Lectures on
Families of Projective Varieties

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1 Beginning remarks on the notion of family

Projective varieties are distributed in *families*, obtained by suitably varying the coefficients of their defining equations. The description of the properties of such families and, in particular, of the properties of their *parameter spaces* is the central theme of these lectures. All varieties and schemes will be assumed to be defined over a fixed algebraically closed field $\mathbf{k}$.

The classical geometers knew some properties of some families of varieties. In particular they knew how to compute their dimension and sometimes they were able to describe the parameter space. The examples classically known are:

(i) *Hypersurfaces in $\mathbb{P}^r$, in particular plane curves.* A hypersurface $Y \subset \mathbb{P}^r$ of degree d is given by an equation:

$$\sum_{i_o+\cdots+i_r=d} a_{i_o,\dots,i_r} X_0^{i_0} \cdots X_r^{i_r} = 0$$

where the indices i_j are nonnegative integers and the $a_{i_o,\dots,i_r}$'s are not all zero. Since two such equations define the same hypersurface if and only if they are proportional, the family of hypersurfaces of degree d in $\mathbb{P}^r$ is parametrized by the $\binom{r+d}{r}$-tuple of coefficients up to proportionality, thus it is parametrized by the projective space $\mathbb{P}^{\binom{r+d}{r}-1}$. In more intrinsic, coordinate-free, terms we can parametrize the hypersurfaces of degree d in $\mathbb{P}^r$ by the space $\mathbb{P}[H^0(\mathbb{P}^r, \mathcal{O}(d))]$.

(ii) *Linear spaces.* The classical geometers knew very well that the family of linear spaces $\Lambda \subset \mathbb{P}^r$ of dimension n in $\mathbb{P}^r$ is parametrized by the *grassmannian* $G(n,r)$, a nonsingular projective variety of dimension $(n+1)(r-n)$. They had a deep understanding of the geometry of the grassmannian, especially in connection with problems arising in enumerative geometry.

(iii) *Curves in* $\mathbb{P}^3$. The families of nonsingular connected curves of given degree d and genus g have been extensively studied by the classical geometers (esp. Halphen and Noether). They knew many examples, how to compute the dimension of such families in several cases, and a rough description of the parameter spaces. For example it was known that every such family has dimension $\geq 4d$, and that equality holds under certain conditions. A precise description of the parameter variety was missing.

(iv) *Singular plane curves.* Since every nonsingular curve can be projected in $\mathbb{P}^2$ as a nodal curve (i.e. with ordinary double points), the families of plane curves of given degree and number of nodes have been studied considerably, especially by Severi and his school. The varieties parametrizing such curves are therefore called *Severi varieties.* The families of curves of given degree and having a fixed number of nodes and ordinary cusps were also studied due to the fact that such curves appear as branch curves of generic projections of surfaces to $\mathbb{P}^2$.

(v) *Surfaces with ordinary singularities in* $\mathbb{P}^3$. Every projective nonsingular algebraic surface can be projected in $\mathbb{P}^3$ and the image in general has a curve consisting generically of double points for the surface, having finitely many triple points, and containing some other singularities of the surface of a very precise type (pinch points). The families of such surfaces were studied because they give information on the local parameters, or *moduli*, on which the surfaces depend abstractly. The only results known were about the dimension of such families, and were not at all complete (Enriques, Segre, Castelnuovo).

Roughly these are all the classes of families of projective varieties known to the classical geometers. I did not mention *Hurwitz varieties* parametrizing branched covers of $\mathbb{P}^1$, because they are of a slightly different nature. For a long time there was no attempt to give a systematic description of families in a global way, i.e. to find a systematic way to describe the family of varieties of a given type. The construction of a parameter variety was proposed by

Bertini, and finally by Van der Waerden and Chow. We will explain their idea next.

2 The Chow variety

We will briefly discuss the construction of the Chow variety referring to the case of curves in $\mathbb{P}^3$ (the general case is analogous). The treatment closely follows the one given in [2].

Consider an irreducible curve $C \subset \mathbb{P}^3$. In $(\mathbb{P}^3)^\vee \times (\mathbb{P}^3)^\vee$ consider the set $V(C)$ of all pairs of planes (π, π') such that $\pi \cap \pi' \cap C \neq \emptyset$. This is clearly a subvariety of dimension 5 in $(\mathbb{P}^3)^\vee \times (\mathbb{P}^3)^\vee$, and it is therefore defined by a bihomogeneous polynomial $f_C(u, v)$ in the variables $u = (u_0, \dots, u_3)$ and $v = (v_0 \dots, v_3)$ which is determined (up to a multiplicative scalar) by C, and is called the *Chow form* of C. By obvious reasons of symmetry, the degree of $f_C(u, v)$ in the two sets of variables is the same and it is easy to see that it coincides with the degree d of C. Therefore to the curve C we associate the point

$$[f_C(u, v)] \in \mathbb{P}(V_{d,d})$$

in the projective space associated to the vector space

$$V_{d,d} := H^0((\mathbb{P}^3)^\vee \times (\mathbb{P}^3)^\vee, \mathcal{O}(d, d))$$

of bihomogeneous polynomials in u, v of bidegree (d, d). We call $[f_C(u, v)]$ the *Chow point* of C.

More generally, if C is an algebraic effective cycle of dimension 1 in $\mathbb{P}^3$, namely if C is the sum of irreducible components $C_1, \dots, C_n$, counted with multiplicities $m_1, \dots, m_n$, (the degree of the cycle is, by definition, $\sum m_i d_i$, where $d_1, \dots, d_n$ are the degrees of $C_1, \dots, C_n$ respectively), then the Chow form of C is, by definition, $f_C = \prod f_{C_i}^{m_i}$, and accordingly the Chow point of C is defined to be $[f_C]$. The crucial facts about Chow points are:

(1) the map from the set of algebraic effective cycles to the set of points of $\mathbb{P}(V_{d,d})$ is injective, namely two distinct cycles have non proportional Chow forms;

(2) a polynomial in $V_{d,d}$ is the Chow form of a cycle of degree d if and only if its coefficients verify a suitable set of homogeneous equations.

In other words the subset $V(d,1,3)$ of $\mathbb{P}(V_{d,d})$ consisting of Chow points of algebraic cycles of dimension 1 and degree d of $\mathbb{P}^3$ is an algebraic variety (by property (2)) and is in fact a parameter space for the family of all cycles of dimension one and degree d in $\mathbb{P}^3$ (by property (1)).

As mentioned above, this construction easily generalizes to algebraic cycles of degree d and dimension n in $\mathbb{P}^r$. Denoting by $V^r(d,\dots,d)$ the space of plurihomogeneous polynomials of multidegree d in $n+1$ sets of $r+1$ variables, we obtain a subvariety

$$V(d,n,r) \subset \mathbb{P}(V^r(d,\dots,d))$$

which is a parameter space for such cycles. This variety is the so called *Chow variety.*

Example 2.1 By the same construction, the Chow variety appears to be a generalization of the Grassmann varieties. In fact it is easy to see that $V(1,n,r)$ is isomorphic to $G(n,r)$ and also that $V(d,r-1,r)$, the Chow variety parametrizing hypersurfaces of degree d in $\mathbb{P}^r$, is isomorphic to the projective space $\mathbb{P}^{\binom{r+d}{r}-1}$ described in Example (i) of §1.

In order to explain how these isomorphisms arise, let us consider the case of the Grassmannian of lines in $\mathbb{P}^r$. Such a line can be assigned in the form $\ell = \langle p_0, p_1 \rangle \subset \mathbb{P}^r$, as generated by two distinct points

$$p_0 = [p_{00}, \dots, p_{0r}], \qquad p_1 = [p_{10}, \dots, p_{1r}]$$

The matrix of their homogeneous coordinates

$$P = \begin{pmatrix} p_{00} & p_{10} \\ p_{01} & p_{11} \\ \cdots & \cdots \\ p_{0r} & p_{1r} \end{pmatrix}$$

is determined up to right-multiplication by an element of $\mathrm{GL}_2(\mathbf{k})$, and the homogeneous coordinates of the corresponding point in $G(1,r) \subset \mathbb{P}^{\binom{r+1}{2}}$ (the so called *Plücker coordinates* of ℓ) are given by the 2×2 minors of the matrix P.

The Chow form of the line ℓ is a bilinear polynomial in the variable coefficients $(u_0,\dots,u_r)$, $(v_0,\dots,v_r)$ of the equations of a pair of hyperplanes

$$H_0 : u_0X_0 + \cdots + u_rX_r = 0, \qquad H_1 : v_0X_0 + \cdots + v_rX_r = 0$$

Such polynomial is obtained by imposing the condition that $\ell \cap H_0 \cap H_1 \neq \emptyset$. Therefore, letting

$$A = \begin{pmatrix} u_0 & u_1 & \dots & u_r \\ v_0 & v_1 & \dots & v_r \end{pmatrix}$$

the Chow form of ℓ is just given by the condition $\det(AP) = 0$, that is:

$$H_0(p_0)H_1(p_1) - H_0(p_1)H_1(p_0) = 0$$

The left side is a sqew-symmetric bilinear form in u, v whose coefficients are just the Plücker coordinates of ℓ, and this circumstance gives us the isomorphism of the Chow variety with the Grassmannian.

The case of hypersurfaces is similar, and is left as an exercise (*hint:* start from the case of plane curves).

An important remark has to be made at this point. Since the Chow variety is a projective variety, it has only finitely many irreducible components. This is what the classical geometers used to express by saying that *the algebraic varieties of given degree and dimension in a fixed projective space fill up finitely many complete irreducible algebraic families* (here "complete" means, of course, not properly contained in some other irreducible algebraic family). This property is what we call today *boundedness.* The classification of varieties in a projective space consisted for them in describing these families, namely their number, dimensions, intersections, etc. For a great deal this is still our concept of classification: the only difference is that, as we shall see, we will use another parameter space.

Another basic remark about the Chow variety is that it is a rather course object. In fact first of all the definition of Chow point makes sense only for purely dimensional cycles. If one considers varieties having more than one component, at least two of which with different dimensions, the construction of the Chow point for these has no meaning, unless one decides to forget all lower dimensional components (and this is what is usually done). In other words the Chow point of a variety is uneffected by lower dimensional components.

Secondly, the Chow variety actually parametrizes cycles and not schemes. In fact the construction of the Chow point does not take into consideration non reduced scheme structure apart from multiplicities of components.

Both the above characteristics could appear to be, at first glance, an advantage of the Chow construction: if we want to study curves in $\mathbb{P}^3$, for instance, why should we look at some complicated scheme structure instead of looking only at reduced curves? The point is that non reduced scheme structures unavoidably come into the picture even when we start looking only at smooth reduced curves. The following example gives an idea of how this occurs.

Example 2.2 Consider the pair of skew lines in $I\!P^3$ defined by the equations

$$\begin{aligned} X_2 &= X_3 &&= 0 \\ X_1 &= X_3 - tX_0 &&= 0 \end{aligned}$$

where $t \neq 0$ is a complex number. The ideal of this reducible curve is generated by the polynomials

$$X_1X_2, X_1X_3, X_2(X_3 - tX_0), X_3(X_3 - tX_0)$$

As t varies, approaching 0, we get a one parameter family of skew lines which approaches the scheme defined by the polynomials

$$X_1X_2, X_1X_3, X_2X_3, X_3^2$$

The support of this scheme is the union of the two lines

$$\begin{aligned} X_2 &= X_3 &&= 0 \\ X_1 &= X_3 &&= 0 \end{aligned}$$

but the scheme itself has an embedded point at their intersection $[0, 0, 0, 1]$, which makes it a non planar scheme. The embedded point keeps track of the fact that the scheme is in fact a limit of non coplanar lines.

Let us consider, on the other hand, the one parameter family of conics given by the equations

$$X_3 = X_1X_2 - tX_0^2 = 0$$

As t approaches 0, the conics of the family approach the conic defined by the equations

$$X_3 = X_1X_2 = 0$$

This is exactly the support of the limit of the two skew lines above, but this time the limit is planar, according to the fact that all the conics of the family are such.

From the Chow variety point of view we have that the Chow points of the curves of the two families above tend, when t goes to 0, to the Chow point of the same cycle, although the geometric situation is completely different in the two cases.

Another unpleasant feature of the Chow variety is that, apart from its existence, it is not easy to say something more about it, and in fact it seems in general a rather untractable object. For instance, even in the simplest cases ($n = 1$ and low d) the following questions appear to be fairly difficult:

(1) What is the dimension of a component of the Chow variety?

(2) Given a point of the Chow variety $V(d, n, r)$ (i.e. a cycle of dimension n and degree d in $\mathbb{P}^r$) what is the tangent space to $V(d, n, r)$ at that point? In particular, is it possible to recognize singular from non singular points of the Chow variety?

Summarizing, the Chow variety shows that the notion of family can be given a general meaning, but the insight obtained is still unsatisfactory. Anyway, this is what the geometers knew on this subject at the time when Grothendieck and the language of schemes came on stage (mid 1950's).

