphi_* \varphi^* N_X \to \mathcal{T} \otimes N_X \to 0. \tag{1.5}$$

Since

$$\chi(\varphi^* N_X) = \chi(\varphi_* \varphi^* N_X)$$
$$\chi(\varphi^* T^1_X) = h^0(\varphi^* T^1_X) = 2\delta$$

and

$$\chi(\mathcal{T} \otimes N_X) = h^0(\mathcal{T} \otimes N_X) = (r-1)\,\delta$$

from (1.4) and (1.5) we deduce

$$\chi(N_X) = \chi(N_\varphi) - (r-3)\,\delta \quad \text{q.e.d.}$$

Let $\mathrm{Hilb} = \mathrm{Hilb}^r_{n, g(X)}$ be the Hilbert scheme parametrizing closed subschemes of $\mathbb{P}^r$ whose Hilbert polynomial is $nt + 1 - g(X)$ (cfr. [11]).

For each subset $I \subseteq \mathrm{Sing}(X)$ we can define in the usual functorial way a subscheme $\mathrm{Hilb}(I)$ of Hilb containing the point corresponding to X and parametrizing a universal family of deformations of X in $\mathbb{P}^r$ which are locally trivial at every $P \in I$. For example

$$\mathrm{Hilb}(\phi) = \mathrm{Hilb}$$

and $\mathrm{Hilb}(\mathrm{Sing}(X))$ parametrizes flat deformations of X in which none of its δ nodes is smoothed.

Every inclusion $I \subseteq J$ corresponds to an inclusion

$$\mathrm{Hilb}(J) \subseteq \mathrm{Hilb}(I).$$

Information on the local structure of $\mathrm{Hilb}(I)$ is obtained by considering the functor of Artin rings:

$$H^I_X(A) = \left\{ \begin{array}{l} \text{flat deformations of } X \subset \mathbb{P}^r \text{ parametrized by } A \\ \text{and locally trivial at all } P \in I \end{array} \right\}$$

for every local Artin $\mathbb{C}$-algebra A. In particular $H^\phi_X = H_X$, the local Hilbert functor (cfr. [19]). From the well known connection between the sheaf T^1_X and the deformation theory of X (cfr. [16]) it follows that $H^0(N^I_X)$ and $H^1(N'_X)$ are the tangent and obstruction spaces respectively of H^I_X. Since clearly H^I_X is prorepresented by $\hat{\mathcal{O}}_{\mathrm{Hilb}(I), p(X)}$, the completed local ring of $\mathrm{Hilb}(I)$ at the point $p(X)$ parametrizing X, we see that $\mathrm{Hilb}(I)$ is a locally closed subscheme of Hilb.

It follows directly from the definitions that $\mathrm{Hilb}(\{P\})$ is a proper closed subscheme of Hilb for all $P \in \mathrm{Sing}(X)$ if and only if X is flatly smoothable in $\mathbb{P}^r$, i.e. if and only if there is an irreducible flat family of curves of $\mathbb{P}^r$ containing X as a member and whose general fibre is smooth.

(1.6) **Proposition.** (i) *For each* $I \subseteq \mathrm{Sing}(X)$ *we have*

$$\chi(n, g(X)) - |I| \leqq \dim_{p(X)} \mathrm{Hilb}(I) \leqq h^0(N_X^I)$$

and the second inequality is an equality if and only if $\mathrm{Hilb}(I)$ *is smooth at the point* $p(X)$ *parametrizing* X; *in particular*

$$H^1(N_X^I) = 0$$

if and only if $\mathrm{Hilb}(I)$ *is smooth of dimension* $\chi(n, g(X)) - |I|$ *at* $p(X)$.

(ii) *If*

$$H^1(N_X') = 0$$

Hilb *is smooth of dimension* $\chi(n, g(X))$ *at* $p(X)$ *and* X *is flatly smoothable in* $\mathbb{P}^r$.

Proof. (i) follows in the usual way from formal deformation theory (cfr. [15], Theorem (4.2.4)).

To prove (ii) let $I \subseteq \mathrm{Sing}(X)$ and consider the exact sequence

$$0 \to N_X' \to N_X^I \to T^1_{X|\mathrm{Sing}(X) \setminus I} \to 0$$

where the last map is the composition

$$N_X^I \to N_X \to T^1_{X|\mathrm{Sing}(X) \setminus I}.$$

From this sequence and from

$$H^1(N_X') = 0$$

it follows that we also have

$$H^1(N_X^I) = 0$$

and therefore that $\mathrm{Hilb}(I)$ is smooth of dimension

$$\chi(n, g(X)) - |I|$$

at the point $p(X)$ by part (i). In particular this is true for Hilb and moreover

$$\dim_{p(X)} \mathrm{Hilb}(\{P\}) = \dim_{p(X)} \mathrm{Hilb} - 1$$

for all $p \in \mathrm{Sing}(X)$; recalling the remark made before, this concludes the proof. q.e.d.

§ 2. The map $\mu_0(D)$

Let C be a projective connected reduced curve of arithmetic genus $g \geqq 2$, with only nodes as singularities, and let D be a divisor on C. Denoting by ω_C the dualizing sheaf of C, we have a natural map

$$\mu_0(D)\colon H^0(D)\otimes H^0(\omega_C(-D))\to H^0(\omega_C).$$

When C is smooth this map plays a role in the study of the relation between the moduli of C and the existence of divisors of given degree and dimension.

We will be especially interested in the case when $\mu_0(D)$ is of maximal rank, i.e. is either surjective or injective.

For the relation between the injectivity of $\mu_0(D)$ and moduli we refer the reader to [2], [6] and [9].

In this section we will only investigate the map $\mu_0(D)$ in some cases, leaving the discussion on moduli for next sections.

If $n=\deg(D)$ and $r+1=h^0(D)$, an obvious necessary condition for $\mu_0(D)$ to be surjective is that the number

$$\rho(g,r,n):=g-(r+1)(g-n+r)=h^0(\omega_C)-h^0(D)\cdot h^0(\omega_C(-D))$$

is non positive. This number is called *the Brill-Noether number of* g, r, n.

In particular we see that in order to have the surjectivity of $\mu_0(D)$ it is necessary that D is a special divisor such that $h^0(D)>0$.
We are mainly interested in the map $\mu_0(D)$ for divisors D which are hyperplane sections of a smooth C in some projective space or such that $|D|$ defines a birational map of a smooth C into $\mathbb{P}^2$. Later on we will also need to consider the map μ_0 associated to hyperplane sections of certain reducible curves. Such curves are constructed as follows.

Let's start with a smooth irreducible C and with a divisor D of degree $n\geqq r+1$ such that $|D|$ is base point free and of dimension $r\geqq 2$. We assume that the map

$$\varphi_D\colon C\to\mathbb{P}^r$$

is an embedding if $r\geqq 3$ and that φ_D is birational onto a curve with only nodes if $r=2$; we call

$$\Gamma:=\varphi_D(C).$$

If $r=2$ let γ be a general line on $\mathbb{P}^2$, so that $\gamma\cap\Gamma$ consists of n distinct points $P_1,\dots,P_n$; let $p_i=\varphi_D^{-1}(P_i)$, $i=1,\dots,n$, and

$$C'=C\cup\gamma$$

with p_1,p_2,p_3 identified with P_1,P_2,P_3. In other words C' is the stable curve obtained by desingularizing all multiple points of

$$\Gamma':=\Gamma\cup\gamma$$

except P_1,P_2,P_3.

