A BRIEF INTRODUCTION TO ALGEBRAIC CURVES

EDOARDO SERNESI
LECTURES DELIVERED AT NERVI, APRIL 12–15, 1984,
TRANSLATED AND UPDATED BY CLAUDIO FONTANARI

1

Even though curves are the most elementary and best known algebraic varieties, nevertheless many crucial related problems still remain widely open.

Algebraic curves can be investigated by following several different approaches (analytic, algebro-geometric, topological, mathematical physics): however, in these lectures we are going to focus on just one point of view, namely, the strictly algebro-geometric one. We always refer to varieties and schemes defined over the complex field $\mathbb{C}$.

We prefer to outline a global overview of algebraic curves rather than address a few specific topics: indeed, all different aspects of the theory are essential in order to properly understand the development of algebraic geometry.

Several results, or even whole parts of the theory, have been known much before than actually proven: indeed, classical geometers made up for the lack of suitable techniques by intuitive geometrical arguments. Several decades (and in certain cases even a century) elapsed before replacing their plausibility arguments with rigorous proofs or finding counterexamples to their claims, in several cases the proof (or the counterexample) is still missing.

By the way, this is one of the reasons stimulating the critical reading of classical authors and the effort to clarify the obscure points of their work.

In the present work we are going to describe some of those arguments, which are not actual proofs but are often enlightening.

Unluckily we are compelled to just touch some important questions and even to completely avoid some relevant arguments, such as the Schottky problem, enumerative geometry, higher rank vector bundles on curves, real algebraic curves, monodromy questions, set-theoretical complete intersections and many other topics. We follow an historical perspective in order to properly understand the development of concepts and methods.

2

As a starting point we choose the year 1851, when the doctoral dissertation [53] by Riemann appeared, followed in 1859 by the other fundamental contribution [54].

Riemann resumed the research started by the analysts, from Cauchy to Puiseux, about algebraic functions of one complex variable, i.e. the functions $y(x)$ implicitly defined by an equation $f(x, y) = 0$ where $f \in \mathbb{C}[x, y]$. By adopting a geometric point of view, Riemann introduced the surfaces named after him (made up by as many copies of $\mathbb{C}$ as the degree of f with respect to one of the variables and suitably glued together in certain "ramification points") and gave a transparent geometric interpretation of those involved analytic theorems. To every surface Riemann associated a topological invariant, the *genus*, and showed that it equals the number of linearly independent abelian integrals of the first kind associated to f, thus establishing a deep connection with the investigations of Legendre, Abel and Jacobi on elliptic functions.

However, in his contributions we find not only the unification of several different branches of his contemporary mathematics, but also a radically new approach to geometry as the investigation of invariant properties under birational transformations. Indeed, Riemann was the first one to adopt this point of view, by showing that the genus is a birational invariant.

On the other hand, we should not forget that Riemann was working in an analytic framework: in his investigations he applied the theory of potential and was led by considerations inspired by fluidodynamics.

Riemann's ideas were resumed by Clebsch, whose aim was to recover the same results and develop them using purely algebraic means. Among other things, he discovered that the genus g of an irreducible plane curve of degree n with δ nodes is equal to the difference with the maximum number of nodes, namely, $g = \frac{1}{2}(n-1)(n-2) - \delta$.

This program, interrupted due to the early death of Clebsch, was carried on in Germany by his student M. Noether and by Brill.

At the beginning of the eighties the theory of plane and space algebraic curves had already been developed with purely algebraic means.

3

As far as plane curves are concerned, the culminating point of that period is the paper [5] by Brill and Noether dating back to 1873, which summarized all knowledge obtained insofar and was the starting point for the investigations of a whole generation of geometers.

There the fundamental concept of *linear series* was introduced.

Roughly speaking, a linear series of dimension r and degree n (denoted by g^r_n) on an irreducible curve C is a set of ∞^r groups of n points of the curve obtained by intersecting it with another curve varying in a linear system of dimension r.