3 The modern notion of family

If we look back at the previous discussion, we see that we did not define what we precisely mean by a family. If we want to go beyond the classical naive point of view we need to give a precise definition.

Definition 3.1 *A* family of projective varieties *is a projective morphism of schemes:*

$$f : \mathcal{X} \to S$$

The members of the family are the fibres of f and S is the parameter scheme. The morphism f is required to satisfy the extra condition of being flat*, or even* smooth *if we want the fibres to be nonsingular.*

A family of closed subschemes of $\mathbb{P}^r$ *is a family f as above which is part of a commutative diagram of morphism:*

$$\begin{tikzcd} \mathcal{X} \arrow[r, hook] \arrow[dr, "f"'] & \mathbb{P}^r \times S \arrow[d] \\ & S \end{tikzcd} \tag{1}$$

realizing $\mathcal{X}$ as a closed subscheme of the product $\mathbb{P}^r \times S$ and where the vertical arrow is the second projection. We will say that f is a family of deformations *of $X \subset \mathbb{P}^r$ if X is a fibre of f in (1), i.e. if there is a* $\mathbf{k}$*-rational point* $s : \mathrm{Spec}(\mathbf{k}) \to S$ *of S and a diagram:*

$$\begin{tikzcd} X \arrow[r] \arrow[d] & \mathcal{X} \arrow[r, hook] \arrow[d, "f"] & \mathbb{P}^r \times S \arrow[dl] \\ \mathrm{Spec}(\mathbf{k}) \arrow[r, "s"] & S & \end{tikzcd} \tag{2}$$

which induces an isomorphism of X with the fibre $\mathcal{X}(s) \subset \mathbb{P}^r$. The family (1) will be called trivial *if $\mathcal{X} = X \times S$ for some closed subvariety $X \subset \mathbb{P}^r$. In this case it will be also called a* trivial deformation of X in $\mathbb{P}^r$.

The condition of flatness is required in order to exclude certain undesirable morphisms f from the definition (we will see examples illustrating this fact). Recall that for families of projective schemes as in (1) flatness implies that the Hilbert polynomial of the fibres is constant; if moreover S is an *algebraic* scheme then flatness is equivalent to the Hilbert polynomial of the fibres being constant.

In the above definition we are free to replace $\mathbb{P}^r$ by any projective scheme Z: we will obtain the definition of family of closed subschemes of Z:

$$\begin{tikzcd} X \arrow[r, hook] \arrow[dr, "f"'] & Z \times S \arrow[d] \\ & S \end{tikzcd} \tag{3}$$

This generalization is very natural and will allow us to consider for example families of projective curves contained in a given surface Z, etc.

Example 3.2 The product $\mathbb{P}^r \times \mathbb{P}^r$ can be viewed as the trivial deformation of $\mathbb{P}^r$ in $\mathbb{P}^r$:

$$\begin{tikzcd} \mathbb{P}^r \times \mathbb{P}^r \arrow[r, equal] \arrow[dr, "pr_2"'] & \mathbb{P}^r \times \mathbb{P}^r \arrow[d, "pr_2"] \\ & \mathbb{P}^r \end{tikzcd}$$

Example 3.3 The family of hypersurfaces of degree d in $\mathbb{P}^r$ is obtained by considering the polynomial ring

$$\mathbf{k}[\underline{a}] = \mathbf{k}[\ldots, a_{i_0,\ldots,i_r}, \ldots]_{i_0+\cdots+i_r=d}$$

and $I\!P^{\binom{d+r}{r}-1} = \mathrm{Proj}(\mathbf{k}[\underline{a}])$. Then

$$\mathcal{X} = V(\sigma) \subset I\!P^r \times I\!P^{\binom{d+r}{r}-1}$$

where $\sigma = \sum a_{i_o,\ldots,i_r} X_0^{i_0} \cdots X_r^{i_r}$ is a section of the invertible sheaf $\mathcal{O}_{I\!P^r \times I\!P^{\binom{d+r}{r}-1}}((d,1)$, called the *universal polynomial.*

Example 3.4 The two families considered in Example (2.2) are respectively:

$$\mathcal{X} = \mathrm{Proj}\left(\frac{\mathbf{k}[t, X_0, \ldots, X_3]}{(X_1X_2, X_1X_3, X_2(X_3 - tX_0), X_3(X_3 - tX_0))}\right) \subset I\!P^3 \times \mathbf{A}^1 \quad (4)$$

and

$$\mathcal{Y} = \mathrm{Proj}\left(\frac{\mathbf{k}[t, X_0, \ldots, X_3]}{(X_3, X_1X_2 - tX_0^2)}\right) \subset I\!P^3 \times \mathbf{A}^1 \quad (5)$$

These two families have no fibres in common because the Hilbert polynomials are different. They are:

$$p(T) = \begin{cases} 2T+2 & \text{for the first family} \\ 2T+1 & \text{for the second family} \end{cases}$$

This example already shows the advantage of the new definition of family versus the approach which uses the Chow variety.

Example 3.5 Consider a pencil of lines in $I\!P^2$:

$$\lambda_0 L_1(X_0, X_1, X_2) - \lambda_1 L_0(X_0, X_1, X_2) = 0$$

where L_0 and L_1 are non-proportional linear forms. This can be viewed as the equation of a hypersurface Λ in

$$I\!P^2 \times I\!P^1 = \mathrm{Proj}(\mathbf{k}[X_0, X_1, X_2]) \times \mathrm{Proj}(\mathbf{k}[\lambda_0, \lambda_1])$$

and the diagram:

$$\begin{tikzcd} \Lambda \arrow[r, hook] \arrow[dr, "f"'] & I\!P^2 \times I\!P^1 \arrow[d] \\ & I\!P^1 \end{tikzcd}$$

represents the family of lines swept by the pencil. The fibres of f are the lines of the pencil. Note that even though the fibres of f are all isomorphic to $I\!P^1$, Λ is not isomorphic to $I\!P^1 \times I\!P^1$. In fact composing the inclusion $\Lambda \subset I\!P^2 \times I\!P^1$ with the first projection we obtain a birational morphism

$$\Lambda \to I\!P^2$$

which identifies Λ with the blow-up of $I\!P^2$ at the base point of the pencil. The morphism f makes Λ isomorphic to the rational ruled surface

$$I\!P(\mathcal{O}_{I\!P^1} \oplus \mathcal{O}_{I\!P^1}(-1))$$

Example 3.6 Consider the families of plane conics (in affine coordinates x, y):

$$C_t : x^2 + y^2 + 2t(x + y) + t^2 = 0$$

and

$$D_u : x^2 + y^2 + 2u(x + y) + 2u^2 = 0$$

Both families are deformations of $C_0 = D_0$, the reducible conic $x^2 + y^2 = 0$.

In the first family C_t is nonsingular for all $t \neq 0$ and all C_t are tangent to the lines $x = 0$ and $y = 0$.

In the second family all the conics D_u are reducible in two distinct lines, meeting at the variable point $(-u, -u)$.

If we remove the term t^2 and the term $2u^2$ from the respective equations, we obtain the same linear pencil:

$$x^2 + y^2 + 2\lambda(x + y) = 0$$

as the linear approximation of both families.

4 In search of the universal family

If we consider a family of projective schemes (1) and a morphism of schemes $\psi : V \to S$, we obtain a new family by taking the *pullback* of (1) by ψ:

$$\begin{tikzcd} \mathcal{X} \times_S V \arrow[r, hook] \arrow[dr] & I\!P^r \times V \arrow[d] \\ & V \end{tikzcd}$$

Therefore from a family (1) we can obtain many new ones by taking pullbacks. For instance, choosing a fibre of (1), i.e. a point of S, is a special case of pullback, corresponding to a morphism $\mathrm{Spec}(\mathbf{k}) \to S$.

The operation of pullback is nothing but a change of parameters: the fibres of the new family are (some of) the fibres of the old one, and we cannot hope to get from (1) by pullback a family containing as fibres subvarieties of $\mathbb{P}^r$ which are not already fibres of (1).

These remarks suggest the following question: can we hope to find a family from which all other families of subvarieties of $\mathbb{P}^r$ can be obtained by pullback?

As it stands, this question is not well-posed: we should impose some restriction on the *type* of subvarieties we want to obtain, otherwise we are asking too much because such family, if it exists, will contain all subvarieties among its fibres. In view of the flatness condition it is natural to restrict the fibres to have a fixed Hilbert polynomial. The new formulation of the question is the following:

Question: Given r and a Hilbert polynomial $p(T)$, does there exist a family of closed subschemes of $\mathbb{P}^r$:

$$\begin{tikzcd} \mathcal{U} \arrow[r, hook] \arrow[dr] & \mathbb{P}^r \times \mathcal{H}^r_{p(T)} \arrow[d] \\ & \mathcal{H}^r_{p(T)} \end{tikzcd} \tag{6}$$

such that every other family (1) of closed subschemes of $\mathbb{P}^r$ having Hilbert polynomial equal to $p(T)$ can be obtained as a pullback of (6) by a *unique* morphism $\varphi : S \to \mathcal{H}^r_{p(T)}$?

Definition 4.1 *If it exists, $\mathcal{H}^r_{p(T)}$ is called the* Hilbert scheme *of $\mathbb{P}^r$ relative to $p(T)$ and (6) is the* universal family*.*

Are we again asking too much? For instance, is the requirement of uniqueness of φ too strong? The answer is NO: in fact $\mathcal{H}^r_{p(T)}$ and the family (6) exist and are uniquely determined by the defining conditions.

Sometimes we will need to consider families having a weaker property as follows.

Definition 4.2 A family

$$\begin{tikzcd} \mathcal{X} \arrow[r, hook] \arrow[dr] & \mathbb{P}^r \times U \arrow[d] \\ & U \end{tikzcd} \tag{7}$$

of closed subschemes of $\mathbb{P}^r$ having Hilbert polynomial $p(T)$ is called *complete* if every other family (1) of closed subschemes of $\mathbb{P}^r$ having Hilbert polynomial equal to $p(T)$ can be obtained as a pullback of it by a morphism $\varphi : S \to U$.

The property of completeness does not require uniqueness of the morphism φ. If a family is universal it is also complete, but the converse is not necessarily true. Given a complete family (7) and a universal family (6) there are a unique morphism $\varphi : U \to \mathcal{H}^r_{p(T)}$, and a morphism $\psi : \mathcal{H}^r_{p(T)} \to U$, not necessarily unique. If also (7) is universal then ψ is unique as well and φ and ψ are isomorphisms inverse to each other. This means that $\mathcal{H}^r_{p(T)}$ is unique up to isomorphism.

Definition 4.2 has a local counterpart (*local completeness*) which is more commonly used. It is formulated by means of the notion of smoothness of a morphism. We will not use it in these lectures. We refer to [6] for details.

Before stating a precise general result of existence of universal families, we will discuss some special cases.

Exercise 4.3 Consider the diagram:

$$\begin{tikzcd} \Delta \arrow[r, hook] \arrow[dr] & \mathbb{P}^r \times \mathbb{P}^r \arrow[d, "pr_2"] \\ & \mathbb{P}^r \end{tikzcd}$$

where Δ is the diagonal. Prove that this is the universal family of $\mathbb{P}^r$ w.r. to the Hilbert polynomial $p(T) = 1$. Then $\Delta \cong \mathbb{P}^r$ is the Hilbert scheme of points of $\mathbb{P}^r$.

Exercise 4.4 Let $\mathbb{P}^r = \mathrm{Proj}(\mathbf{k}[X_0, \ldots, X_r])$, and let the dual space be $\mathbb{P}^{r\vee} = \mathrm{Proj}(\mathbf{k}[u_0, \ldots, u_r])$. Prove that the hypersurface

$$H : u_0X_0 + \cdots + u_rX_r = 0$$

of $\mathbb{P}^r \times \mathbb{P}^{r\vee}$ defines a universal family of hyperplanes of $\mathbb{P}^r$:

$$\begin{tikzcd} H \arrow[r, hook] \arrow[dr, "\pi"'] & \mathbb{P}^r \times \mathbb{P}^{r\vee} \arrow[d, "pr_2"] \\ & \mathbb{P}^{r\vee} \end{tikzcd}$$

Therefore the Hilbert scheme of $\mathbb{P}^r$ w.r. to $p(T) = \binom{T+r-1}{r-1}$ is the dual projective space $\mathbb{P}^{r\vee}$. The bilinear polynomial $u_0X_0 + \cdots + u_rX_r$ is a section of

$$\mathcal{O}_{\mathbb{P}^r \times \mathbb{P}^{r\vee}}(1,1) := pr_1^*\mathcal{O}_{\mathbb{P}^r}(1) \otimes pr_2^*\mathcal{O}_{\mathbb{P}^{r\vee}}(1)$$

called the *universal section*.