If $r\geqq 3$ let H be a general hyperplane in $\mathbb{P}^r$. $H\cap\Gamma$ consists of n distinct points in general position $P_1,\dots,P_n$; let $p_i=\varphi_D^{-1}(P_i)$, $i=1,\dots,n$. Let moreover γ be a smooth rational curve of degree $r-1$ in H containing $P_1,\dots,P_{r+1}$ and not $P_{r+2},\dots,P_n$; it is easy to show that such a γ exists (cfr. [8]). Let

$$C'=C\cup\gamma$$

with $p_1, \ldots, p_{r+1}$ identified with $P_1, \ldots, P_{r+1}$; C' is a stable curve isomorphic to

$$\Gamma' := \Gamma \cup \gamma.$$

In all cases $r \geqq 2$ the curve C' has arithmetic genus

$$g' = g + r.$$

The divisor D' on C' defined by a general hyperplane section of Γ' has degree

$$n' = n + r - 1$$

and

$$h^0(C', \mathcal{O}(D')) \geqq r + 1.$$

(2.1) **Lemma.** $h^0(C', \mathcal{O}(D')) = r + 1$; *equivalently*

$$h^1(C', \mathcal{O}(D')) = h^1(C, \mathcal{O}(D)) + 1.$$

Proof. Letting $L = \mathcal{O}_{\mathbb{P}^{r-1}}(1) \otimes \mathcal{O}_\gamma$ we have an exact sequence of sheaves on C'

$$0 \to L(-P_1 \ldots - P_{r+1}) \to \mathcal{O}_{C'}(D') \to \mathcal{O}_C(D) \to 0 \tag{2.2}$$

from which the lemma follows immediately. q.e.d.

We have the following

(2.3) **Proposition.** *Under the above assumptions, if $\mu_0(D)$ is of maximal rank then $\mu_0(D')$ is also of maximal rank.*

Proof. Obviously $\mu_0(D')$ is of maximal rank if

$$h^0(\omega_{C'}(-D')) \leqq 1.$$

Since by Lemma (2.1)

$$h^0(\omega_{C'}(-D')) = h^0(\omega_C(-D)) + 1$$

we may assume that

$$h^0(\omega_C(-D)) \geqq 1.$$

From this assumption it follows that C is non hyperelliptic of genus $g \geqq 3$. From the exact sequence of sheaves on C'

$$0 \to \mathcal{O}_C(-P_1 \ldots - P_{r+1}) \to \mathcal{O}_{C'} \to \mathcal{O}_\gamma \to 0$$

one deduces an exact sequence

$$0 \to \omega_\gamma \to \omega_{C'} \to \omega_C(P_1 + \ldots + P_{r+1}) \to 0 \tag{2.4}$$

and from this an isomorphism

$$H^0(\omega_{C'}) \xrightarrow{\sim} H^0(\omega_C(P_1 + \ldots + P_{r+1})).$$

After tensoring (2.4) by $\mathcal{O}(-D')$ we deduce an isomorphism

$$H^0(\omega_{C'}(-D')) \xrightarrow{\sim} H^0(\omega_C(-D + P_1 + \ldots + P_{r+1})).$$

We also have an isomorphism induced by restriction (lemma (2.1))

$$H^0(D') \xrightarrow{\sim} H^0(D).$$

Altogether these give rise to a commutative diagram

$$\begin{array}{ccc} H^0(D')\otimes H^0(\omega_{C'}(-D') & \xrightarrow{\mu_0(D')} & H^0(\omega_{C'}) \\ \downarrow & & \downarrow \\ H^0(D)\otimes H^0(\omega_C(-D+P_1+\ldots+P_{r+1})) & \xrightarrow{\mu_0''} & H^0(\omega_C(P_1+\ldots+P_{r+1})) \end{array}$$

in which the vertical maps are isomorphisms and μ_0'' is the natural map.

Therefore it suffices to show that μ_0'' has maximal rank. The space $H^0(\omega_C(P_1+\ldots+P_{r+1}))$ defines the complete linear system on C

$$|K+P_1+\ldots+P_{r+1}|$$

where K is a canonical divisor; since $r+1\geqq 3$ this system embeds C in $\mathbb{P}^{g+r-1}$. The image of μ_0'' corresponds to a linear system of hyperplanes which cut on C the minimal sum

$$|D|+|K-D+P_1+\ldots+P_{r+1}|$$

and whose base locus is a certain linear space $B\subset\mathbb{P}^{g+r-1}$. Since

$$g=h^0(K)=h^0(K+P_1+\ldots+P_{r+1})-r$$

and $|K|$ is base point free, the linear span

$$\langle P_1,\ldots,P_{r+1}\rangle$$

of $P_1,\ldots,P_{r+1}$ in $\mathbb{P}^{g+r-1}$ has dimension $r-1$ and satisfies

$$\langle P_1,\ldots,P_{r+1}\rangle\cap C=\{P_1,\ldots,P_{r+1}\}.$$

Claim: $B\cap\langle P_1,\ldots,P_{r+1}\rangle=\phi$.

Assume that

$$\dim(B\cap\langle P_1,\ldots,P_{r+1}\rangle)=:\nu\geqq 0.$$

Note that by construction

$$h^0(D-P_1\ldots-P_{r+1})=h^0(D-P_1-\ldots-\hat{P}_i-\ldots-P_{r+1})$$

or equivalently

$$h^0(K-D+P_1+\ldots+P_{r+1})=1+h^0(K-D+P_1+\ldots+\hat{P}_i+\ldots+P_{r+1}),$$
$$i=1,\ldots,r+1.$$

This means that $P_1, \dots, P_{r+1}$ are not base points of

$$|K-D+P_1+\dots+P_{r+1}|;$$

since $|D|$ has no base points it follows that $P_1, \dots, P_{r+1} \notin B$.

Therefore after possibly reordering the P_i's we have

$$\langle B, P_1, \dots, P_{r-1-\nu}\rangle = \langle B, P_1, \dots, P_{r+1}\rangle.$$

By the genericity of the divisor

$$P_1+\dots+P_n \in |D|$$

we can find $E' \in |D|$ such that

$$P_1, \dots, P_{r-1-\nu} \in \operatorname{Supp}(E')$$

and

$$P_{r-\nu}, \dots, P_{r+1} \notin \operatorname{Supp}(E').$$

Moreover let

$$E'' \in |K-D+P_1+\dots+P_{r+1}|$$

be such that

$$P_1, \dots, P_{r+1} \notin \operatorname{Supp}(E'').$$

We then see that

$$\langle \operatorname{Supp}(E'+E'')\rangle \supseteq \langle B, P_1, \dots, P_{r-1-\nu}\rangle = \langle B, P_1, \dots, P_{r+1}\rangle$$

and therefore

$$P_{r-\nu,\dots,r+1} \in \langle \operatorname{Supp}(E'+E'')\rangle \cap C.$$

But this is a contradiction because

$$P_{r-\nu}, \dots, P_{r+1} \notin \operatorname{Supp}(E') \cup \operatorname{Supp}(E'') = \langle \operatorname{Supp}(E'+E'')\rangle \cap C$$

and the claim follows.

Consider the commutative diagram

$$\begin{array}{ccc} H^0(D)\otimes H^0(K-D) & \xrightarrow{\mu_0(D)} & H^0(K) \\ \downarrow & & \downarrow \chi \\ H^0(D)\otimes H^0(K-D+P_1+\dots+P_{r+1}) & \xrightarrow{\mu_0''} & H^0(K+P_1+\dots+P_{r+1}). \end{array}$$

$\operatorname{Im}(\chi)$ and $\operatorname{Im}(\chi)\cap\operatorname{Im}(\mu_0'')$ correspond to the linear systems of hyperplanes of $\mathbb{P}^{g+r-1}$ containing

$$\langle P_1, \dots, P_{r+1}\rangle \quad \text{and} \quad \langle B, P_1, \dots, P_{r+1}\rangle$$

respectively. Since

$$\operatorname{Im}(\chi\circ\mu_0(D)) \subseteq \operatorname{Im}(\chi)\cap\operatorname{Im}(\mu_0'')$$

we have

$$rk(\mu_0(D)) = rk(\chi \circ \mu_0(D)) \leqq \dim[\mathrm{Im}(\chi) \cap \mathrm{Im}(\mu_0'')] \tag{2.5}$$
$$= g + r - 1 - \dim\langle B, P_1, \ldots, P_{r+1}\rangle$$
$$= g + r - 1 - \dim B - r = rk(\mu_0'') - r.$$

If $\mu_0(D)$ is surjective this implies that μ_0'' too is surjective and proves the proposition in this case.