More precisely, a *divisor* on the irreducible plane curve C is a formal finite linear combination $D = \sum_p n_p p$ of points $p \in C$ with integral coefficients, where a singular point (assumed for the sake of simplicity to be an ordinary one, i.e. with distinct tangents) accounts for as many distinct points as the number of tangent directions. The *degree* of D is $\deg(D) = \sum_p n_p$. The divisor D is *effective* (denoted by $D \geq 0$), if $n_p \geq 0$ for every p.

With the sum defined as the natural formal one, divisors form an abelian group $\mathrm{Div}(C)$, where the zero is the null divisor.

If $D_1, D_2 \in \mathrm{Div}(C)$ then $D_1 \geq D_2$ means $D_1 - D_2 \geq 0$. For every plane curve such that C is not one of its irreducible components, the divisor cut out on C can be defined in a natural way: it turns out to be effective and of degree equal to the product of the degrees of the two curves.

This way we associate to a linear system of plane curves a set of effective divisors, the so-called *linear series cut out by the system.* If all divisors of this g^r_n are $\geq$ than the same effective divisor E, this is said to be a *fixed divisor* of the series and by subtracting E from every divisor of the g^r_n we obtain a g^r_{n-v}, where $v = \deg(E)$.

For instance, if C has degree n, then the lines passing through a point $p \in \mathbb{P}^2$ cut out a g^1_n. If $p \in C$ then $E = p$ is a fixed divisor (in this case p is said to be a *fixed point*) of the series and by subtracting p we obtain a g^1_{n-1}. If $p \in C$ is an s-uple point, then by subtracting the s tangent directions in p we obtain a g^1_{n-s}.

Two effective divisors are *linearly equivalent* if there exists a linear series containing both of them (in particular, they have the same degree). More generally, any two divisors D_1 and D_2 are linearly equivalent (denoted by $D_1 \sim D_2$) if $D_1 - D_2 = E_1 - E_2$, where E_1 and E_2 are effective and linearly equivalent.

The set of all effective divisors linearly equivalent to a certain D form a linear series $|D|$ which is not contained in a bigger series, hence is said to be a *complete linear series.*

If $D \sim 0$ then $|D|$ is a g^0_0. Every linear series is contained in a unique complete linear series.

For instance, the g^2_3 cut out by the lines on a cubic with one node is not complete, since it is contained in the g^3_3 cut out on C by the conics through the node and another point. This g^3_3 is complete since a g^r_n with $r > n$ cannot exist (indeed, there are ∞^n n-tuples of points on a curve and no more).

The sum between divisors is compatible with linear equivalence, hence linear series can be summed and subtracted.

Brill and Noether investigated linear series by exploiting two essential tools.

The first one, of geometric flavour, is the theorem stating that every irreducible plane curve can be birationally transformed into one carrying only ordinary multiple points by applying to it a finite number of quadratic transformations of the plane (i.e. transformations defined up to a change of homogeneous coordinates as $x_0 = y_1 y_2$, $x_1 = y_0 y_2$, $x_2 = y_0 y_1$): this theorem is due to Noether [46] and independently to Kronecker (unpublished). Its relevance is due to the fact that linear series are preserved under a birational transformation, hence it allows to replace an irreducible curve with arbitrary singularities by a much simpler one.

The other tool, of more algebraic flavour, is the theorem by Noether which is known as $AF + B\Phi$ Theorem, and he called Fundamentalsatz, proven for the first time in [47] and subsequently refined and improved by the same author. This theorem establishes the possibility of expressing the equation of a curve $f = 0$, passing in a suitable way through the intersection of two curves $F = 0$ and $\Phi = 0$, in the form $f = AF + B\Phi$.

In the approach of Brill and Noether, abelian integrals and the topological considerations by Riemann were replaced by the study of linear series and their properties. The genus g was characterized by the dimension of the so-called *canonical series*, which is the complete g^{g-1}_{2g-2} cut out on C by the plane curves of degree $n-3$ *adjoint* to C. If $g=0$ then the canonical series is empty, i.e. there are no adjoint curves of degree $n-3$. For every $g \geq 1$ the canonical series is the unique g^{g-1}_{2g-2} on C.