Example 4.5 (Hypersurfaces) The family of Example 3.3 is the universal family of $\mathbb{P}^r$ relative to the Hilbert polynomial of hypersurfaces of degree d, which is

$$p(T) = \binom{T+r}{r} - \binom{T+r-d}{r} = \frac{d}{(r-1)!}T^{r-1} + \cdots$$

For the proof of this fact we need first to check that every closed subscheme of $\mathbb{P}^r$ with Hilbert polynomial $p(T)$ is a hypersurface of degree d. We refer to [6], p. 207, for the proof. Therefore every family of closed subschemes of $\mathbb{P}^r$ having Hilbert polynomial $p(T)$ is a family of hypersurfaces of degree d.

For simplicity assume that we have such a family parametrized by an affine algebraic scheme $S = \mathrm{Spec}(A)$. It can be given by an equation $F(X_0, \ldots, X_r) = 0$ where

$$F(X_0, \ldots, X_r) = \sum_{i_o+\cdots+i_r=d} \alpha_{i_o,\ldots,i_r} X_0^{i_0} \cdots X_r^{i_r}$$

with $\alpha_{i_o,\ldots,i_r} \in A$. Then there is induced a morphism $\varphi : S \to \mathbb{P}^{\binom{d+r}{r}-1}$ by the rule:

$$s \mapsto [\ldots, \alpha_{i_o,\ldots,i_r}(s), \ldots]$$

The pullback of the family 3.3 by φ is the given family. The uniqueness follows from the fact that the fibres of 3.3 are in 1–1 correspondence with the hypersurfaces of degree d.

Therefore we see that the Hilbert scheme of hypersurfaces of degree d in $\mathbb{P}^r$ exists and is isomorphic to $\mathbb{P}^{\binom{d+r}{r}-1}$.

Example 4.6 Let's consider the case $r = 2$ and $p(T) = 2$, i.e. families of subschemes of length 2, in particular pairs of points in $\mathbb{P}^2$. Every subscheme $X \subset \mathbb{P}^2$ of length 2 is the complete intersection of a line and a conic:

$$L(X_0, X_1, X_2) = Q(X_0, X_1, X_2) = 0 \tag{8}$$

such that L is not a component of Q. In case the conic Q is tangent to the line L we obtain a subscheme of length 2 supported at one point. We can describe a family parametrizing all $X \subset \mathbb{P}^2$ of degree 2 as follows.

Consider $\mathbb{P}^2_u = \mathrm{Proj}(\mathbf{k}[u_0, u_1, u_2])$, $\mathbb{P}^5_a = \mathrm{Proj}(\mathbf{k}[a_{00}, a_{01}, a_{02}, a_{11}, a_{12}, a_{22}])$ and the closed subscheme $W \subset \mathbb{P}^2_u \times \mathbb{P}^5_a$ defined by

$$W = \{([u],[a]) : \mathrm{rk}(A) \leq 3\}$$

where

$$A = \begin{pmatrix} a_{00} & a_{01} & a_{02} & a_{11} & a_{12} & a_{22} \\ u_0 & u_1 & u_2 & 0 & 0 & 0 \\ 0 & u_0 & 0 & u_1 & u_2 & 0 \\ 0 & 0 & u_0 & 0 & u_1 & u_2 \end{pmatrix}$$

Then $([u],[a]) \in W$ if and only if the line

$$L : u_0X_0 + u_1X_1 + u_2X_2 = 0$$

is a component of the conic

$$Q : \sum a_{ij}X_iX_j = 0$$

Let $S = (\mathbb{P}^2_u \times \mathbb{P}^5_a)\backslash W$ and consider the closed subscheme $\mathcal{X} \subset \mathbb{P}^2 \times S$ defined by

$$\begin{aligned} & u_0X_0 + u_1X_1 + u_2X_2 = 0 \\ & \textstyle\sum a_{ij}X_iX_j = 0 \end{aligned} \tag{9}$$

We obtain a flat family of subschemes of degree 2 of $\mathbb{P}^2$:

$$\begin{tikzcd} \mathcal{X} \arrow[r, hook] \arrow[dr, "f"'] & \mathbb{P}^2 \times S \arrow[d] \\ & S \end{tikzcd} \tag{10}$$

Since any pair of polynomials as in (8) occurs in (9) for some $([u],[a]) \in S$ we deduce that the family (10) contains all subschemes of degree 2 of $\mathbb{P}^2$. It follows easily from this fact that (10) is a complete family. But the family is not universal because there are pairs of different points of S having the same fibre (we leave it to the reader to check this).

Exercise 4.7 Generalizing the construction of Example 4.6 describe a complete family of complete intersections $X \subset \mathbb{P}^r$ of dimension $0 \leq n = r-2$ and multidegree $\mathbf{d} = (d_1, d_2)$.

5 Grassmannians

In §1 we mentioned the grassmannians $G(n,r)$ as examples of classically well known parameter spaces. In §2 we proved (in a special case) that $G(n,r)$ coincides with the Chow variety of linear spaces $V(1,n,r)$. In this section we will reconsider the grassmannians with the purpose of understanding how close they are to the Hilbert scheme $\mathcal{H}^r_{\binom{T+n}{n}}$. Let's start by recalling the description of the grassmannian by means of an affine cover.

A linear subspace $\Lambda \subset I\!P^r$ of dimension n can be assigned as

$$\Lambda = \langle p_0, \dots, p_n \rangle$$

the span of $n+1$ independent points whose homogeneous coordinates can be displayed in a $(n+1)(r+1)$ matrix

$$M = \begin{pmatrix} p_{00} & p_{01} & \cdots & p_{0r} \\ p_{10} & p_{11} & \cdots & p_{1r} \\ \cdots & \cdots & \cdots & \cdots \\ p_{n0} & p_{n1} & \cdots & p_{nr} \end{pmatrix}$$

This matrix is determined by Λ up to multiplying by $A \in GL_{n+1}(\mathbf{k})$ on the left. Assume that, corresponding to a multiindex $I = \{i_0, \dots, i_n\}$, $0 \le i_0 < \cdots < i_n \le r$, the square submatrix M_I, consisting of the columns belonging to I, is invertible. Then $(M_I^{-1}M)_I = \mathbf{I}_{n+1}$ and the remaining $(n+1)(r+1)$ entries are uniquely determined by Λ and, conversely, they determine Λ, so that they can be taken as local coordinates of Λ. Since each Λ has $\det(M_I) \neq 0$ for some I, we see that $G(n,r)$ is covered by $\binom{r+1}{n+1}$ copies of the affine space $\mathbf{A}^{(n+1)(r-n)}$, each of which we denote by U_I as I varies among all the multiindices. If $\Lambda \in U_I \cap U_K$, then the change of coordinates from U_I to U_K is given by left-multiplying by M_K^{-1} the matrix M associated to Λ and having $M_I = \mathbf{I}_{n+1}$.

We can suitably modify the previous argument, as follows, to give a description of the *scheme structure* on $G(n,r)$.

Let $I = \{i_0, \dots, i_n\}$, $0 \le i_0 < \cdots < i_n \le r$ be a multiindex. Then

$$U_I = \mathrm{Spec}(\mathbf{k}[u_{ij}])$$

where the u_{ij}'s are indeterminates, and

$$0 \le i \le n, \qquad 0 \le j \le r, \qquad j \notin I$$

Define a $(n+1)\times(r+1)$ matrix by

$$M(I) = (m_{ij})$$

by

$$m_{ij} = \begin{cases} u_{ij} & \text{if } j \notin I \\ \delta_{im} & \text{if } j = i_m \end{cases}$$

where δ_{im} is the Kronecker symbol. Let $K = \{k_0, \dots, k_n\}$, $0 \le k_0 < \cdots < k_n \le r$, be another multiindex,

$$U_K = \operatorname{Spec}(\mathbf{k}[v_{ih}])$$

with

$$0 \le i \le n, \qquad 0 \le h \le r, \qquad h \notin K$$

Let $u_K = \det(M(I)_K) \in \mathbf{k}[u_{ij}]$ and $v_I = \det(M(K)_I)$. consider the open subsets

$$U_{IK} := \operatorname{Spec}\left(\mathbf{k}[u_{ij}]_{u_K}\right) \subset U_I, \qquad U_{KI} := \operatorname{Spec}\left(\mathbf{k}[v_{ih}]_{v_I}\right) \subset U_K$$

Then we obtain an isomorphism

$$U_{IK} \xrightarrow{\sim} U_{KI}$$

by means of the isomorphism

$$\mathbf{k}[v_{ih}]_{v_I} \xrightarrow{\sim} \mathbf{k}[u_{ij}]_{u_K}$$

obtained by mapping each v_{ih} to the corresponding entry of the matrix $M(I)_K^{-1}M(I)$. This gives the scheme structure on the grassmannian. As a consequence of this construction we see that $G(n,r)$ *is nonsingular of dimension* $(n+1)(r-n)$ *and rational.*

After choosing an ordering on the set of multiindices $I \subset \{0, \dots, r\}$ such that $|I| = n+1$ we can map

$$G(n,r) \longrightarrow \mathbb{P}^{\binom{r+1}{n+1}-1}$$

by the rule

$$\Lambda \longmapsto [\dots, \det(M_I), \dots]$$

This is the *Plücker embedding.* It can be proved that this is a closed embedding. In particular $G(n,r)$ *is a projective variety.* The coordinates

$[\ldots, \det(M_I), \ldots]$ of the Plücker image of a space Λ are called *Plücker coordinates* of Λ.

It will be convenient to consider, more generally, the grassmannian $G_n(V)$ of $(n+1)$-dimensional vector subspaces of a $\mathbf{k}$-vector space V of dimension $r+1$ or, equivalently, of n-dimensional linear subspaces of the projective space $\mathbb{P} = \mathbb{P}(V)$.

The simplest case of grassmannian is a projective space $\mathbb{P} = \mathbb{P}(V) = G_0(V)$. Since $V^\vee = H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(1))$, we have a surjection (evaluation of sections):

$$ev : V^\vee \otimes \mathcal{O}_{\mathbb{P}} \longrightarrow \mathcal{O}_{\mathbb{P}}(1) \longrightarrow 0 \tag{11}$$

such that for each $p \in \mathbb{P}$ the induced surjection

$$ev(p) : V^\vee \longrightarrow \mathcal{O}_{\mathbb{P}}(1)(p) \cong \mathbf{k}$$

has kernel consisting of the linear forms on V vanishing on p. The map ev describes the invertible sheaf $\mathcal{O}(1)$ as a *tautological quotient* of the trivial sheaf $V^\vee \otimes \mathcal{O}_{\mathbb{P}}$. Therefore the surjection (11) gives an equivalent (and better) way to describe $\mathbb{P}(V)$ as the set of 1-dimensional *quotients* of the *dual* of V. Equivalently, $\mathbb{P}(V)$ can be viewed as the set of *stars of codimension 1 of hyperplanes.*

On the other hand such description is equivalent to the scheme-theoretic definition:

$$\mathbb{P}(V) = \mathrm{Proj}(S^*(V^\vee))$$

where $S^*(V^\vee) = \oplus_{\rho \geq 0} S^\rho(V^\vee)$ is the symmetric algebra of $V^\vee$.

A similar convention is used for the projective bundle $\mathbb{P}(\mathcal{F})$ associated to a locally free sheaf $\mathcal{F}$ on a scheme X. In particular, if we take $X = \mathbb{P}$ and $\mathcal{F} = V \otimes \mathcal{O}_{\mathbb{P}}$ we obtain

$$\mathbb{P} \times \mathbb{P} = \mathbb{P}(V \otimes \mathcal{O}_{\mathbb{P}}) = \mathrm{Proj}[S^*(V^\vee)] \times \mathbb{P}$$

and the surjection (11) induces a section

$$\mathbb{P} = \mathbb{P}(\mathcal{O}_{\mathbb{P}}(-1)) \xrightarrow{\sigma} \mathbb{P} \times \mathbb{P}$$

of the second projection $p_2 : \mathbb{P} \times \mathbb{P} \to \mathbb{P}$, which identifies $\sigma(\mathbb{P})$ with the diagonal Δ. In this way we recover the universal family of Exercise 4.3.