If $\mu_0(D)$ is injective but not surjective, we must show that μ_0'' is injective; this amounts to show, by (2.5), that

$$rk(\mu_0'') \geqq rk(\mu_0(D)) + r + 1$$

or equivalently that we have a strict inclusion

$$\mathrm{Im}(\chi \circ \mu_0(D)) \subsetneqq \mathrm{Im}(\chi) \cap \mathrm{Im}(\mu_0'').$$

For this purpose we will consider the map

$$\mu_0(D - P_1 - \ldots - P_{r+1})\colon H^0\left(D - \sum_{j=1}^{r+1} P_j\right) \otimes H^0\left(K - D + \sum_{j=1}^{r+1} P_j\right) \to H^0(K).$$

Since we have

$$\mathrm{Im}\left(\chi \circ \mu_0\left(D - \sum_{j=1}^{r+1} P_j\right)\right) \subseteq \mathrm{Im}(\chi) \cap \mathrm{Im}(\mu_0'')$$

in view of the injectivity of χ the proposition will follow if we can show that

$$\mathrm{Im}\left(\mu_0\left(D - \sum_{j=1}^{r+1} P_j\right)\right) \nsubseteq \mathrm{Im}(\mu_0(D)). \tag{2.6}$$

We now view C canonically embedded in $\mathbb{P}^{g-1}$ and we denote by $\langle - \rangle$ *the linear span in* $\mathbb{P}^{g-1}$ of $-$. The minimal sum

$$|D| + |K - D|$$

is cut on C by the hyperplanes of $\mathbb{P}^{g-1}$ containing a certain linear space A of dimension

$$\rho(g, r, n) - 1 \geqq 0.$$

A is contained in all the $(n-r-1)$-dimensional linear spaces $\langle \mathrm{Supp}(E')\rangle$, $E' \in |D|$; in particular

$$A \subseteq \langle P_1, \ldots, P_n\rangle.$$

Since

$$h^0(P_{r+1} + \ldots + P_n) = h^0(D - P_1 - \ldots - P_r) = 1$$

we see that

$$\langle P_{r+1}, \ldots, P_n\rangle = \langle P_1, \ldots, P_n\rangle$$

and $\langle P_{r+1}, \ldots, \hat{P}_i, \ldots, P_n \rangle$ is a hyperplane of $\langle P_1, \ldots, P_n \rangle$ for all $i=r+1, \ldots, n$. Therefore A is not contained in $\langle P_{r+1}, \ldots, \hat{P}_i, \ldots, P_n \rangle$ for *some* i; from the fact that we can interchange any two of the P_i's by moving $P_1 + \ldots + P_n$ in $|D|$ it follows that A is not contained in $\langle P_{r+1}, \ldots, \hat{P}_i, \ldots, P_n \rangle$ for *all* i. In particular A is not contained in $\langle P_{r+2}, \ldots, P_n \rangle$. This implies that the linear system

$$P_{r+2} + \ldots + P_n + |K - P_{r+2} - \ldots - P_n| = \left| D - \sum_{j=1}^{r+1} P_j \right| + \left| K - D + \sum_{j=1}^{r+1} P_j \right|$$

is not contained in $|D| + |K - D|$, or equivalently that (2.6) is true. q.e.d.

Next proposition describes a class of smooth curves in $\mathbb{P}^3$ with $\mu_0(D)$ of maximal rank.

(2.7) **Proposition.** *Let Γ be a smooth irreducible non degenerate curve in $\mathbb{P}^3$, and D a plane section of Γ. Assume that*

$$h^0(\Gamma, \mathcal{O}(D)) = 4$$

and that Γ is contained either in a smooth quadric or in a smooth cubic surface. Then $\mu_0(D)$ has maximal rank.

Proof. Suppose that $\Gamma \subset Q$, a smooth quadric. Let l_1, l_2 be two intersecting lines on Q, H a plane section of Q and let

$$\Gamma \sim \alpha l_1 + \beta l_2, \ \alpha, \beta \text{ positive integers, on } Q.$$

We have, denoting by K a canonical divisor on Γ,

$$\mathcal{O}_\Gamma(K) \cong \mathcal{O}_\Gamma((\alpha-2)\, l_1 + (\beta-2)\, l_2)$$
$$\mathcal{O}_\Gamma(K-D) \cong \mathcal{O}_\Gamma((\alpha-3)\, l_1 + (\beta-3)\, l_2).$$

The restriction map

$$H^0(Q, \mathcal{O}_Q(H)) \to H^0(\Gamma, \mathcal{O}_\Gamma(D))$$

is bijective by hypothesis, and one easily checks that the restriction maps

$$H^0(Q, \mathcal{O}_Q(\alpha-2)\, l_1 + (\beta-2)\, l_2) \to H^0(\Gamma, \mathcal{O}_\Gamma(K))$$
$$H^0(Q, \mathcal{O}_Q(\alpha-3)\, l_1 + (\beta-3)\, l_2) \to H^0(\Gamma, \mathcal{O}_\Gamma(K-D))$$

are bijective as well.

It follows that we have a commutative diagram

$$\begin{array}{ccc} H^0(Q, \mathcal{O}_Q(H)) \otimes H^0(Q, \mathcal{O}_Q((\alpha-3)\, l_1 + (\beta-3)\, l_2)) & \to & H^0(\Gamma, \mathcal{O}(D)) \otimes H^0(\Gamma, \mathcal{O}(K-D)) \\ \downarrow{\scriptstyle \mu} & & \downarrow{\scriptstyle \mu_0(D)} \\ H^0(Q, \mathcal{O}_Q((\alpha-2)\, l_1 + (\beta-2)\, l_2)) & \to & H^0(\Gamma, \mathcal{O}(K)) \end{array}$$

in which the horizontal maps are bijective. Therefore it suffices to prove that the map μ has maximal rank. We may assume that D is a special divisor and therefore that $\alpha, \beta \geqq 3$. In this case μ is in fact surjective. This follows from Castelnuovo's vanishing theorem (cfr. [17]) since the sheaf $\mathcal{O}_Q((\alpha-3)l_1+(\beta-3)l_2)$ is 1-regular; in fact

$$H^1(\mathcal{O}_Q((\alpha-3)l_1+(\beta-3)l_2))=H^2(\mathcal{O}_Q((\alpha-4)l_1+(\beta-4)l_2))=0.$$

If $\Gamma \subset S$ a smooth cubic surface with plane section H, we have the following bijective restriction maps:

$$H^0(S, \mathcal{O}_S(H)) \to H^0(\Gamma, \mathcal{O}_\Gamma(D))$$

(bijective by hypothesis),

$$H^0(S, \mathcal{O}_S(\Gamma-H)) \to H^0(\Gamma, \mathcal{O}_\Gamma(K))$$

(bijective because $H^0(S, \mathcal{O}_S(-H))=H^1(S, \mathcal{O}_S(-H))=0$),

$$H^0(S, \mathcal{O}_S(\Gamma-2H)) \to H^0(\Gamma, \mathcal{O}_\Gamma(K-D))$$

(bijective because $H^0(S, \mathcal{O}_S(-2H))=H^1(S, \mathcal{O}_S(-2H))=0$).

Since $\mu_0(D)$ has maximal rank if D is nonspecial or Γ is a canonical curve of genus 4, we may assume that Γ is not contained in a quadric surface. It is easy to check that in this case the sheaf $\mathcal{O}_S(\Gamma-2H)$ is 1-regular. The surjectivity of $\mu_0(D)$ follows now as in the first part of the proof. q.e.d.

§ 3. The number of moduli of a family

Fix integers $r \geqq 2$, $g \geqq 3$, $n \geqq 3$.

If $r \geqq 3$ (respectively $r=2$) let V be an open irreducible subset of $\mathrm{Hilb}^r_{n,g}$ parametrizing smooth irreducible nondegenerate curves of genus g and degree n (respectively let V be an irreducible component of $\mathcal{V}_{n,g}$).

If $r \geqq 3$ the proper and smooth family parametrized by V defines a natural morphism

$$\pi: V \to \mathcal{M}_g$$

of V into the moduli space of curves of genus g.

In the case $r=2$ a natural morphism

$$\pi: V \to \mathcal{M}_g$$

is also defined. In fact the family of plane nodal curves

$$\begin{array}{c} \mathcal{G} \subset \mathbb{P}^2 \times V \\ \downarrow \\ V \end{array}$$

parametrized by V can be "simultaneously desingularized", i.e. there is a diagram of proper morphisms

$$\begin{tikzcd} \mathscr{C} \arrow[r, "\Phi"] \arrow[dr, "f"'] & \mathscr{G} \subset \mathbb{P}^2 \times V \\ & V \end{tikzcd}$$

where f is a proper smooth morphism of curves and Φ is fibrewise the normalization map. The map Φ is the blow up of $\mathscr{G}$ along its singular locus. The morphism π is defined functorially by the family f.