If $|D|$ is a complete g^r_n and K is a canonical divisor (i.e. a divisor such that $|K|$ is the canonical series), then the series $|K-D|$ is called *residue* to D: it is a g^{i-1}_{2g-2-n}, where $i \geq 0$ is the *index of speciality* of D and D is said to be *special* if $i>0$. Since $\dim(|K-D|) \leq \dim(|K|)$ we have $i \leq g$ and equality holds if and only if $D \sim 0$.

In the terminology of Brill and Noether the *Riemann-Roch Theorem*, allowing to compute the dimension of spaces of meromorphic functions with prescribed singularities on a fixed Riemann surface, became the relationship

$$r - i = n - g$$

among the genus g and the characters of a complete g^r_n.

In order to exploit modern sheaf theory, we introduce the normalization $\nu : C \to X$ of the plane curve X and we recall that every $D \in \mathrm{Div}(C)$ defines an invertible sheaf $\mathcal{O}(D)$ over C and, if D is effective, a section $s \in \Gamma(C, \mathcal{O}(D))$ up to a nonzero constant such that $D = \{s = 0\}$. This way, a linear series of dimension r containing D corresponds to a vector subspace V of dimension $r+1$ of $\Gamma(C, \mathcal{O}(D))$ cointaining s. If $V = \Gamma(C, \mathcal{O}(D))$ then we obtain a complete linear series.

Linearly equivalent divisors define isomorphic invertible sheaves. Thus invertible sheaves of fixed degree (up to isomorphism) correspond to linear equivalence classes of divisors and those carrying sections to complete linear series of fixed degree. The sum of divisors induces tensor product of sheaves. The null and the canonical series correspond respectively to the structural sheaf $\mathcal{O}$ and to the so-called canonical sheaf ω.

In this language the study of linear series is translated into that of invertible sheaves and their sections.

For instance, let us consider the Riemann-Roch Theorem. From the point of view of sheaf theory, its proof splits into two parts. The first ingredient is

Theorem 1. (Serre duality) *For every divisor D on C there is a nondegenerate bilinear form*

$$H^1(D) \times H^0(\omega(-D)) \to H^1(\omega) \cong \mathbb{C}$$

thus inducing an isomorphism $H^1(D) = H^0(\omega(-D))^*$.

In particular, we have $g = h^0(\omega) = h^1(\mathcal{O})$ and special divisors define special invertible sheaves L, i.e. with $h^1(L) > 0$, where we have denoted by $h^j()$ the dimension of $H^j()$.

The second part of the proof relies on the exact sequence of sheaves over C associated to every effective divisor D of degree n:

$$0 \to \mathcal{O} \to \mathcal{O}(D) \to \mathcal{O}_D(D) \to 0$$

hence we obtain $h^0(D) - h^1(D) = 1 - h^1(\mathcal{O}) + n$ and, by applying Serre duality,

$$h^0(D) - h^0(\omega(-D)) = n - g + 1$$

which is precisely the Riemann-Roch Theorem.

Update. In [87] Hartshorne introduced the notion of *generalized divisor* on an integral, projective *Gorenstein* curve C (recall that C is Gorenstein if and only if ω_C is invertible). According to [87], a generalized divisor on C is just a fractional ideal of C, namely, a nonzero subsheaf of the constant sheaf of the function field of C which is a coherent $\mathcal{O}_C$-module.

Even though two generalized divisors can never be added unless at least one of them is a honest Cartier divisor, nevertheless both the Riemann-Roch Theorem and Serre duality hold for generalized divisors (see [87], Theorem 1.3 and Theorem 1.4).

4

The link between linear series and the geometry of C is given by their relationship with the maps of C in $\mathbb{P}^2$, $\mathbb{P}^3$, or in any $\mathbb{P}^r$.