The description of grassmannians is similar. One can describe the grassmannian $G_n(V)$ of $(n+1)$-dimensional *subspaces* of V as the set of $(n+1)$-dimensional *quotients* of $V^\vee$, or, what is the same, as the set of *stars of codimension* $n+1$ *of hyperplanes.*

In analogy with the case of projective space, this description suggests that on the grassmannian $G = G_n(V)$ there should be a locally free sheaf $\mathcal{F}$ of rank $n+1$ and a surjection:

$$ev : V^\vee \otimes \mathcal{O}_G \longrightarrow \mathcal{F} \longrightarrow 0$$

such that for each $\lambda \in G$ the kernel of the induced surjection

$$V^\vee \longrightarrow \mathcal{F}(\lambda) \cong \mathbf{k}^{n+1}$$

is the star of hyperplanes vanishing on the space parametrized by λ. $\mathcal{F}$ will be called the *tautological quotient sheaf.*

An explicit construction of this sheaf can be made as follows. Let's consider the case of $G(n,r)$ to which we can always reduce after choosing a basis of V. For each multiindex $I = \{i_0, \ldots, i_n\}$, $0 \le i_0 < \cdots < i_n \le r$, we can associate to the matrix $M(I)$ a homomorphism of free $\mathbf{k}[u_{ij}]$-modules:

$$\mathbf{k}[u_{ij}]^{r+1} \xrightarrow{M(I)} \mathbf{k}[u_{ij}]^{n+1}$$

which defines a homomorphism of free $\mathcal{O}_{U_I}$-modules:

$$\eta_I : \mathcal{O}_{U_I}^{r+1} \longrightarrow \mathcal{O}_{U_I}^{n+1}$$

Given another multiindex $K = \{k_0, \ldots, k_n\}$, $0 \le k_0 < \cdots < k_n \le r$, and the corresponding

$$\eta_K : \mathcal{O}_{U_K}^{r+1} \longrightarrow \mathcal{O}_{U_K}^{n+1}$$

the homomorphisms $(\eta_I)_{|U_{IK}}$ and $(\eta_K)_{|U_{KI}}$ are made compatible with the glueing of U_{IK} with U_{KI} by the following diagram:

$$\begin{tikzcd}
\mathbf{k}[v_{ih}]_{v_I}^{r+1} \arrow[r, "M(K)"] \arrow[d] & \mathbf{k}[v_{ih}]_{v_I}^{n+1} \arrow[d, "M(K)_I^{-1}"] \\
\mathbf{k}[u_{ij}]_{u_K}^{r+1} \arrow[r, "M(I)"] & \mathbf{k}[u_{ij}]_{u_K}^{n+1}
\end{tikzcd} \tag{12}$$

This implies that the η_I's glues together to a surjection:

$$\mathcal{O}_G^{r+1} \xrightarrow{ev} \mathcal{F} \longrightarrow 0$$

where $\mathcal{F}$ is the locally free sheaf of rank $n+1$ obtained by the glueings of diagram (12).

Taking projective bundles we obtain

$$\begin{tikzcd} I\!\!P(\mathcal{F}^\vee) \arrow[r, hook] \arrow[drr] & I\!\!P(\mathcal{O}_G^{r+1}) \arrow[r, equal] & I\!\!P^r \times G(n,r) \arrow[d] \\ & & G(n,r) \end{tikzcd} \tag{13}$$

Proposition 5.1 *(13) is the universal family of linear subspaces of dimension n of $I\!\!P^r$. In particular $G(n,r) = \mathcal{H}^r_{\binom{T+n}{n}}$.*

Proof. Let

$$\begin{tikzcd} \mathcal{Y} \arrow[r, hook] \arrow[dr] & I\!\!P^r \times S \arrow[d, "q"] \\ & S \end{tikzcd} \tag{14}$$

be a family of n-subspaces of $I\!\!P^r$. Let $\mathcal{I}_\mathcal{Y} \subset \mathcal{O}_{I\!\!P^r \times S}$ be the ideal sheaf of $\mathcal{Y}$. Then we have a surjection of locally free sheaves on S:

$$\begin{tikzcd} \Phi : \mathcal{O}_S^{r+1} \arrow[r] \arrow[d, equal] & q_*\mathcal{O}_\mathcal{Y}(1) \arrow[r] & 0 \\ p_*\mathcal{O}_{I\!\!P^r \times S}(1) \end{tikzcd}$$

and $q_*\mathcal{O}_\mathcal{Y}(1)$ is locally free of rank $n+1$. for each multiindex $I = \{i_0, \ldots, i_n\} \subset \{1, \ldots, r\}$, consider the composition

$$\Phi_I : \mathcal{O}_S^{n+1} \hookrightarrow \mathcal{O}_S^{r+1} \xrightarrow{\Phi} q_*\mathcal{O}_\mathcal{Y}(1)$$

where the first arrow is the inclusion corresponding to I. Let $S_I \subset S$ be the (possibly empty) open subset where Φ_I is surjective (i.e. an isomorphism). Then $\{S_I\}$ is an open cover of S. On S_I we can consider the composition:

$$\Phi_I^{-1} \cdot \Phi : \mathcal{O}_S^{r+1} \longrightarrow \mathcal{O}_S^{n+1}$$

This morphism is defined by a $(n+1)\times(r+1)$ matrix $N(I)$ with elements in $\Gamma(S_I,\mathcal{O}_S)$ such that $N(I)_I = \mathbf{I}_{n+1}$. Therefore the matrix $N(I)$ defines a morphism $\varphi_I : S_I \to U_I$. It is clear that the morphisms φ_I are compatible with the glueings of the U_I's, so that they patch together defining a morphism $\varphi : S \to G(n,r)$. It is easy to check that

$$\varphi^*\mathcal{F} = q_*\mathcal{O}_{\mathcal{Y}}(1)$$

and from this fact to deduce that the family (14) is the pullback of (13) by φ. The uniqueness follows from the uniqueness of the matrices $N(I)$. $\square$

6 Existence of the Hilbert schemes

The examples of Hilbert schemes considered so far are concrete and natural, and give some evidence for the existence of a universal family in general. But things can get more complicated if we consider families of more general subschemes. It would be difficult to find a universal family in general by direct and constructive methods. The standard original proof of existence, due to Grothendieck with refinements by Mumford, uses a different approach, which essentially reduces the construction to the special case of grassmannians. We shall outline their approach in this section.

The main difference between the Chow variety and the Hilbert scheme is that the Chow variety uses linear spaces of appropriate codimension that intersect the subvarieties we want to parametrize, while the Hilbert scheme uses hypersurfaces containing the subvarieties. The main ingredients of the construction of $\mathcal{H}^r_{p(T)}$ are:

(i) *Given a Hilbert polynomial $p(T)$ there is a* uniform *m such that for every subscheme $X \subset \mathbb{P}^r$ having Hilbert polynomial $p(T)$, the Hilbert function of X equals $p(n)$ for all $n \geq m$ and moreover the homogeneous ideal of X is generated in degrees $\leq m$.* Therefore every such X is uniquely determined by the subspace

$$\Lambda_X := H^0(\mathbb{P}^r, \mathcal{I}_X(m)) \subset H^0(\mathbb{P}^r, \mathcal{O}(m))$$

which has dimension

$$\dim(\Lambda_X) = \binom{m+r}{r} - p(T)$$

independent of X. Therefore each X defines a point of the grassmannian $G(N-p(T),N)$, where $N=\binom{m+r}{r}-1$. This procedure identifies the set of all subschemes $X\subset I\!\!P^r$ with a given Hilbert polynomial with a subset of a grassmannian. But so far this subset has no scheme structure.

(ii) *There is an integer $c\geq 1$ such that a necessary and sufficient condition for a subspace $\Lambda\subset H^0(I\!\!P^r,\mathcal{O}(m))$ of codimension $p(m)$ to be of the form $\Lambda=\Lambda_X$ for some X is that the map:*

$$\mu_k:\Lambda\otimes H^0(I\!\!P^r,\mathcal{O}(k))\longrightarrow H^0(I\!\!P^r,\mathcal{O}(m+k))$$

has rank $\leq\binom{m+k+r}{r}-p(m+k)$ for all $1\leq k\leq c$. These conditions on the maps μ_k can be expressed by the vanishing of appropriate minors of matrices representing them, and they give rise to polynomial equations involving the Plücker coordinates of Λ. They are the equation of the locus considered in (i). We thus obtain the Hilbert scheme as a closed subscheme of a grassmannian. In particular we obtain that $\mathcal{H}^r_{p(T)}$ is projective.

The previous procedure, as well as the construction of the universal family, involve some delicate technical points. We will not give any details, referring the reader to [6] and references therein.

A straightforward variant of the above arguments leads to a proof of the existence of a generalized version of the Hilbert scheme, as follows.

Theorem 6.1 *Let $Z\subset I\!\!P^r$ be a projective scheme. For every numerical polynomial $p(T)$ there is a universal family of closed subschemes of Z:*

$$\begin{tikzcd} \mathcal{U} \arrow[r, hook] \arrow[dr] & Z\times\mathcal{H}^Z_{p(T)} \arrow[d] \\ & \mathcal{H}^Z_{p(T)} \end{tikzcd}$$

having Hilbert polynomial $p(T)$. The scheme $\mathcal{H}^Z_{p(T)}$ is projective and it is a closed subscheme of $\mathcal{H}^r_{p(T)}$; it is called the Hilbert scheme of Z.

It is sometimes convenient to consider the disjoint union of all the Hilbert schemes of $I\!\!P^r$:

$$\mathcal{H}^r:=\coprod_{p(T)}\mathcal{H}^r_{p(T)}$$

which is a scheme locally of finite type. It will be called *the Hilbert scheme of $I\!P^r$*. Similarly, for a closed subscheme $Z \subset I\!P^r$ we will consider

$$\mathcal{H}^Z = \coprod_{p(T)} \mathcal{H}^Z_{p(T)}$$

and we will call it *the Hilbert scheme of Z*. Note that $\mathcal{H}^Z$ is independent of the embedding $Z \subset I\!P^r$, because changing the embedding only changes the Hilbert polynomials indexing its components.

7 The tangent space

What really makes the Hilbert scheme useful is the fact that, in principle, we can compute its tangent spaces and other local invariants.

We fix r and a Hilbert polynomial $p(T)$. Consider the universal family

$$\begin{tikzcd} \mathcal{U} \arrow[r, hook] \arrow[dr] & I\!P^r \times \mathcal{H}^r_{p(T)} \arrow[d] \\ & \mathcal{H}^r_{p(T)} \end{tikzcd} \tag{15}$$

Let $X \subset I\!P^r$ be a closed subscheme and let $[X] \in \mathcal{H}^r$ be the point parametrizing X. We want to compute the tangent space $T_{[X]}\mathcal{H}^r$.

Denote by

$$\mathrm{Spec}(\mathbf{k}[\epsilon]) = \mathrm{Spec}(\mathbf{k}[t]/(t^2)$$

Then $\mathrm{Spec}(\mathbf{k}[\epsilon])$ is an algebraic scheme having only one point 0 and a 1-dimensional Zariski tangent space. If V is an algebraic scheme and $v \in V$ is a closed point, we will denote by $\mathrm{Mor}(\mathrm{Spec}(\mathbf{k}[\epsilon]), V)_v$ the *set* of morphisms $\theta : \mathrm{Spec}(\mathbf{k}[\epsilon]) \to V$ such that $\theta(0) = v$.

Lemma 7.1 *Given V and v as above,* $\mathrm{Mor}(\mathrm{Spec}(\mathbf{k}[\epsilon]), V)_v$ *has a natural structure of* $\mathbf{k}$*-vector space and there is a canonical isomorphism of* $\mathbf{k}$*-vector spaces:*

$$T_vV \cong \mathrm{Mor}(\mathrm{Spec}(\mathbf{k}[\epsilon]), V)_v$$

We refer to [6], p. 285 for the proof.

Definition 7.2 *If $X \subset Z$ is a closed embedding, and if $\mathcal{I} \subset \mathcal{O}_Z$ is the ideal sheaf of X in Z, the sheaf of $\mathcal{O}_X$-modules $\mathcal{I}/\mathcal{I}^2$ is called the* conormal sheaf *of X in Z. The* normal sheaf *of X in Z is*

$$N_{X/Z} := Hom_{\mathcal{O}_Z}(\mathcal{I}, \mathcal{O}_X) = Hom_{\mathcal{O}_X}(\mathcal{I}/\mathcal{I}^2, \mathcal{O}_X)$$

If it is locally free then $\mathcal{I}/\mathcal{I}^2$ (resp. $N_{X/Z}$) is called the conormal bundle *(resp.* normal bundle*).*

Proposition 7.3 *Let $X \subset Z$ be a closed embedding of projective schemes. Then there is a natural identification*

$$T_{[X]}\mathcal{H}^Z = H^0(X, N_{X/Z})$$

Proof. According to Lemma 7.1,

$$T_{[X]}\mathcal{H}^Z = \mathrm{Mor}(\mathrm{Spec}(\mathbf{k}[\epsilon]), \mathcal{H}^Z)_{[X]}$$

By the universal property of the Hilbert scheme, elements of $\mathrm{Mor}(\mathrm{Spec}(\mathbf{k}[\epsilon]), \mathcal{H}^Z)_{[X]}$ correspond in a 1-1 way to families:

$$\begin{tikzcd} \mathcal{X} \arrow[r, hook] \arrow[dr] & Z \times \mathrm{Spec}(\mathbf{k}[\epsilon]) \arrow[d] \\ & \mathrm{Spec}(\mathbf{k}[\epsilon]) \end{tikzcd} \tag{16}$$

whose fibre over the closed point $\mathrm{Spec}(\mathbf{k}) \to \mathrm{Spec}(\mathbf{k}[\epsilon])$ is $X \subset Z$. These families are called *first order deformations* of X in Z.