Recall that $\mathscr{M}_g$ is a reduced and irreducible variety of dimension $3g-3$.

(3.1) **Definition.** *V has general moduli* (resp. *special moduli*) if

$$\dim \pi(V) = 3g-3 \quad (\text{resp. } \dim \pi(V) < 3g-3).$$

The number of moduli of V is $\dim \pi(V)$. We say that *V has the expected number of moduli* if

$$\dim \pi(V) = \min(3g-3,\, 3g-3+\rho(g,r,n)).$$

The following two propositions give criteria for a family as above to have the expected number of moduli in terms of the map μ_0 introduced in § 2.

(3.2) **Proposition.** *Let Γ be a nodal plane irreducible curve of degree n and genus g, $\varphi: C \to \Gamma$ the normalization of Γ and D the pullback on C of a line section of Γ.*

Then Γ is parametrized by a point of a component V of $\mathscr{V}_{n,g}$ which has the expected number of moduli if the following conditions are satisfied:

(i) $h^0(C, \mathcal{O}(D)) = 3$;

(ii) *$\mu_0(D)$ has maximal rank.*

In particular if (i) *and* (ii) *are satisfied with $\mu_0(D)$ injective then V has general moduli.*

Proof. Let

$$\begin{tikzcd} \mathscr{C} \arrow[r, "\Phi"] \arrow[dr, "f"'] & \mathscr{G} \subset \mathbb{P}^2 \times V \\ & V \end{tikzcd}$$

be the simultaneous desingularization of the family of nodal curves parametrized by V and let $0 \in V$ be the point parametrizing

$$C \xrightarrow{\varphi} \Gamma \subset \mathbb{P}^2 \times V.$$

It is well known that the map

$$\xi(C): H^0(N_\varphi) \to H^1(\Theta_C),$$

deduced from the exact sequence (1.1) on C, is the Kodaira-Spencer map of the family f (cf. [13]).

Since N_φ is an invertible sheaf of degree $3n+2g-2$, we have

$$H^1(N_\varphi)=0;$$

from the exact sequence (1.1) it follows that

$$rk(\xi(C))=3g-3-h^1(T_C).$$

Using the "Euler sequence" on C

$$0\to\mathcal{O}_C\to\mathcal{O}(D)\underset{\mathbb{C}}{\otimes}H^0(D)\to T_C\to 0$$

we see that

$$H^1(T_C)\cong\ker\mu_0(D)^*$$

and therefore

$$rk(\xi(C))=\min(3g-3,\,3g-3+\rho(g,2,n)).$$

By upper semicontinuity properties (i) and (ii) are satisfied by all curves of the family f parametrized by a certain open set U containing $0\in V$. It then follows that the Kodaira-Spencer map has constant rank equal to $rk(\xi(C))$ at all points of U.

From general facts of deformation theory we may conclude that

$$\dim\pi(V)=rk(\xi(C))$$

and therefore V has the expected number of moduli.

The last assertion of the proposition follows from the fact that $\mu_0(D)$ injective is equivalent to

$$H^1(T_C)=0$$

i.e. to the fact that $\xi(C)$ is surjective. q.e.d.

(3.3) **Proposition.** *Let $\Gamma\subset\mathbb{P}^r$, $r\geqq 3$, be a smooth irreducible non degenerate curve of degree n and genus g and D a hyperplane section divisor on Γ. Assume that the following conditions are satisfied:*

(i) $h^0(\Gamma,\mathcal{O}(D))=r+1$;
(ii) $\mu_0(D)$ *has maximal rank*
(iii) $H^1(\Gamma,N_\Gamma)=0$

then Γ is parametrized by a smooth point of $\mathrm{Hilb}^r_{n,g}$ *belonging to an open set V with the following properties:*

(iv) *V is smooth of dimension $\chi(n,g)$;*

(v) *all closed points of V parametrize smooth irreducible non degenerate curves of genus g and degree n.*

(vi) *V has the expected number of moduli.*

In particular if (i), (ii), (iii) *are satisfied with $\mu_0(D)$ injective then V has general moduli.*

Proof. From (iii) and from proposition (1.6)(i) it follows that $\mathrm{Hilb}^r_{n,g}$ is smooth of dimension $\chi(n,g)$ at the point 0 parametrizing Γ. The existence of V satisfying (iv) and (v) follows from this. Since by uppersemicontinuity conditions (i), (ii) and (iii) are satisfied by all curves parametrized by some open neighborhood of $0 \in V$, property (vi) can be proved as in the previous proposition. Similarly for the last assertion of the proposition. q.e.d.

§ 4. Plane curves: existence of families with the expected number of moduli

In this section we will discuss the case of plane curves. We will show the existence of irreducible components of $\mathscr{V}_{n,g}$ with the expected number of moduli for all n, g such that

$$n-2 \leqq g \leqq \binom{n-1}{2}, \quad n \geqq 5.$$

When $\rho(g,2,n) \geqq 0$, equivalently $g \leqq 3n/2-3$, this is well known as a particular case of the results of [9]. Note that if V is a component of $\mathscr{V}_{n,g}$ then certainly we have

$$\text{(4.1)} \qquad \begin{aligned} &\text{number of moduli of } V \leqq \dim V - \dim \operatorname{Aut}(\mathbb{P}^2) = 3n+g-9 \\ &= 3g-3+\rho(g,2,n). \end{aligned}$$

If $\rho(g,2,n)<0$ this means that V has *at most* the expected number of moduli. From (4.1) it follows that in order for V to have the expected number of moduli it is sufficient that a general point of V parametrizes a curve Γ which is birationally, but not projectively, equivalent to only finitely many curves of the family. In other words it is sufficient that the normalization C of Γ has only finitely many linear systems of degree n and dimension 2.

This condition was classically known to be satisfied in the case $g=\binom{n-1}{2}$ i.e. in the case of smooth plane curves, and $n \geqq 4$; it is in fact classical that a smooth plane curve of degree $n \geqq 4$ has a unique linear system of degree n and dimension $\geqq 2$ (cfr. [24]).

The existence of components of $\mathscr{V}_{n,g}$ having the expected number of moduli has also been proved independently by M.R.M. Coppens [3] in the cases

$$2n-4 \leqq g \leqq \binom{n-1}{2}$$

(4.2) **Theorem.** *For all n, g such that*

$$\text{(4.3)} \qquad n-2 \leqq g \leqq \binom{n-1}{2}, \quad n \geqq 5,$$

there is an irreducible component V of $\mathscr{V}_{n,g}$ whose general point parametrizes a nodal curve Γ with the following properties:

(1) *the lines cut a complete linear system* $|D|$ *on the normalization* C *of* Γ;
(2) $\mu_0(D)$ *has maximal rank*;
(3) V *has the expected number of moduli.*

Proof. By proposition (3.2) property (3) follows from properties (1) and (2); therefore it will suffice to prove the existence of a component V of $\mathscr{V}_{n,g}$ having properties (1) and (2).

For every fixed $a \in \mathbb{Z}$ the existence of V will be proved for all couples (n, g) such that (4.3) and

$$g = 2n - 5 - a \tag{4.4}$$

are satisfied.

The proof is by induction on n. Suppose that we have proved the existence of V for a given couple (n, g) satisfying (4.3) and (4.4); we will deduce the existence of a component of $\mathscr{V}_{n+1,g+2}$ having properties (1) and (2).