First order deformations of X in Z can be obtained by glueing together first order deformations of the open sets of an affine covering of X. Therefore we will study the affine case first. We need the following

Lemma 7.4 *Let A be a $\mathbf{k}[\epsilon]$-algebra, and let $A_0 = A/\epsilon A = A \otimes_{\mathbf{k}[\epsilon]} \mathbf{k}$. Then A is flat over $\mathbf{k}[\epsilon]$ if and only if there is an isomorphism $\epsilon A \cong A_0$.*

Proof of Lemma 7.4. We have an exact sequence

$$\begin{tikzcd} 0 \arrow[r] \arrow[d, equal] & \mathrm{Tor}_1^{\mathbf{k}[\epsilon]}(A, \mathbf{k}) \arrow[r] & A \otimes_{\mathbf{k}[\epsilon]} (\epsilon) \arrow[r, "\beta"] \arrow[d, equal] & A \arrow[r] & A \otimes_{\mathbf{k}[\epsilon]} \mathbf{k} \arrow[r] \arrow[d, equal] & 0 \\ \mathrm{Tor}_1^{\mathbf{k}[\epsilon]}(A, \mathbf{k}[\epsilon]) & & A \otimes_{\mathbf{k}[\epsilon]} \mathbf{k} \arrow[d, equal] & & A_0 & \\ & & A_0 & & & \end{tikzcd}$$

obtained from

$$0 \to (\epsilon) \to \mathbf{k}[\epsilon] \to \mathbf{k} \to 0$$

after tensoring by $\otimes_{\mathbf{k}[\epsilon]} A$.

Assume that A is $\mathbf{k}[\epsilon]$-flat. Then $\mathrm{Tor}_1^{\mathbf{k}[\epsilon]}(A, \mathbf{k}) = 0$. Since $\mathrm{Im}(\beta) = \epsilon A$ we deduce that $\epsilon A \cong A_0$.

Conversely, assume that $\epsilon A \cong A_0$. Then β is an isomorphism and this implies that $\mathrm{Tor}_1^{\mathbf{k}[\epsilon]}(A, \mathbf{k}) = 0$. Therefore A is $\mathbf{k}[\epsilon]$-flat. □

Let's assume that $X = \mathrm{Spec}(A_0)$, $Z = \mathrm{Spec}(B_0)$ *are affine*, with $A_0 = B_0/I_0$ for some ideal $I_0 \subset B_0$. Then

$$Z \times \mathrm{Spec}(\mathbf{k}[\epsilon]) = \mathrm{Spec}(B_0 \otimes_{\mathbf{k}} \mathbf{k}[\epsilon]) = \mathrm{Spec}(B_0[\epsilon])$$

and a first order deformation of X in Z is given by a closed subscheme

$$\mathcal{X} = \mathrm{Spec}(A) \subset \mathrm{Spec}(B_0[\epsilon])$$

defined by an ideal $I \subset B_0[\epsilon]$ such that A is flat over $\mathbf{k}[\epsilon]$. By applying Lemma 7.4 we obtain the following commutative and exact diagram:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & I \arrow[r] \arrow[d] & I_0 \arrow[d] & \\
0 \arrow[r] & \epsilon B_0 \arrow[r] \arrow[d, "\gamma"] & B_0[\epsilon] \arrow[r] \arrow[d] & B_0 \arrow[r] \arrow[d] & 0 \quad (*) \\
0 \arrow[r] & \epsilon A_0 \arrow[r] \arrow[d] & A \arrow[r] \arrow[d] & A_0 \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd} \tag{17}$$

So the first order deformations of X in Z are in 1-1 correspondence with the ideals I as above.

Assume moreover that $I_0 = (f)$ *is principal*, generated by a non 0-divisor f. Given I, let $F = f + \epsilon g \in I$ be a lifting of f and let $A' = B_0[\epsilon]/(F)$. Since clearly F is not a 0-divisor in $B_0[\epsilon]$, because otherwise f would be one in B_0, we have an exact sequence:

$$0 \longrightarrow B_0[\epsilon] \xrightarrow{F} B_0[\epsilon] \longrightarrow A' \longrightarrow 0$$

Tensoring by $\otimes_{\mathbf{k}[\epsilon]}\mathbf{k}$ we obtain:

$$
\begin{tikzcd}
0 \arrow[r] & \mathrm{Tor}_1^{\mathbf{k}[\epsilon]}(A', \mathbf{k}) \arrow[r] & B_0 \arrow[r, "f"] & B_0 \arrow[r] & A_0 \arrow[r] & 0
\end{tikzcd}
$$

and, since f is not a 0-divisor, we deduce that $\mathrm{Tor}_1^{\mathbf{k}[\epsilon]}(A', \mathbf{k}) = 0$, and therefore A' is flat over $\mathbf{k}[\epsilon]$. Therefore, using Lemma 7.4 again, we deduce that we have a commutative diagram:

$$
\begin{tikzcd}
0 \arrow[r] & \epsilon A_0 \arrow[r] \arrow[d, equal] & A' \arrow[r] \arrow[d] & A_0 \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \epsilon A_0 \arrow[r] & A \arrow[r] & A_0 \arrow[r] & 0
\end{tikzcd}
$$

which shows that $A' \cong A$. Therefore *the ideal I is principal as well.*

Now observe that, given I, the generator $F = f + \epsilon g$ which lifts f is determined only up to multiplication by a unit $U = u + \epsilon a$ of $B_0[\epsilon]$. Since UF must be a lifting of f, $u = 1$ and $a \in B_0$ is arbitrary. Therefore:

$$UF = (1 + \epsilon a)(f + \epsilon g) = f + \epsilon(g + af)$$

This means that we can replace g by any other element of the class $g + I_0$ to obtain a generator of I lifting f. If we take another generator uf of I_0, then a lifting generating I is of the form $uf + \epsilon(ug + af)$.

Therefore: given I we are given a homomorphism $\varphi : I_0 \to A_0$ defined by $\varphi(f) = g + I_0$. Conversely, such a homomorphism defines an ideal $I \subset B_0[\epsilon]$ by $I = (f + \epsilon g)$, where $g + I_0 = \varphi(f)$, and I defines a first order deformation of X in Z. The conclusion is that we have a canonical 1-1 correspondence between first order deformations of X in Z and $\mathrm{Hom}_{B_0}(I_0, A_0) = \mathrm{Hom}_{A_0}(I_0/I_0^2, A_0)$.

If we do not assume any more that I_0 is principal we can proceed as follows. Taking the pushout of the row $(*)$ by γ we replace diagram (17) by the following one:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & I_0 \arrow[r, equal] \arrow[d, "\alpha"] & I_0 \arrow[d] & \\
0 \arrow[r] & \epsilon A_0 \arrow[r] \arrow[d, equal] & E \arrow[r] \arrow[d] & B_0 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \epsilon A_0 \arrow[r] \arrow[d] & A \arrow[r] \arrow[d] & A_0 \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

From this diagram we see that we have a bijection between the set of first order deformations of X in Z and the homomorphisms $\alpha : I_0 \to E$ making the above diagram commutative. Letting $\alpha_0 : I_0 \to E$ be the homomorphism corresponding to the trivial deformation, we can associate to each α the homomorphism

$$\alpha - \alpha_0 : I_0 \to \epsilon A_0$$

thus obtaining a 1-1 natural correspondence:

$$\begin{tikzcd}
\{\text{first order deformations of } X \text{ in } Z\} \arrow[r, leftrightarrow] & \mathrm{Hom}_{B_0}(I_0, A_0) \arrow[d, equal] \\
 & \mathrm{Hom}_{A_0}(I_0/I_0^2, A_0)
\end{tikzcd}$$

Now, globalizing this analysis we deduce that there is a natural 1-1 correspondence between $H^0(X, N_{X/Z})$ and the set of first order deformations of X in Z, and this proves the proposition. □

Proposition 7.3 implies that $h^0(X, N_{X/Z})$ is an upper bound for the dimension of $\mathcal{H}^Z$. The following important result gives a lower bound as well.

Theorem 7.5 *Given a closed embedding of projective schemes $X \subset Z$ such that X is a local complete intersection in Z, the dimension of $\mathcal{H}^Z$ at the point $[X]$ satisfies ths following inequalities:*

$$h^0(X, N_{X/Z}) - h^1(X, N_{X/Z}) \leq \dim_{[X]}\mathcal{H}^Z \leq h^0(X, N_{X/Z}) \tag{18}$$

When X is a local complete intersection in Z we say that X *is regularly embedded in* Z or that $X \subset Z$ is a *regular embedding*. In this case $\mathcal{I}/\mathcal{I}^2$ and $N_{X/Z}$ are both locally free of rank equal to the codimension of X in Z (see [6], Appendix D). Note the following immediate consequence:

Corollary 7.6 *In the situation of the theorem, the second inequality holds in (18) if and only if $\mathcal{H}^Z$ is nonsingular of dimension $h^0(X, N_{X/Z})$ at $[X]$. This is true in particular if*

$$H^1(X, N_{X/Z}) = 0$$

For the proof of Theorem 7.5 some more refined algebraic machinery is required. We refer to [6] for details.

8 About the normal sheaf

Theorem 7.5 and its corollary show that, given $X \subset Z$ a closed embedding of projective schemes, the computation of the cohomology of $N_{X/Z}$ is very important if we want to study the local properties of $\mathcal{H}^Z$ at $[X]$. In this section we will describe some properties of the conormal and normal sheaves.

The starting point is the following exact sequence associated to any closed embedding $j : X \subset Z$ of (not necessarily projective) schemes defined by an ideal sheaf $\mathcal{I} \subset \mathcal{O}_Z$:

$$\mathcal{I}/\mathcal{I}^2 \xrightarrow{\delta} \Omega^1_{Z|X} \longrightarrow \Omega^1_X \longrightarrow 0 \tag{19}$$

This is the *conormal sequence* of X in Y. If X is nonsingular then Ω^1_X is locally free. The left arrow δ is not in general injective, and its kernel is supported on the locus where j is not a regular embedding. Therefore, if j is a regular embedding then δ is injective and $\mathcal{I}/\mathcal{I}^2$ is locally free. If moreover X is nonsingular then Z is also nonsingular and the sequence (19) dualizes as follows:

$$0 \longrightarrow T_X \longrightarrow T_{Z|X} \longrightarrow N_{X/Z} \longrightarrow 0 \tag{20}$$

This is the *normal sequence* of X in Z. This sequence turns out to be very useful for the computation of the cohomology of $N_{X/Z}$. In general, even if $X \subset Y$ is not a regular embedding and we make no assumptions on X and Y, the sequence (20) is exact except possibly at $N_{X/Z}$.

If we have closed embeddings

$$X \overset{j}{\hookrightarrow} Y \overset{i}{\hookrightarrow} Z$$

corresponding to ideal sheaves

$$\mathcal{I}_{X/Y} \subset \mathcal{O}_Y, \qquad \mathcal{I}_{X/Z} \subset \mathcal{O}_Z, \qquad \mathcal{I}_{Y/Z} \subset \mathcal{O}_Z$$

then we obtain an exact sequence

$$0 \longrightarrow \mathcal{I}_{Y/Z} \longrightarrow \mathcal{I}_{X/Z} \longrightarrow \mathcal{I}_{X/Y} \longrightarrow 0$$

which, tensored by $\otimes_{\mathcal{O}_Z}\mathcal{O}_X$, gives the exact sequence:

$$(\mathcal{I}_{Y/Z}/\mathcal{I}^2_{Y/Z}) \otimes \mathcal{O}_X \longrightarrow \mathcal{I}_{X/Z}/\mathcal{I}^2_{X/Z} \longrightarrow \mathcal{I}_{X/Y}/\mathcal{I}^2_{X/Y} \longrightarrow 0 \tag{21}$$

and dualizing we obtain:

$$N_{X/Y} \longrightarrow N_{X/Z} \longrightarrow N_{Y/Z} \otimes \mathcal{O}_X \longrightarrow 0 \tag{22}$$

Lemma 8.1 *If i and j are regular embeddings then the sequences (21) and (22) are both also exact on the left.*

Proof. See [6], Lemma D.1.3 p. 306. □

Example 8.2 Let E be a locally free sheaf of rank e on a scheme Z. If $\sigma \in H^0(Z, E)$ is such that its zero-scheme $X := V(\sigma)$ in non-empty and has everywhere codimension e, then there is an isomorphism:

$$N_{X/Z} \cong E_{|X} \tag{23}$$

In fact from the assumption about codimension it follows that $X \subset Z$ is a regular embedding, so that the conormal sheaf $\mathcal{I}/\mathcal{I}^2$ is locally free of rank e. By the definition of X there is a homomorphism:

$$\mathcal{E}^\vee \xrightarrow{\sigma} \mathcal{O}_Z$$

whose image is $\mathcal{I}$. Tensoring by $\mathcal{O}_X$ we obtain a surjective homomorphism

$$\sigma_{|X} : E^\vee \otimes \mathcal{O}_X \to \mathcal{I}/\mathcal{I}^2$$

of locally free sheaves of rank e on X, which must therefore by an isomorphism. By dualizing we obtain (23).