Let

$$\varphi_D: C \to \Gamma$$

be the normalization of Γ and

$$\Gamma' = \Gamma \cup \gamma$$

where γ is a general line in $\mathbb{P}^2$. Let

$$\Gamma \cap \gamma = \{P_1, \ldots, P_n\},$$

$p_i = \varphi_D^{-1}(P_i)$, $i = 1, \ldots, n$, and C' be the stable curve obtained by desingularizing all the singular points of Γ' except P_1, P_2, P_3. C' has genus $g+2$ and the pullback D' of a line section of Γ' is a divisor of degree $n+1$ on C'. By Lemma (2.1)

$$h^0(C', \mathcal{O}(D')) = 3.$$

Note that C is not hyperelliptic if D is a special divisor (i.e. if $n-2 < g$) and if $n-2 = g$ we may assume that C is not hyperelliptic by the genericity of Γ in the family. Therefore by Proposition (2.3) the map $\mu_0(D')$ is of maximal rank. From Severi's theory we deduce that there is a family of plane nodal curves of degree $n+1$

$$\begin{array}{c} \mathscr{G}' \subset \mathbb{P}^2 \times S \\ \big\downarrow \pi \\ S \end{array}$$

parametrized by a smooth curve S and a point $0 \in S$ whose fibre is

$$\Gamma' \subset \mathbb{P}^2$$

and such that all fibres

$$\mathscr{G}'(s) \subset \mathbb{P}^2$$

$s \in S$, $s \neq 0$, are *irreducible* nodal curves of genus $g+2$, i.e. belonging to $\mathscr{V}_{n+1,g+2}$, whose nodes specialize to the nodes of Γ' except P_1, P_2, P_3. We now consider the following commutative diagram

$$\begin{tikzcd} \mathscr{C}' \arrow[rr, "\Phi"] \arrow[dr] & & \mathscr{G}' \subset \mathbb{P}^2 \times S \arrow[dl, "\pi"] \\ & S & \end{tikzcd}$$

where ϕ' is the blow-up of $\mathscr{G}'$ along the closure of the singular locus of $\pi^{-1}(S \setminus \{0\})$. By construction the fibre over $0 \in S$ of this diagram is just

$$\varphi' \colon C' \to \Gamma' \subset \mathbb{P}^2 \times S$$

and for all $s \in S$, $s \neq 0$,

$$\Phi'(s) \colon \mathscr{C}'(s) \to \mathscr{G}'(s) \subset \mathbb{P}^2$$

is the normalization map.

We may apply the theorems of semicontinuity to deduce that for a sufficiently general $s \in S$ the divisor $D'(s)$, pullback on $\mathscr{C}'(s)$ of a line section of $\mathscr{G}'(s)$ has the same properties of the divisor D' on C'. In particular there is a Γ'' parametrized by a point of $\mathscr{V}_{n+1,g+2}$ having properties (1) and (2). By semicontinuity the irreducible component of $\mathscr{V}_{n+1,g+2}$ containing that point has the required properties. This completes the proof of the inductive step.

To prove the starting step of the induction assume first $a \geqq 2$.

In this case the induction starts at

$$(n, g) = (a+3, a+1);$$

in particular $g = n-2$. It is an elementary well known fact that every smooth irreducible curve C of genus $g \geqq 3$ can be mapped birationally onto a plane nodal curve Γ of degree $g+2$ by a complete linear system $|D|$. Such a Γ clearly satisfies conditions (1) and (2) and therefore is parametrized by a point of a component of $\mathscr{V}_{g+2,g}$ having the required properties.

If $a = 1$ the initial step of the induction is

$$(n, g) = (5, 4)$$

and it is immediate to check properties (1) and (2) for an irreducible quintic with two nodes; the theorem follows in this case as above.

If $a = 0$ the induction starts at

$$(n, g) = (5, 5).$$

Property (1) is obvious for a quintic with one node Γ; the surjectivity of $\mu_0(D)$ can be easily checked using the base point free pencil trick (cfr. [20]). Again the theorem is proved in this case.

Let's assume now that $a \leqq -1$. It is easy to check that in this case the induction starts at a couple (n, g) such that

(4.5) $$\binom{n-1}{2} - g \leqq n-5.$$

Let Γ be an irreducible nodal plane curve of degree n and genus g satisfying (4.5) (recall that $\mathscr{V}_{n,g} \neq \phi$, see § 1), and let

$$\varphi_D \colon C \to \Gamma$$

be the normalization of Γ. Denote by $\mathscr{U} \subset \mathcal{O}_{\mathbb{P}^2}$ the ideal of plane curves adjoint to Γ. Inequality (4.5) implies that the

$$\delta = \binom{n-1}{2} - g$$

nodes of Γ impose independent conditions to the curves of any order $\geqq n-5$. Therefore

$$h^1(C, \mathcal{O}(D)) = h^0(\mathbb{P}^2, \mathscr{U}(n-4)) = \binom{n-2}{2} - \delta = g - n + 2,$$

equivalently

$$h^0(C, \mathcal{O}(D)) = 3;$$

hence Γ satisfies condition (1).

As before the theorem will be completely proved if we will show that $\mu_0(D)$ is surjective.

We have

$$h^1(C, \mathcal{O}(2D)) = h^0(\mathbb{P}^2, \mathscr{U}(n-5)) = \binom{n-3}{2} - \delta = g - 2n + 5 = -a \geqq 1$$

and by Riemann-Roch theorem

$$h^0(C, \mathcal{O}(2D)) = 6.$$

The theorem is now a consequence of the following

(4.6) **Lemma.** *Let Γ be an irreducible plane nodal curve of degree n and genus g for some n and g, and*

$$\varphi_D \colon C \to \Gamma$$

its normalization. Assume that

$$h^0(C, \mathcal{O}(D)) = 3$$
$$h^0(C, \mathcal{O}(2D)) = 6$$

and that $2D$ is a special divisor. Then $\mu_0(D)$ is surjective.

Proof. Let $P\in C$ be a general point and K a canonical divisor on C. We have a commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & H^0(D-P)\otimes H^0(K-D) \arrow[r] \arrow[d, "\mu_0'"] & H^0(D)\otimes H^0(K-D) \arrow[r] \arrow[d, "\mu_0(D)"] & H^0(D\otimes\mathcal{O}_P)\otimes H^0(K-D) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & H^0(K-P) \arrow[r] \arrow[d] & H^0(K) \arrow[r] \arrow[d] & H^0(K\otimes\mathcal{O}_P) \arrow[r] \arrow[d] & 0 \\
 & \operatorname{coker}(\mu_0') \arrow[r] & \operatorname{coker}[\mu_0(D)] \arrow[r] & 0 &
\end{tikzcd}$$

The first and the second columns are exact by definition; the third column is exact because D is special and P is general. The first and second row are exact because

$$H^1(D-P)\cong H^1(D)$$

and

$$H^1(K-P)\cong H^1(K)$$

respectively. The third row is exact by linear algebra. Therefore it suffices to show that μ_0' is surjective.

Since $|D-P|$ is a base point free pencil we have, by the base point free pencil trick:

$$\begin{aligned}\dim\operatorname{Im}(\mu_0') &= \dim H^0(D-P)\otimes H^0(K-D)-h^0(K-2D+P)\\ &=2(g-n+2)-h^0(K-2D)=2(g-n+2)-(g-2n+5)\\ &=g-1=h^0(K-P)\quad\text{q.e.d.}\end{aligned}$$

§ 5. Smoothing certain reducible curves in $\mathbb{P}^r$, $r\geqq 3$

In this section we prove that certain reducible curves in $\mathbb{P}^r$ can be flatly smoothed; this result will be applied to obtain an inductive construction of smooth curves similar to the construction of irreducible plane nodal curves given in the proof of theorem (4.2).

We need some preliminary remarks.

Let γ and Γ be two smooth irreducible curves in $\mathbb{P}^r$ such that

$$\Gamma'=\Gamma\cup\gamma$$

is connected and

$$\{P_1,\dots,P_\delta\}=\Gamma\cap\gamma$$

are ordinary double points of Γ'. We denote by $\mathcal{I}_{\gamma/\Gamma'}$, $\mathcal{I}_{\Gamma/\Gamma'}$ the ideal sheaves of γ and Γ respectively in $\mathcal{O}_{\Gamma'}$ and by $\mathcal{I}_\gamma,\mathcal{I}_\Gamma,\mathcal{I}_{\Gamma'}$ those of γ,Γ,Γ' respectively in $\mathcal{O}_{\mathbb{P}^r}$.

Note that $\mathcal{I}_{\Gamma/\Gamma'}$ has support in γ and we have a natural isomorphism

$$\mathcal{I}_{\Gamma/\Gamma'}\cong\mathcal{O}_\gamma\left(-\sum_{j=1}^{\delta}P_j\right)\subset\mathcal{O}_\gamma\subset\mathcal{O}_{\Gamma'}.$$

Similarly we have

$$\mathscr{I}_{\gamma/\Gamma'} \cong \mathcal{O}_\Gamma\left(-\sum_{j=1}^{\delta} P_j\right) \subset \mathcal{O}_\Gamma \subset \mathcal{O}_{\Gamma'}.$$

(5.1) **Lemma.** *There are exact sequences on Γ'*

(i) $0 \to \mathscr{I}_{\Gamma/\Gamma'} \otimes N_{\Gamma'} \to N'_{\Gamma'} \xrightarrow{\psi} N_\Gamma \to 0$

(ii) $0 \to N_\gamma\left(-\sum_{j=1}^{\delta} P_j\right) \to \mathscr{I}_{\Gamma/\Gamma'} \otimes N_{\Gamma'} \to T^1_{\Gamma'} \to 0.$

Proof. It is easy to check that the surjective restriction

$$T_{\Gamma'} \to T_\Gamma$$

maps $\Theta_{\Gamma'}$ into Θ_Γ and therefore induces the map ψ. We have a commutative diagram

$$\begin{tikzcd} & N'_{\Gamma'} \arrow[r, "\psi"] \arrow[d] & N_\Gamma \arrow[d] \arrow[r] & 0 \\ 0 \arrow[r] & \mathscr{I}_{\Gamma/\Gamma'} \otimes N_{\Gamma'} \arrow[r, "j"] & N_{\Gamma'} \arrow[r] & N_{\Gamma'} \otimes \mathcal{O}_\Gamma \arrow[r] & 0 \end{tikzcd}$$

where the lower row is exact and the vertical maps are injective; we get an inclusion

$$\ker(\psi) \subseteq \mathscr{I}_{\Gamma/\Gamma'} \otimes N_{\Gamma'}.$$

In order to show that this is an isomorphism it suffices to check that

$$j(\mathscr{I}_{\Gamma/\Gamma'} \otimes N_{\Gamma'}) \subseteq N'_{\Gamma'}.$$

This follows from the very definition of the map

$$N_{\Gamma'} \to T^1_{\Gamma'}$$

whose kernel is $N'_{\Gamma'}$, and from the fact that $j(\mathscr{I}_{\Gamma/\Gamma'} \otimes N_{\Gamma'})$ is supported only on one branch of Γ' locally at each singular point. This proves (i) and also proves that there is an exact sequence

$$0 \to N_\Gamma \to N_{\Gamma'} \otimes \mathcal{O}_\Gamma \to T^1_{\Gamma'} \to 0.$$

After interchanging the role of γ and Γ we obtain an exact sequence

$$0 \to N_\gamma \to N_{\Gamma'} \otimes \mathcal{O}_\gamma \to T^1_{\Gamma'} \to 0$$

and from this we deduce sequence (ii) tensoring by $\mathcal{O}_\gamma\left(-\sum_{j=1}^{\delta} P_j\right)$. q.e.d.

By applying Prop. (1.6)(ii), from the above lemma one immediately deduces that if

$$H^1(N_\Gamma)=0=H^1\left(N_\gamma\left(-\sum_{j=1}^{\delta} P_j\right)\right)$$

then Γ' is smoothable. Bur for our purposes we will need to smooth certain curves which do not satisfy these conditions. Therefore we now prove the following

(5.2) **Theorem.** *Let $\Gamma\subset\mathbb{P}^r$ be a smooth irreducible non degenerate curve of genus g and degree n such that*

$$H^1(\Gamma, N_\Gamma)=0,$$

H a general hyperplane and

$$H\cap\Gamma=\{P_1,\dots,P_n\}.$$

Let $1\leqq\delta\leqq\min(n,r+2)$, γ a smooth irreducible rational curve of degree $r-1$ contained in H but not in a hyperplane of H and containing $P_1,\dots,P_\delta$ but not $P_{\delta+1},\dots,P_n$; let

$$\Gamma'=\Gamma\cup\gamma.$$

Then $\mathrm{Hilb}^r_{n+r-1,g+\delta-1}$ is smooth of dimension $\chi(n+r-1,g+\delta-1)$ at the point parametrizing Γ' and Γ' is flatly smoothable in $\mathbb{P}^r$.

Proof. By Proposition (1.6)(ii) it suffices to prove that

$$H^1(N'_{\Gamma'})=0.$$

From the exact sequence (5.1)(i) we see that for this purpose we only need to show that

$$H^1(\mathscr{I}_{\Gamma/\Gamma'}\otimes N_{\Gamma'})=0.$$

Denote by $N_{\gamma/H}$ the normal bundle of γ in H. We have

$$N_{\gamma/H}\cong M\oplus\dots\oplus M \qquad (r-2 \text{ copies}) \tag{5.3}$$

where M is an invertible sheaf of degree $r+1$ (cfr. [18]).

The inclusions $\gamma\subset H\subset\mathbb{P}^r$ induce an exact sequence

$$0\to N_{\gamma/H}\to N_\gamma\to L\to 0$$

where $\deg L=r-1$; since

$$Ext^1(L,N_{\gamma/H})=H^1(\gamma,N_{\gamma/H}\otimes L^{-1})=0$$

by (5.3), the above sequence splits and therefore

$$N_\gamma\cong N_{\gamma/H}\oplus L.$$

Consider the exact sequence

$$0\to N_\gamma\xrightarrow{\eta} N_{\Gamma'}\otimes\mathcal{O}_\gamma\to T^1_{\Gamma'}\to 0 \tag{5.4}$$

deduced from (5.1)(ii) after tensoring by $\mathcal{O}\left(\sum_{j=1}^{\delta} P_j\right)$. The middle sheaf is locally free of rank $r-1$ and therefore, since γ is rational,

$$N_{\Gamma'} \otimes \mathcal{O}_\gamma \cong M_1 \oplus \ldots \oplus M_{r-1}$$

where $M_1, \ldots, M_{r-1}$ are invertible sheaves.

Claim:

$$\eta = \begin{pmatrix} 1 & & 0 \\ & 1 & \\ 0 & & \sigma \end{pmatrix} : M \oplus \ldots \oplus M \oplus L \to M_1 \oplus \ldots \oplus M_{r-1}$$

where $\sigma \in H^0\left(\gamma, \mathcal{O}_\gamma\left(\sum_{j=1}^{\delta} P_j\right)\right)$.

Proof of the claim: at each point different from $P_1, \ldots, P_\delta$ η is the identity and therefore the matrix of η is a diagonal matrix

$$\begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_{r-1} \end{pmatrix}$$

with $\sigma_1 \ldots \sigma_{r-1} \in H^0\left(\gamma, \mathcal{O}_\gamma\left(\sum_{j=1}^{\delta} P_j\right)\right)$ by the exact sequence (5.4). For each $i = 1, \ldots, \delta$ we may find $f_1, \ldots, f_{r-2}$, $u, v \in \mathcal{O}_{\mathbb{P}^r, P_i}$ such that

$$\mathcal{I}_{H,P_i} = (u) \subset \mathcal{O}_{\mathbb{P}^r, P_i}$$
$$\mathcal{I}_{\gamma,P_i} = (f_1, \ldots, f_{r-2}, u) \subset \mathcal{O}_{\mathbb{P}^r, P_i}$$
$$\mathcal{I}_{\Gamma,P_i} = (f_1, \ldots, f_{r-2}, v) \subset \mathcal{O}_{\mathbb{P}^r, P_i}$$
$$\mathcal{I}_{\Gamma',P_i} = (f_1, \ldots, f_{r-2}, uv) \subset \mathcal{O}_{\mathbb{P}^r, P_i}.$$

The map

$$\mathrm{Hom}(\mathcal{I}_\gamma, \mathcal{O}_\gamma) = N_\gamma \to N_{\Gamma'} \otimes \mathcal{O}_\gamma = \mathrm{Hom}(\mathcal{I}_{\Gamma'}, \mathcal{O}_\gamma)$$

is defined by the inclusion $\mathcal{I}_{\Gamma'} \subset \mathcal{I}_\gamma$ and therefore locally at P_i it is defined by the matrix

$$\begin{pmatrix} 1 & & 0 \\ & 1 & \\ 0 & & v \end{pmatrix}$$

This proves the claim.