It easy to show that if E is globally generated then the general section $\sigma \in H^0(E)$ has a zero-scheme of pure codimension e.

A special case is obtained by taking $Z = \mathbb{P}^r$, $E = \mathcal{O}(d_1) \oplus \cdots \oplus \mathcal{O}(d_c)$ for integers $1 \leq d_1 \leq \cdots \leq d_c$. A section $\sigma \in H^0(E)$ is nothing but a c-tuple of homogeneous polynomials $(F_1(X), \ldots, F_c(X))$ in $X = (X_0, \ldots, X_r)$ of multidegree $(d_1, \ldots, d_c)$. If $X = V(\sigma)$ has pure codimension c then it is a complete intersection of equations:

$$F_1 = \cdots = F_c = 0$$

In this case the normal bundle of X is

$$N_{X/\mathbb{P}^r} = \oplus_{i=1}^c \mathcal{O}_X(d_i)$$

For example, for a hypersurface $X \subset \mathbb{P}^r$ of degree d we have

$$H^0(X, N_{X/\mathbb{P}^r}) = H^0(X, \mathcal{O}_X(d))$$

which has dimension $\binom{d+r}{r} - 1$, equal to the dimension of the Hilbert scheme (the projective space $\mathbb{P}^{\binom{d+r}{r}-1}$), as expected. It is also true that the Hilbert scheme of complete intersections of any multidegree is nonsingular, but we cannot prove it using Corollary 7.6 because in general $H^1(X, N_{X/\mathbb{P}^r}) \neq 0$.

Example 8.3 Consider a nonsingular projective variety $X \subset \mathbb{P}^r$, and the diagram:

$$\begin{tikzcd}
 & & 0 \arrow[d] & & \\
 & & \mathcal{O}_X(-1) \arrow[d] & & \\
 & & \mathcal{O}^{r+1} \arrow[d] \arrow[dr] & & \\
0 \arrow[r] & T_X(-1) \arrow[r] & T_{\mathbb{P}^r|X}(-1) \arrow[r] \arrow[d] & N_{X/\mathbb{P}^r}(-1) \arrow[r] & 0 \\
 & & 0 & &
\end{tikzcd}$$

The horizontal sequence is (20) twisted by $\mathcal{O}(-1)$ and restricted to X, and the vertical sequence is the Euler sequence twisted by $\mathcal{O}(-1)$ and restricted to X. The diagonal homomorphism is clearly surjective, showing that $N_{X/\mathbb{P}^r}(-1)$ is globally generated. Then it follows from standard facts on ample vector bundles that $N_{X/\mathbb{P}^r} = N_{X/\mathbb{P}^r}(-1) \otimes \mathcal{O}(1)$ is ample.

9 Classification of projective curves and Hilbert schemes

The problem of classifying nonsingular projective curves in a given projective space $\mathbb{P}^r$ is classical. It can be divided in two parts:

i) Find all values of (d, g) for which there exist connected nonsingular non-degenerate curves ${}_gC^d \subset \mathbb{P}^r$ of degree d and genus g.

ii) For those values (d, g) for which such curves exist, describe the family which parametrizes them.

Part (i) is known as *the gap problem.* Of course in $\mathbb{P}^2$ every nonsingular curve of degree d has genus $\binom{d-1}{2}$. Therefore the gap problem is interesting only if $r \geq 3$. In a fundamental paper of 1882 Halphen considered the case of curves in $\mathbb{P}^3$. His results where completed and clarified by Gruson and Peskine in a series of papers dating from 1977. It turns out that there exist connected nonsingular non-degenerate curves ${}_gC^d \subset \mathbb{P}^3$ for every $d \geq 3$. The genus of such a curve satisfies the inequality:

$$g \leq \pi_0(3, d) := \begin{cases} \left(\frac{d}{2} - 1\right)^2 & \text{if } d \text{ is even} \\ \left(\frac{d-1}{2} - 1\right)\left(\frac{d+1}{2} - 1\right) & \text{if } d \text{ is odd} \end{cases} \tag{24}$$

The maximum is attained bu curves on a nonsingular quadric. Since the genus of nonsingular curves on a quadric is known the next, and more difficult, result is the following:

If ${}_gC^d \subset \mathbb{P}^3$ does not lie on a quadric then

$$g \leq \pi_1(3, d) := 1 + \frac{d(d-3)}{6} \tag{25}$$

Moreover for all (d, g) as in (25) such a ${}_gC^d \subset \mathbb{P}^3$ exists.

This result is important because it characterizes the *gaps*, i.e. those pairs (d,g) such that $g \leq \pi_0(3,d)$ and such that curves ${}_gC^d \subset I\!P^3$ do not exist. It suffices to choose (d,g) such that

$$\pi_1(3,d) < g < \pi_0(3,d)$$

and such that such a curve cannot exist on a quadric. Then first example is $(d,g) = (9,11)$; here $\pi_0(3,d) = 12$.

The upper bound $\pi_0(3,d)$ was generalized by Castelnuovo to all $r \geq 3$ in a famous paper of 1889. He proved that if a ${}_gC^d \subset I\!P^r$ exists then:

$$g \leq \pi_0(r,d) := \binom{m}{2}(r-1) + m\epsilon \tag{26}$$

where m and ϵ are defined by the equality:

$$d - 1 = m(r-1) + \epsilon, \qquad 0 \leq \epsilon \leq r-2$$

The number $\pi_0(r,d)$ is called *Castelnuovo bound* and curves attaining this bound are called *Castelnuovo curves.* They were completely described by Castelnuovo who showed that they are generalizations to $I\!P^r$ of curves lying on quadric surfaces in $I\!P^3$: in fact a Castelnuovo curve in $I\!P^r$ necessarily lies on a rational ruled surface of degree $r-1$. The gap problem has not been completely solved in full generality. Partial results are due to several authors.

Part (ii) of the classification problem is related to the structure of the Hilbert schemes of $I\!P^r$. It can be rephrased as the problem of describing, for a given (d,g) for which the gap problem can be solved, those irreducible components of $\mathcal{H}^r_{dt+1-g}$ whose general point parametrizes a ${}_gC^d \subset I\!P^r$. Let's denote by $U^r_{d,g} \subset \mathcal{H}^r_{dt+1-g}$ the union of such components.

This problem turns out to be too ambitious even for $r = 3$. We know very few general results regarding such families, while our concrete knowledge of families of projective curves is scattered. Classical work on families of curves in $I\!P^3$ goes back to M. Noether, while the case of curves in any $I\!P^r$ was considered especially by Severi. We can ask questions regarding either local or global properties of such families.

Let's consider *local properties.* The first information one would like to know is about the dimension and type of singularities of $U^r_{d,g}$. A curve ${}_gC^d$ such that $[{}_gC^d]$ is a singular (resp. nonsingular) point of $U^r_{d,g}$ is called *obstructed* (resp. *unobstructed*).

If $C = {}_gC^d \subset I\!P^r$ is non-degenerate, nonsingular connected then we can use the diagram of Example 8.3 and obtain:

$$\begin{aligned} \chi(N_{C/I\!P^r}) &= (r+1)\chi(\mathcal{O}_C(1)) - \chi(\mathcal{O}_C) - \chi(T_C) \\ &= (r+1)d + (r-3)(1-g) \end{aligned} \tag{27}$$

This number gives a lower bound for the dimension of $U^r_{d,g}$ at $[C]$ (Theorem 7.5). The right hand side of (27) is called the *expected dimension* of $U^r_{d,g}$ at $[C]$. Note that for a curve in $I\!P^3$ we get

$$\chi(N_{C/I\!P^3}) = 4d$$

which is independent of g. The following are some classes of curves having $H^1(N_{C/I\!P^r}) = 0$, for which therefore $U^r_{d,g}$ is nonsingular of the expected dimension.

Example 9.1 Suppose that $H^1(C, \mathcal{O}_C(1)) = 0$, i.e. that $C = {}_gC^d \subset I\!P^r$ is *non-special*. Consider the surjection

$$\mathcal{O}_C^{r+1} \longrightarrow N_{C/I\!P^r} \longrightarrow 0$$

deduced from the diagram of Example 8.3. We obtain

$$h^1(C, N_{C/I\!P^r}) \leq (r+1)h^1(C, \mathcal{O}_C(1)) = 0$$

Therefore $H^1(N_{C/I\!P^r}) = 0$. The following are particular classes of non-special curves:

- $C \subset I\!P^r$ is a rational curve of degree d. Then

$$h^0(C, N) = (r+1)d + r - 3$$

- $C \subset I\!P^r$ is a nonsingular curve of degree d and genus 1. Then

$$h^0(C, N) = (r+1)d$$

- $C \subset I\!P^r$ is a nonsingular curve of genus g and degree $d \geq 2g-1$.

Example 9.2 For a given nonsingular non-degenerate $C = {}_gC^d \subset \mathbb{P}^r$ consider the subspace

$$V^\vee = \mathrm{Im}[H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}}(1)) \longrightarrow H^0(C, \mathcal{O}_C(1))]$$

Note that $\mathbb{P}^r = \mathbb{P}(V)$. The restriction to C of the Euler sequence can be written as follows:

$$0 \longrightarrow \mathcal{O}_C \longrightarrow V \otimes \mathcal{O}_C(1) \longrightarrow T_{\mathbb{P}|C} \longrightarrow 0 \tag{28}$$

It induces a map:

$$H^1(C, \mathcal{O}_C)) \longrightarrow V \otimes H^1(C, \mathcal{O}_C(1)) \tag{29}$$

whose dual is just the cup-product map:

$$\mu_0(V^\vee) : V^\vee \otimes H^0(C, \omega_C(-1)) \longrightarrow H^0(C, \omega_C)$$

$\mu_0(V^\vee)$ is called the *Petri map* of $C \subset \mathbb{P}^r$. If $\mu_0(V^\vee)$ is injective then (29) is surjective and from (28) we deduce that $H^1(C, T_{\mathbb{P}|C}) = 0$. But then from the exact sequence (20) we see that $H^1(C, N_{C/\mathbb{P}}) = 0$. Therefore *if the Petri map of $C \subset \mathbb{P}^r$ is injective then $H^1(C, N_{C/\mathbb{P}^r}) = 0$.*

The most important examples of curves with injective Petri map are the *canonical curves* ${}_gC^{2g-2} \subset \mathbb{P}^{g-1}$, equivalently denoted by ${}_{r+1}C^{2r} \subset \mathbb{P}^r$. In fact in this case we have $\mathcal{O}_C(1) \cong \omega_C$, $V = H^0(C, \omega_C)$ and

$$\mu_0 : H^0(C, \omega_C) \otimes H^0(C, \mathcal{O}_C) \longrightarrow H^0(C, \omega_C)$$

is just the identity map. It follows that $H^1(C, N) = 0$ and

$$h^0(C, N) = (g+4)(g-1) = (r+5)r$$

Unfortunately (or fortunately?) most curves in $\mathbb{P}^r$ do not belong to any of the classes described in Examples 9.1 and 9.2. The scheme $U^r_{d,g}$ tends to be singular and not of the expected dimension in general. Note for example that the expected dimension (27) is negative when $r \geq 4$ and g is large with respect to d: in such cases necessarily $h^1(C, N) \neq 0$. The complete intersections in $\mathbb{P}^r$ with $r \geq 3$ and of multidegree $(d_1, \ldots, d_{r-1})$ with $d_1 \leq \cdots \leq d_{r-1}$ and such that $\sum_{j=2}^{r-1} d_j \geq r+1$ are the most typical examples of nonsingular curves having $h^1(C, N) \neq 0$. This fact follows from the fact that $N = \oplus \mathcal{O}_C(d_j)$

(see Example 8.2). Nevertheless it can be proved that complete intersection curves are unobstructed.

Several examples are known of obstructed nonsingular curves. We refer to [6] for more details.

Our knowledge of *global properties* of the schemes $U^r_{d,g}$ is very limited. Perhaps the most important starting point in this area is a classical problem, dating back to Severi. Let

$$\rho(g,r,d) := g - (r+1)(g-d+r)$$

be the so called *Brill-Noether number*. The we may ask:

- *If* $\rho(g,r,d) \geq 0$ *is* $U^r_{d,g}$ *irreducible?*

Note that if L is an invertible sheaf of degree d on a connected nonsingular curve C of genus g then we may consider the *Petri map of* L:

$$\mu_0(L) : H^0(C,L) \otimes H^0(C,\omega_C L^{-1}) \longrightarrow H^0(C,\omega_C)$$

Then $\rho(g,r,d) \geq 0$ is a necessary condition for the injectivity of $\mu_0(L)$. If $C \subset \mathbb{P}^r$ and $L = \mathcal{O}_C(1)$ then the injectivity of $\mu_0(L)$ implies that $h^1(C, T_{\mathbb{P}|C}) = 0$ (as shown in Example 9.2) and, by the exact sequence (20), it follows that $H^1(C,N) = 0$ and that the coboundary map

$$H^0(C,N) \longrightarrow H^1(C,T_C)$$

is surjective. This turns out to mean that there is a unique component U of $U^r_{d,g}$ containing $[C]$, that U is generically nonsingular and that the rational map

$$m : U \dashrightarrow M_g$$

with values in the moduli space of curves of genus g is dominant. Since M_g is known to be irreducible, one might hope that U is the only irreducible component of $U^r_{d,g}$. This is essentially the content of the above problem. We know today that, as stated, the problem has a negative answer. In fact there are examples of J. Harris of components of $U^r_{d,g}$ not dominating M_g and with $\rho(g,r,d) \geq 0$; these components consist generically of *non linearly normal curves*. On the other hand Fulton and Lazarsfeld have proved that if $\rho(g,r,d) \geq 0$ there is a unique component U of $U^r_{d,g}$ dominating M_g; this component consists generically of linearly normal curves. What remains unknown is whether the component U is the unique one generically consisting of linearly normal curves. Some interesting irreducibility results have been proved by L. Ein for $r = 3, 4$ and $d \geq g + r$.