From the claim it follows that

$$M_i \cong M, \qquad i = 1, \ldots, r-2$$
$$M_{r-1} \cong L\left(\sum_{j=1}^{\delta} P_j\right).$$

We may conclude that

$$\mathcal{I}_{\Gamma/\Gamma'} \otimes N_{\Gamma'} \cong N_{\Gamma'} \otimes \mathcal{O}_\gamma\left(-\sum_{j=1}^{\delta} P_j\right) \cong M\left(-\sum_{j=1}^{\delta} P_j\right)^{\oplus r-2} \oplus L$$

and therefore

$$H^1(\mathscr{I}_{\Gamma/\Gamma'} \otimes N_{\Gamma'})=0$$

because all summands have degree $\geqq -1$. q.e.d.

(5.5) **Corollary.** *Let Γ be a smooth irreducible nondegenerate curve $\Gamma \subset \mathbb{P}^r$, $r \geqq 3$, of degree n and genus g. Then*

(i) *if $H^1(\Gamma, N_\Gamma)=0$ then for each $1 \leqq \delta \leqq \min(n, r+2)$ there exists a smooth irreducible nondegenerate curve X in $\mathbb{P}^r$ of degree $n+r-1$ and genus $g+\delta-1$ such that*

$$H^1(X, N_X)=0;$$

(ii) *if*

$$H^1(\Gamma, N_\Gamma)=0 \quad \textit{and} \quad h^0(\Gamma, \mathcal{O}(D_\Gamma))=r+1$$

where D_Γ is a hyperplane section of Γ, then for each $r \leqq \delta \leqq \min(n, r+2)$ there exists a smooth irreducible nondegenerate curve X in $\mathbb{P}^r$ of degree $n+r-1$ and genus $g+\delta-1$ such that

$$H^1(X, N_X)=0 \quad \textit{and} \quad h^0(X, \mathcal{O}(D_X))=r+1$$

where D_X is a hyperplane section of X;

(iii) *if*

$$H^1(\Gamma, N_\Gamma)=0, \qquad h^0(\Gamma, \mathcal{O}(D_\Gamma))=r+1, \qquad n \geqq r+1$$

and $\mu_0(D_\Gamma)$ is of maximal rank, then there exists a smooth irreducible nondegenerate curve X in $\mathbb{P}^r$ of degree $n+r-1$ and genus $g+r$ such that

$$H^1(X, N_X)=0, \qquad h^0(X, \mathcal{O}(D_X))=r+1$$

and $\mu_0(D_X)$ is of maximal rank.

Proof. (i) follows immediately from Theorem (5.2): X can be taken as a sufficiently small smooth deformation of $\Gamma' = \Gamma \cup \gamma$.

(ii) If we set

$$L=\mathcal{O}_H(1) \otimes \mathcal{O}_\gamma$$

we have an exact sequence

$$0 \to L\left(-\sum_{j=1}^{\delta} P_j\right) \to \mathcal{O}(D_{\Gamma'}) \to \mathcal{O}(D_\Gamma) \to 0$$

which shows that

$$h^0(D_{\Gamma'})=r+1 \quad \text{if} \quad \delta \geqq r.$$

By uppersemicontinuity we also have

$$h^0(X, \mathcal{O}(D_X))=r+1$$

when X is taken as above.

(iii) If $\delta=r+1$ by Proposition (2.3) $\mu_0(D_{\Gamma'})$ is of maximal rank and by semicontinuity $\mu_0(D_X)$ has the same property, if X is as before. q.e.d.

§ 6. Curves in $\mathbb{P}^r$, $r \geqq 3$: existence of families with the expected number of moduli

In § 4 we have observed that every component of $\mathscr{V}_{n,g}$ has *at most* the expected number of moduli.

In $\mathbb{P}^r$, $r \geqq 3$, the situation is different. One can easily find examples of components of $\mathrm{Hilb}^r_{n,g}$, for some n, g, which have a number of moduli larger than expected.

For instance, let Γ be a smooth complete intersection of $r-1$ hypersurfaces of degrees $n_1 \geqq \ldots \geqq n_{r-1} \geqq 2$. Then Γ has degree $n = n_1 \ldots n_{r-1}$ and genus

$$g = (1/2)\, n\,(n_1 + \ldots + n_{r-1} - r - 1) + 1$$

and

$$N_\Gamma \cong \mathcal{O}_\Gamma(n_1 D) \oplus \ldots \oplus \mathcal{O}_\Gamma(n_{r-1} D).$$

The map $\mu_0(D)$ coincides in this case with the natural map

$$H^0(\Gamma, \mathcal{O}(D)) \otimes H^0(\Gamma, \mathcal{O}((d-1)\, D)) \to H^0(\Gamma, \mathcal{O}(dD))$$

where $d = n_1 + \ldots + n_{r-1} - r - 1$, and it is surjective because Γ is projectively normal. Therefore, by Euler sequence,

$$h^1(T_\Gamma) = -\rho(g, r, n).$$

On the other hand suppose that $n_1 \geqq r+1$. Then

$$h^1(N_\Gamma) \neq 0$$

and, by the exact sequence (1.2),

$$\begin{aligned} \operatorname{rank}(H^0(N_\Gamma) \to H^1(\Theta_\Gamma)) &= h^1(\Theta_\Gamma) - h^1(T_\Gamma) + h^1(N_\Gamma) \\ &= 3g - 3 + \rho(g, r, n) + h^1(N_\Gamma) > 3g - 3 + \rho(g, r, n). \end{aligned}$$

One can check that $\mathrm{Hilb}^r_{n,g}$ is smooth at the point parametrizing Γ and there is an open irreducible subset V containing that point and parametrizing smooth complete intersections of the same multidegree as Γ (cfr. [21]); it follows that V has a number of moduli equal to

$$rk[H^0(N_\Gamma) \to H^1(\Theta_\Gamma)]$$

and therefore larger than expected.

Note that if $r = 3$ every smooth irreducible nondegenerate curve of degree

$$n = 2a, \qquad a \geqq 4$$

and genus

$$g = a(a-2) + 1$$

is the complete intersection of a quadric and a surface of degree a. This shows that in general there are values of $r \geqq 3, n, g$ such that there exist smooth irreducible nondegenerate curves of degree n and genus g in $\mathbb{P}^r$, but there are no components of $\mathrm{Hilb}^r_{n,g}$ having the expected number of moduli.

For every given $r \geqq 3$ what is the set of (n,g) for which there exists an open set of a component of $\mathrm{Hilb}^r_{n,g}$ having the expected number of moduli is to my knowledge unknown. I do not even know of any component of the Hilbert scheme having a number of moduli strictly *smaller* than expected.

One should note that for $r \geqq 4$ the set of (n,g) for which there exist smooth irreducible nondegenerate curves of degree n and genus g in $\mathbb{P}^r$ is not known (this set is known for $r=3$, cfr. [10]). A partial answer to this problem has been given by Gieseker in [7], where he shows the existence of such curves for all $r \geqq 4, n$ and g such that

$$g \leqq \frac{r+1}{r-1} n - \frac{(r+1)(r+3)}{r-1}.$$

We will now prove an existence theorem for smooth curves and for families with the expected number of moduli.

(6.1) **Theorem.** *Let $r \geqq 3$. For all n, g such that*

$$n - r \leqq g \leqq \frac{r(n-r)-1}{r-1}, \qquad n \geqq r+1 \tag{6.2}$$

there is a smooth irreducible nondegenerate curve $\Gamma \subset \mathbb{P}^r$ of genus g and degree n embedded by a complete linear system $|D|$ and such that

$$h^1(\Gamma, N_\Gamma) = 0$$

and $\mu_0(D)$ has maximal rank. The curve Γ is parametrized by a smooth point of an irreducible open subset V of $\mathrm{Hilb}^r_{n,g}$ *having dimension $\chi(n,g)$ and the expected number of moduli.*

Proof. The second part follows from the first in view of Propositions (1.6)(i) and (3.3). Therefore it suffices to prove the existence of Γ.

The curve Γ exists if $n=g+r$, $g \geqq 1$. In fact given any smooth irreducible curve X of genus g there is on X a nonspecial very ample divisor D of degree $g+r$ (cfr. [12]); let Γ be the image of X in $\mathbb{P}^r$ by the map defined by $|D|$. From the Euler sequence on Γ and the nonspeciality of D we get

$$H^1(\Gamma, T_\Gamma) = 0$$

and therefore

$$H^1(\Gamma, N_\Gamma) = 0$$

using the exact sequence (1.2). From the nonspeciality of D it follows also that $\mu_0(D)$ is of maximal rank.

To prove the theorem for any n, g satisfying (6.2), let

$$a = rn - g(r-1) - r^2.$$

We may obtain the given couple (n,g) from $(a+r, a)$ with a finite number of substitutions of the form

$$(m,h) \mapsto (m+r-1, h+r).$$

This follows observing that

$$g-a=r(g-n+r)$$

and

$$n-(a+r)=(r-1)(g-n+r).$$

If the theorem is true for a couple (m,h) it is also true for the couple $(m+r-1, h+r)$ by Corollary (5.5)(iii); the theorem is therefore true for (n,g) because it holds for $(a+r,a)$ by the first part of the proof (note that $a \geqq 1$ by (6.2)). q.e.d.

The inequality

$$\rho(g,r,n) \geqq 0$$

is equivalent to

$$g \leqq \frac{r+1}{r}(n-r).$$

Since

$$\frac{r+1}{r}(n-r) < \frac{r(n-r)-1}{r-1} \quad \text{for } n \geqq 2r$$

the above theorem implies that for all $r \geqq 3, n, g$ such that

$$\rho(g,r,n) \geqq 0, \qquad n \geqq 2r$$

a general curve C of genus g can be realized as a smooth non degenerate curve of degree n in $\mathbb{P}^r$ or, what is the same, that C has a very ample invertible sheaf L of degree n such that

$$h^0(C,L)=r+1.$$

If moreover

$$\rho(g,r,n)>0$$

it is known that the scheme $W_n^r(C)$ of linear systems of degree n and dimension $\geqq r$ on C is reduced irreducible of dimension $\rho(g,r,n)$ (cfr. [5]). Since the set of $L \in W_n^r(C)$ which are very ample is open, it follows that this set being not empty, is also dense in $W_n^r(C)$. Note that if $g-n+r>0$ then $n \geqq 2r$ by Clifford theorem, while for $g-n+r=0$ the existence of smooth curves is well known. We therefore have the following

(6.3) **Corollary.** *Let* $r \geqq 3$. *If*

$$\rho(g,r,n)>0$$

and C is a general curve of genus g, the set of $L \in W_n^r(C)$ which are very ample is open and dense in $W_n^r(C)$.

If

$$\rho(g,r,n)=0$$

and C is a general curve of genus g there exists at least one very ample $L \in W_n^r(C)$.

As already mentioned in the introduction, this result has been proved by Eisenbud and Harris in [4], using different techniques and in a more general form.

They give more precise information on the projective embedding defined by a general $L \in W_n^r(C)$ on a general curve C of genus g, if

$$\rho(g, r, n) > 0.$$

Moreover for

$$\rho(g, r, n) = 0$$

they prove that *every* $L \in W_n^r(C)$ is very ample on such a curve C.

§ 7. Curves in $\mathbb{P}^3$: a more general existence theorem

In this section we improve Theorem (6.1) in the case $r=3$. The method of proof is the same, but the result is better because we can find a better starting point for the induction in this case.

(7.1) **Theorem.** *For all n, g such that*

$$n-3 \leqq g \leqq 3n-18, \qquad n \geqq 9 \tag{7.2}$$

there exists a smooth irreducible nondegenerate curve $\Gamma \subset \mathbb{P}^3$ of genus g and degree n embedded by a complete linear system $|D|$ such that

$$H^1(\Gamma, N_\Gamma) = 0$$

and $\mu_0(D)$ is of maximal rank. The curve Γ is parametrized by a smooth point of an irreducible open subset V of $\mathrm{Hilb}^3_{n,g}$ *having dimension $4n$ and the expected number of moduli.*

Proof. As in the proof of Theorem (6.1), it suffices to show the existence of Γ, since the last part follows from Propositions (1.6)(i) and (3.3).

Let S be a smooth cubic surface in $\mathbb{P}^3$, H a plane section of S and Γ a smooth irreducible nondegenerate curve on S. Denote by D a plane section divisor on Γ. Suppose that

$$h^0(\Gamma, \mathcal{O}(D)) = 4. \tag{7.3}$$

Then by Proposition (2.7) $\mu_0(D)$ is of maximal rank of Γ. Suppose moreover that

$$h^0(S, \mathcal{O}_S(\Gamma - 4H)) = 0. \tag{7.4}$$

Then by the exact sequence

$$0 \to \mathcal{O}_S(-4H) \to \mathcal{O}_S(\Gamma - 4H) \to \mathcal{O}_\Gamma(K - 3D) \to 0$$

where K is a canonical divisor on Γ, we have

$$h^1(\mathcal{O}_\Gamma(3D)) = h^0(\mathcal{O}_\Gamma(K - 3D)) = 0.$$

From this and from the exact sequence

$$0 \to \mathcal{O}_\Gamma(K + D) \to N_\Gamma \to \mathcal{O}_\Gamma(3D) \to 0$$

($\mathcal{O}_\Gamma(K+D)$ is the normal bundle of Γ in S and $\mathcal{O}_\Gamma(3D)$ is the restriction to Γ of the normal bundle of S in $\mathbb{P}^3$) it follows that

$$H^1(\Gamma, N_\Gamma)=0.$$

Let now $E_1, \ldots, E_6$ be disjoint lines on S and L a smooth irreducible rational curve of S such that

$$(L^2)=1, \quad (L\cdot E_i)=0, \quad i=1,\ldots,6.$$

Consider the following linear systems on S, for $\delta \geqq 9$:

$$\begin{aligned}
\Lambda_1(\delta)&=|\delta L-(\delta-7)E_1-3E_2-3E_3-3E_4-3E_5-3E_6|\\
\Lambda_2(\delta)&=|\delta L-(\delta-7)E_1-4E_2-3E_3-3E_4-2E_5-2E_6|\\
\Lambda_3(\delta)&=|\delta L-(\delta-7)E_1-4E_2-3E_3-3E_4-3E_5-2E_6|\\
\Lambda_4(\delta)&=|\delta L-(\delta-7)E_1-4E_2-4E_3-3E_4-3E_5-2E_6|\\
\Lambda_5(\delta)&=|\delta L-(\delta-7)E_1-4E_2-4E_3-3E_4-2E_5-2E_6|\\
\Lambda_6(\delta)&=|\delta L-(\delta-7)E_1-4E_2-4E_3-4E_4-2E_5-2E_6|.
\end{aligned}$$

It is easy to check they all contain smooth irreducible nondegenerate curves. If $\Gamma \in \Lambda_i(\delta)$ is such a curve, it is likewise easy to verify that Γ satisfies both conditions (7.3) and (7.4). By the first part of the proof this means that the theorem is proved for all n, g which are respectively the degree and the genus of one of the linear systems $\Lambda_i(\delta)$. Their values are readily computed to be the following, with $\delta \geqq 9$:

$$\begin{aligned}
n&=2\delta-8, & g&=6\delta-42;\\
n&=2\delta-9, & g&=6\delta-45;\\
n&=2\delta-8, & g&=6\delta-43;\\
n&=2\delta-9, & g&=6\delta-46;\\
n&=2\delta-8, & g&=6\delta-44;\\
n&=2\delta-9, & g&=6\delta-47.
\end{aligned}$$

The above expressions give all values of n, g satisfying the following inequalities:

$$(7.5) \qquad 3n-20 \leqq g \leqq 3n-18, \quad n \geqq 9.$$

Let now (n, g) be any couple of integers satisfying inequalities (7.2). If

$$g \leqq (3/2)(n-3)$$

the theorem is true by Theorem (6.1). Therefore we may assume that

$$(3/2)(\mathrm{n}-3) < g \leqq 3n-18, \quad n \geqq 9.$$

Since

$$g-(3/2)(n-3)=g+3-(3/2)(n-1)$$

and

$$(3n-18-g)+3=3(n+2)-18-(g+3)$$

we see that the couple (n, g) can be obtained from one satisfying (7.5) after a finite number of substitutions of the form

$$(n, g) \mapsto (n+2, g+3).$$

The theorem is now a consequence of Corollary (5.5). q.e.d.

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