On the opposite side we know that if $\rho < 0$ then $U^r_{d,g}$ is in general reducible and it may have an arbitrarily large number of irreducible components.

Other important global questions about $U^r_{d,g}$ concern its birational properties. To fix ideas let's assume $\rho(g, r, d) \geq 0$ and let U be the unique component dominating M_g. It is customary to say that curves of U have *general moduli.* Since we have the dominant morphism m, there is some relation between the geometry of U and that of M_g and, since U is a much more concrete object, one might hope to understand the geometry of M_g from that of U. From this point of view several results are known, especially for low values of g. For example it has been proved in this way (or using variants of this idea) that M_g is unirational for $g \leq 14$ and rationally connected for $g = 15$ (Severi, Sernesi, Chang-Ran, Verra, Bruno-Verra).

Of course one can ask similar questions about $U^r_{d,g}$ in more general cases, but little is known. For example one may ask about unirationality, rational connectedness or uniruledness of components of $U^r_{d,g}$. Questions of this type could be used to deduce birational properties of more general (and interesting) moduli spaces.

Example 9.3 Let $g \geq 3$ and consider the scheme $\mathcal{K}_g := U^{g-1}_{2g-2,g}$. It is an irreducible and generically nonsingular variety of dimension $(g + 4)(g - 1)$ (Example 9.2). Because of the uniqueness of the canonical linear series on every curve, every non-hyperelliptic curve C of genus g can be embedded in $\mathbb{P}^{g-1}$ in a unique way up to projective transformations. In other words C corresponds to a unique orbit of $\mathrm{PGL}(g)$ inside $\mathcal{K}_g$. The geometry of $\mathcal{K}_g$ is therefore intimately related to the geometry of $\mathcal{M}_g$, the *moduli space of curves of genus g*, which parametrizes *abstract curves* of genus g. It is suggestive to think of $\mathcal{K}_g$ and of $\mathcal{M}_g$ as spaces whose geometry reflects the complexity of the *number* g. The first case $g = 3$ has been studied classically. Despite the fact that plane nonsingular quartics are just a linear system of curves, their geometry is incredibly rich and complicated, and at the same time extremely elegant. This reflects in some sense the properties of the number 3. No other $\mathcal{K}_g$'s have been studied in such a detail, even though we know some of their properties, especially for low values of g.

10 Symmetric products of curves and their generalizations

Given a projective variety Z, the properties of its Hilbert schemes often reflect the geometry of Z itself and this can be frequently used to study the geometry of Z and conversely. The most classical example of this phenomenon are the Hilbert schemes of a projective nonsingular connected curve C, whose genus we denote by g.

A few preliminary general observations. If $z \in Z$ is a (closed) point of a projective algebraic variety, we have

$$N_{z/Z} = T_z Z$$

In other words, Z and $\mathcal{H}_1^Z$ have the same tangent space at z. This is obvious also because there is a canonical identification $\mathcal{H}_1^Z = Z$ with the universal family given by the diagonal:

$$\begin{tikzcd} \Delta \arrow[r, hook] \arrow[d, "pr_1"] & Z \times Z \\ Z & \end{tikzcd}$$

This a first confirmation of what we stated at the beginning. The next cases to consider are the Hilbert schemes $\mathcal{H}_n^Z$ parametrizing closed subschemes of finite length n. If $X \subset Z$ is such a subscheme consisting of n nonsingular points then

$$N_{X/Z} = \bigoplus_{x \in X} T_x Z$$

In particular $h^0(N_{X/Z}) = n\dim(Z)$. Since certainly $\dim_{[X]}\mathcal{H}_n^Z = n\dim(Z)$, this means in particular that $\mathcal{H}_n^Z$ is nonsingular at a point $[X]$ as above. But this will not be the case for an arbitrary non-reduced X even if Z is nonsingular. Let's consider the simplest case of curves, where we can more easily understand what happens.

Let C be a projective nonsingular curve of genus g. A closed subscheme $D \subset C$ of length n is a local complete intersection in C, i.e. an effective Cartier divisor of degree n, which can be written as $D = \sum n_i p_i$, where $\sum n_i = n$, $n_i \geq 1$. The normal sheaf is $N_{D/C} = \mathcal{O}_D(D)$. Since D is a 0-dimensional scheme, we certainly have $H^1(D, N_{D/X}) = 0$. and it follows that $\mathcal{H}_n^C$ is nonsingular at $[D]$ because D is a local complete intersection in C (Corollary 7.6). Since we have an isomorphism $\mathcal{O}_D(D) \cong \mathcal{O}_D$ we have $h^0(\mathcal{O}_D(D)) = n$. Therefore $\mathcal{H}_n^C$ *is everywhere nonsingular of dimension* n.

Consider the n-fold symmetric product of C:

$$C_n := C^n/\sigma_n$$

the quotient of the n-fold cartesian product by the natural action of the symmetric group on n letters. Using local coordinates it is easy to show that C_n is nonsingular of dimension n. Its points are in natural 1-1 correspondence with the effective Cartier divisors of degree n. We have a natural *bijective* morphism:

$$C_n \longrightarrow \mathcal{H}_n^C$$

which must therefore be an isomorphism. The conclusion is that $\mathcal{H}_n^C$ *can be naturally identified with* C_n, which is projective irreducible and nonsingular of dimension n.

It is not difficult to realize that the various C_n contain essentially all the possible geometrical information about C. For example, the linear systems of degree n are projective spaces sitting inside C_n. For every $n > 0$ there is a natural map:

$$\alpha_n : C_n \longrightarrow \mathrm{Pic}^n(C)$$

whose target is the n-th Picard variety, which set-theoretically consists of the isomorphism classes of invertible sheaves of degree n on C. It is non-canonically isomorphic to the *jacobian variety*, which can be identified with $\mathrm{Pic}_0(C)$, and is an abelian variety (a projective algebraic group) of dimension g with origin the class of the trivial sheaf $[\mathcal{O}_C]$. It is also denoted by $J(C)$. The morphism α_n is called the *Abel map*. It acts by sending $D \mapsto [\mathcal{O}_C(D)]$. Therefore the fibres of α_n are precisely the complete linear systems of divisors of degree n. The structure of α_n varies considerably as n and g change.

If $g = 0$, i.e. $C = \mathbb{P}^1$ then $\mathrm{Pic}_n(\mathbb{P}^1) = \{[\mathcal{O}(n)]\}$ consists of one element. All divisors of degree n are linearly equivalent, thus $(\mathbb{P}^1)_n = \mathbb{P}^n$. A geometrical realization of this identification can be obtained by embedding $\mathbb{P}^1 \subset \mathbb{P}^n$ as a rational normal curve: the effective divisors are then identified with the hyperplane sections, i.e. with the dual projective space $(\mathbb{P}^n)^\vee$.

If $g \geq 1$ and $n \geq 2g - 1$ then all divisors of degree n are non-special, and all complete linear systems of degree n have the same dimension $n - g$: α_n is surjective because by Riemann-Roch every invertible sheaf L of degree n

has at least one non-zero section. In this case α_n is a fibration with fibres isomorphic to $\mathbb{P}^{n-g}$. For example, when $g = 1$ and $n = 1$ it follows that α_1 is an isomorphism, i.e. $C \cong J(C)$.

If $g \le n \le 2g-2$ then α_n is still surjective again because every invertible sheaf L of degree n has at least one non-zero section. But in this case the structure of α_n reflects the existence of special divisors of degree n and is more complicated. If $1 \le n \le g-1$ then α_n cannot be surjective for dimension reasons, and is birational onto its image.

For example

$$\alpha_2 : C_2 \longrightarrow J(C)$$

is an embedding unless C is hyperelliptic. More generally, α_n is not an embedding for some $2 \le n \le g-1$ precisely if there is a linear series of dimension $r \ge 1$ and degree n.

The most important symmetric product is C_{g-1}. Its image

$$\alpha_{g-1}(C_{g-1}) =: \Theta \subset \mathrm{Pic}_{g-1}(C)$$

is a divisor, called the *theta divisor*. It is an ample divisor. The pair $(\mathrm{Pic}_{g-1}(C), \Theta)$ is the *polarized jacobian* of C. This object has been studied extensively since about 150 years. For example, the singularities of Θ are related to the geometry of the linear systems of degree $g-1$ on C in a very precise manner, by the following:

Theorem 10.1 *If $D \in C_{g-1}$ then*

$$\mathrm{mult}_{\alpha(D)}(\Theta) = h^0(C, \mathcal{O}_C(D))$$

The following is also true:

Theorem 10.2 (Torelli) *The polarized jacobian uniquely determines C.*

There are several proofs of this theorem today, and they all reveal the deep interplay between the geometry of C and that of the polarized jacobian. We refer to [1] for an exposition of two proofs of the Theorem of Torelli. Theorem 10.2 is the prototype of the property we want to discuss, namely that a Hilbert scheme of a variety can be used to recover completely the variety, because to give the polarized jacobian is essentially the same thing as giving C_{g-1}.

Theorem 10.2 has been generalized in many ways, because it states a natural property one would like to reproduce in several situations. Namely, it is natural to ask whether an algebraic variety can be reconstructed from data associated to it which a priori contain less information. Results answering questions of this kind are called *Torelli type theorems*. More generally it is interesting to understand how strong are the informations coming from the families of subvarieties of certain types. Questions and results of this type can range from very elementary to very deep ones.

Example 10.3 The classical work of Enriques and Castelnuovo on the classification of algebraic surfaces was based on the study of families of curves on surfaces, in particular of linear systems. For example, the existence of a linear pencil of rational curves on a (projective algebraic nonsingular connected) surface Y implies the rationality of Y.

The direct generalization of this result to higher dimensional varieties is known to be false. For example a nonsingular cubic threefold $Y \subset \mathbb{P}^4$ is covered by a linear system of rational surfaces, namely its hyperplane sections. Nevertheless Y is known to be non-rational. The non-rationality of cubic threefolds was proved by Clemens and Griffiths in 1972, solving a long-standing classical problem known as the *Luroth problem*. The solution came out from the study of $\mathcal{H}^Y_{t+1}$, the Hilbert scheme of lines of Y. This Hilbert scheme turns out to be a projective nonsingular surface, which was classically studied by Fano, and for this reason it is called the *Fano surface of lines* of Y, and usually denoted by $F(Y)$. The Fano surface has a very rich geometry which faithfully reflects the geometry of Y. In fact the Torelli theorem holds for $F(Y)$: namely $F(Y)$ uniquely determines Y. This theorem was first proved by Beauville, and then reconsidered by Tyurin and others. It has been generalized to other 3-folds analogous to the cubic (certain *Fano threefolds*) and has been disproved in several situations too.

In general, the study of curves, especially of rational curves, that can exist on a variety, is one of the central tools in higher dimensional algebraic geometry.

Coming back to the beginning of this section, the Hilbert scheme of 0-dimensional subschemes of given degree n of a nonsingular projective variety Z can be quite complicated. The first next case is the one of surfaces. In this case a general theorem due to Fogarty states that H^Z_n *is nonsingular and irreducible of dimension* $2n$ *for every nonsingular projective irreducible* *surface* Y *and for every* $n \geq 1$ (see [6] for a proof). The study of these Hilbert schemes has progressed significantly in the last two decades, especially in the case when Y is a K3 surface. The reason is that, when Y is a complex algebraic K3 surface, the Hilbert schemes H_n^Z are *symplectic varieties*. By definition a symplectic variety carries an everywhere non-vanishing regular 2-form. This property has been discovered by Beauville.

When Y is a nonsingular variety of dimension $d \geq 3$ then the Hilbert schemes $\mathcal{H}_n^Y$ are in general singular and reducible. This was first discovered by Iarrobino in the 80's (see [6] for more details).

11 Severi varieties

We saw that the Hilbert scheme of plane curves of degree d can be naturally identified with the projective space

$$\Sigma_d := \mathbb{P}[H^0(\mathbb{P}^2, \mathcal{O}(d))] \cong \mathbb{P}^{\frac{d(d+3)}{2}}$$

A very important and interesting problem is to describe the loci in Σ_d consisting of points which parametrize singular curves with assigned "types" of singularities. In particular one can consider reduced curves having an assigned number δ of ordinary double points (called *nodes*) and κ of ordinary cusps. The loci parametrizing such curves have a natural functorially defined scheme structure (see definition below), and with this structure they are called *Severi varieties* or *Severi schemes* and denoted by $\mathcal{V}_d^{\delta,\kappa}$. In case $\kappa = 0$ we will denote the corresponding Severi variety by $\mathcal{V}_d^{\delta}$: it parametrizes curves of degree d having δ nodes and no other singularities.

Set-theoretic description. The geometrical configuration where everything takes place is described by the following diagram:

$$\begin{tikzcd} Z \arrow[r, hook] \arrow[dr] & \mathcal{C} \arrow[r, hook] \arrow[d, "\pi_2"] \arrow[drr, "\pi_1"] & \mathbb{P}^2 \times \Sigma_d & \\ & \Sigma_d & & \mathbb{P}^2 \end{tikzcd}$$

where $\mathcal{C}$ is the hypersurface of $\mathbb{P}^2 \times \Sigma_d$ defined by the universal equation:

$$F(X_0, X_1, X_2) = \sum_{i_0+i_1+i_2=d} a_{i_o,i_1,i_2} X_0^{i_0} X_1^{i_1} X_2^{i_2} = 0$$

Inside $\mathcal{C}$ we have the *critical locus* Z, defined by the equations:

$$\frac{\partial F}{\partial X_0} = \frac{\partial F}{\partial X_1} = \frac{\partial F}{\partial X_2} = 0 \tag{30}$$

Lemma 11.1 *(i) Z is irreducible, nonsingular and rational of codimension 3 in $\mathbb{P}^2 \times \Sigma_d$.*

(ii) π_2 maps Z birationally onto its image $W \subset \Sigma_d$, which is an irreducible rational hypersurface parametrizing all singular curves of degree d.

Proof. Left as an exercise (*hint:* consider the the projection $\pi_1 : Z \to \mathbb{P}^2$). $\square$

The equation of the hypersurface W is given by a polynomial in the coefficients a_{i_o,i_1,i_2} obtained from the equations (30) after elimination of the variables X_0, X_1, X_2. It follows that W *has degree* $3(d-1)$.

Z contains an open non-empty subset Z_1 whose set of closed points is

$$\{(p,s) : \mathcal{C}(s) \text{ has only nodes as singularities and } p \text{ is a node of } \mathcal{C}(s)\}$$

Let

$$W_1 := \pi_2(Z_1)$$

which is a dense open subset of W, and parametrizes all singular curves of degree d having only nodes as singularities. A general point of W_1 parametrizes a curve of degree d having one node and no other singularities. Therefore the points of $W \backslash W_1$ parametrize all singular curves of degree d which have at least one non-nodal singular point. The picture is the following:

$$\begin{tikzcd}
Z_1 \arrow[r, hook] \arrow[d, "\pi"'] & Z \arrow[r, hook] \arrow[d] & \mathcal{C} \arrow[d, "\pi_2"] \\
W_1 \arrow[r, hook] & W \arrow[r, hook] & \Sigma_d
\end{tikzcd}$$

The fibre $\pi^{-1}(s)$ over a point $s \in W_1$ is the scheme of nodes of the curve $\mathcal{C}(s)$. It follows that the morphism π is finite, birational and unramified. A more accurate analysis shows that W_1 has normal crossing singularities and π is the normalization morphism of W_1. A point $s \in W_1$ has multiplicity δ equal to the cardinality of $\pi^{-1}(s)$, i.e. equal to the number of nodes of the curve $\mathcal{C}(s)$. Therefore we can say that the Severi variety $\mathcal{V}_d^\delta$ is supported on the locus of points of W_1 having exactly multiplicity δ, in symbols:

$$\mathrm{Supp}(\mathcal{V}_d^\delta) = \mathrm{Sing}_\delta(W_1)$$

It is possible to deduce some properties of $\mathcal{V}_d^\delta$ from this description. In particular, the fact that W_1 has normal crossing singularities implies that $\mathcal{V}_d^\delta$ *has codimension* $\leq \delta$ *in* Σ. We refer to §4.7.3 of [6] for more details.

Scheme structure. The structure of scheme on $\mathcal{V}_d^\delta$ is defined by means of a universal property, as follows.

Theorem 11.2 *For every d, δ, κ here is a uniquely determined locally closed subscheme $\mathcal{V}_d^{\delta,\kappa} \subset \Sigma_d$ such that, for each family*

$$\begin{array}{ccc} \mathcal{X} & \hookrightarrow & \mathbb{P}^2 \times S \\ {\scriptstyle f}\downarrow & & \\ S & & \end{array}$$

of plane curves of degree d such that S is an algebraic scheme and all closed fibres $\mathcal{X}(s)$ have exactly δ nodes, κ cusps and no other singularities, the characteristic morphism $\chi : S \to \Sigma_d$ induced by the universal property of π_2 factors through $\mathcal{V}_d^{\delta,\kappa}$; in symbols:

$$\begin{tikzcd} S \arrow[rr, "\chi"] \arrow[dr] & & \Sigma \\ & \mathcal{V}_d^{\delta,\kappa} \arrow[ur, hook] & \end{tikzcd}$$

$\mathcal{V}_d^{\delta,\kappa}$ *is called the* Severi variety *of curves of degree d with δ nodes and κ cusps.*

A consequence is that every family f as in the statement can be obtained by pulling back the restriction to $\mathcal{V}_d^{\delta,\kappa}$ of the universal family π_2. This is the universal property of the Severi varieties.

We refer to [6], §4.7.2, for the proof of Theorem 11.2, which is long and technical.

Local properties. In this subsection we will restrict our attention to the Severi varieties $\mathcal{V}_d^\delta$ of nodal curves. Suppose that $C \subset I\!P^2$ is such a curve, so that $[C] \in \mathcal{V}_d^\delta$. The tangent space to Σ_d at the point $[C]$ is

$$T_{[C]}\Sigma_d = \frac{H^0(I\!P^2, \mathcal{O}(d))}{\langle C \rangle} = H^0(C, \mathcal{O}_C(d)) = H^0(N_{C/I\!P^2}) \tag{31}$$

where, with an abuse of notation, we have identified the curve C with its equation. These equalities can be interpreted as follows. A tangent vector to Σ_d at $[C]$ is a first order deformation of C. If $C(X_0, X_1, X_2) = 0$ is an equation of C, then a first order deformation is given by

$$C(X_0, X_1, X_2) + \epsilon g(X_0, X_1, X_2) = 0$$

where g is *any* homogeneous polynomial of degree d. Two such polynomials g and g' define the same deformation if and only if $g - g' = \alpha f$ for some $0 \neq \alpha \in \mathbf{k}$. This accounts for the first equality. The second and the third follow from the exact sequence

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{O}_{I\!P^2} \arrow[r, "C"] & \mathcal{O}_{I\!P^2}(d) \arrow[r] & \mathcal{O}_C(d) \arrow[r] \arrow[d, equal] & 0 \\ & & & N_{C/I\!P^2} & \end{tikzcd}$$

The tangent space to $\mathcal{V}_d^\delta$ at $[C]$ is therefore a subspace of $H^0(N_{C/I\!P^2})$ which is represented by a subspace of $H^0(I\!P^2, \mathcal{O}(d))$, i.e. by a linear system of plane curves of degree d. The curves passing through the singular points of C are called *adjoint* to C. The linear system of curves adjoint to C is identified with the subspace

$$H^0(I\!P^2, \mathcal{I}_\Delta(d)) \subset H^0(I\!P^2, \mathcal{O}(d))$$

where

$$\Delta := \{n_1, \ldots, n_\delta\}$$

is the scheme of nodes of C.

Lemma 11.3 $\Delta := \mathrm{Sing}(C)$ *imposes independent conditions to the curves of degree* $\geq d - 2$.

Proof. Assume first that C is irreducible, and let $\nu : \tilde{C} \to C$ be the normalization. Then

$$h^0(I\!P^2, \mathcal{I}_\Delta(d-3)) \geq \binom{d-1}{2} - \delta = g(\tilde{C}) \tag{32}$$

Since the adjoint curves of degree $d-3$ pullback to divisors of degree

$$d(d-3) - 2\delta = 2g(\tilde{C}) - 2$$

on $\tilde{C}$, they define a linear series on $\tilde{C}$ whose dimension cannot exceed $g(\tilde{C})-1$. Therefore (32) must be an equality, and in fact the adjoints of degree $d-3$ pullback to canonical divisors on $\tilde{C}$. In particular Δ imposes independent conditions to the curves of degree $\geq d-3$.

Assume now that $C = C_1 \cup C_2$ is reducible, with $\deg(C_i) = d_i$. Assume moreover that C_1 and C_2 are both nonsingular, so that $\Delta = C_1 \cap C_2$ and $\delta = d_1 d_2$. Then we have an exact sequence:

$$0 \longrightarrow \mathcal{O}(-d_1 - d_2 + k) \longrightarrow \mathcal{O}(-d_1 + k) \oplus \mathcal{O}(-d_2 + k) \longrightarrow \mathcal{I}_\Delta(k) \longrightarrow 0$$

which implies

$$h^1(\mathcal{I}_\Delta(k)) \leq h^2(\mathcal{O}(-d_1 - d_2 + k))$$

and therefore $h^1(\mathcal{I}_\Delta(k)) = 0$ if $k \geq d_1 + d_2 - 2 = d - 2$. This proves the assertion in this case.

The general case can be proved by combining the first and the second part of the proof, and is left to the reader. $\square$

The following proposition gives an explicit description of the tangent space to the Severi variety at $[C]$.

Proposition 11.4 *For any nodal curve C of degree d we have a natural identification*

$$T_{[C]}\mathcal{V}_d^\delta = \frac{H^0(I\!P^2, \mathcal{I}_\Delta(d))}{\langle C \rangle} \tag{33}$$

and $\mathcal{V}_d^\delta$ is nonsingular at $[C]$. In particular $\mathcal{V}_d^\delta$ is a nonsingular subvariety of Σ_d of dimension:

$$\dim(\mathcal{V}_d^\delta) = \dim(\Sigma_d) - \delta = \frac{d(d+3)}{2} - \delta = 3d + g - 1$$

where the last equality holds in case C is irreducible, with $g = \binom{d-1}{2}$ the genus of C.

Proof. (outline) Let $n \in \Delta$ be a node of C. Up to projectivity we may assume that in local coordinates $n = (0,0)$ is the origin and C has local equation

$$f(x,y) = xy + \tilde{f}(x,y)$$

where $\tilde{f}(x,y) \in m^3 = (x,y)^3$. A first order deformation $f + \epsilon g$ of C is tangent to $\mathcal{V}_d^\delta$ if and only if it is of the form:

$$\begin{aligned} f(x,y,\epsilon) &= (x - \alpha\epsilon)(y - \beta\epsilon) + \varphi(x,y,\epsilon) \\ &= xy - (\alpha y + \beta x)\epsilon + \varphi(x,y,\epsilon) \end{aligned}$$

where $\varphi \in (x - \alpha\epsilon, y - \beta\epsilon)^3$ and $\varphi(x,y,0) = \tilde{f}(x,y)$. This immediately implies that $g(x,y) \in m$. In other words the curve $g = 0$ contains n. On the global level this translates into the equality (33).

The equality (33) and Lemma 11.3 imply in particular that $\mathcal{V}_d^\delta$ has codimension $\geq \delta$ in Σ. But since we have already seen that its codimension is also $\leq \delta$, we deduce that $\mathcal{V}_d^\delta$ is nonsingular of codimension δ. $\square$

The properties of nonsingularity and codimension of the Severi varieties of nodal curves stated in Proposition 11.4 do not extend to the varieties $\mathcal{V}_d^{\delta,\kappa}$. In other words Severi varieties of plane curves with nodes *and cusps* can be singular and of dimension larger that the one one would expect from a naive count of parameters.

Global properties. In general the Severi varieties $\mathcal{V}_d^\delta$ are reducible. This already happens for $d = 4$ and $\delta = 3$: $\mathcal{V}_4^3$ has two components, one consisting of irreducible quartics with 3 nodes, and the other of quartic reducible in a line and a nonsingular cubic intersecting transversally.

In 1915 Severi gave a controversial argument to show that *for all δ and d the variety $\mathcal{V}_d^\delta$ has a unique irreducible component consisting of irreducible curves.* His argument contains a gap, and the problem of proving the assertion of Severi remained open until J. Harris proved it in 1986 [3]. Other proofs have been found later by Z. Ran and by R. Treger.

The interest of the Severi statement is that it asserts the irreducibility of the variety of plane irreducible curves of any degree and number of nodes, regardless to their relation with moduli. It is even more remarkable that the Severi varieties $\mathcal{V}_d^{\delta,\kappa}$ are in general reducible. The properties of Severi varieties of nodal curves contrast with the case of space nonsingular irreducible curves, whose families are in general reducible, as we saw in §9.

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