For instance, the g^2_n cut out on a plane curve X, hence on C, by the lines in $\mathbb{P}^2$ allows to recover the map $\nu : C \to X$. Indeed, let the g^2_n correspond to a vector subspace $V \subseteq H^0(L)$ of dimension 3 for a certain invertible sheaf L: V may be identified with the vector space of linear forms on $\mathbb{P}^2$, hence $\mathbb{P}^2$ identifies with the projective space $\mathbb{P}(V^*)$ of 2-dimensional subspaces of V. Hence we have $\nu(p) = \{s \in V : s(p) = 0\}$.

It is straightforward to generalize the above remark in order to interpret any base-point free g^r_n as a map $C \to \mathbb{P}^r$, where the g^r_n is induced by intersecting the image with hyperplanes. In a similar fashion, intersections with quadrics, cubics, and so on, define corresponding linear series.

The same curve can be realized in different ways in a projective space according to its linear series (notice however that a g^r_n does not necessarily induce an immersion of C in $\mathbb{P}^r$). Therefore, understanding either linear series on a curve or curves in a projective space are just two aspects of the same problem.

This point of view was applied by Noether and Halphen to the investigation of space curves, i.e. contained in a $\mathbb{P}^3$, in their important papers [48] and [25], published in the same year and winning ex-aequo the Steiner prize in 1882.

Halphen and Noether addressed the problem of classifying space curves according to their degree and genus, thus obtaining several important results. In particular, Halphen tried to characterize the set of pairs (n, g) such that there exist nonsingular and irreducible space curves of degree n and genus g. His main result, which he stated with an incomplete proof, has been proven by Gruson and Peskine [24] only in 1978. It claims that:

Theorem 2. (1) *A nonsingular irreducible non-plane space curve C of degree n has genus*

$$g \le \pi(3,n) = \begin{cases} \frac{(n-2)^2}{4} & \text{if } n \text{ is even} \\ \frac{(n-1)(n-3)}{4} & \text{if } n \text{ is odd} \end{cases}$$

(2) *If*

$$1 + n(n-3)/6 < g \le \pi(3,n)$$

then C lies on a quadric (and not all values of g satisfying these inequalities can be obtained, but there are some gaps).

(3) *For every* $0 \le g \le 1 + n(n-3)/6$ *there exist nonsingular irreducible non-plane space curves of genus* g *and degree* n.

The proof by Gruson and Peskine relies on the explicit construction of curves with prescribed degree and genus either on a nonsingular cubic surface or on a quartic surface with a double line.

The investigation of curves in $\mathbb{P}^r$ was pursued in the same period especially in Italy. Here a new geometric school was emerging, with Cremona (first in Bologna, then in Milan and Rome), Betti (in Pisa) who had been the first disseminator of Riemann's ideas in Italy, Beltrami (in Bologna), Battaglini (in Naples) and others.

First with Veronese and then with C. Segre the study of hyperspace geometry was extensively developed. Along these lines, new interesting results about algebraic curves emerged. C. Segre, and a few years later his student G. Castelnuovo, rephrased the theory of linear series on curves purely in terms of projective geometry.

The attempt to free the theory from Noether's Fundamentalsatz led Castelnuovo to give a new proof of the Riemann-Roch Theorem relying on enumerative geometry.

He also determined the maximum genus $\pi(r,n)$ for a curve of degree n in $\mathbb{P}^r$, thus generalizing the corresponding formula given by Halphen in the case $r = 3$ and characterized curves of maximal genus, now called *Castelnuovo curves.* From that period, the papers [59], [6], and [7], are especially important; it is worth mentioning also the interesting [14] by Fano (his master thesis written under the guidance of Castelnuovo), whose investigations should be resumed.

For an arbitrary r, the classification of nonsingular curves in $\mathbb{P}^r$, with respect to the genus and the degree as Halphen did for $r = 3$, was never completed. We point out the contributions by Gieseker [19] and Harris [27] to this problem.

Update. The classification of all possible genera in the admissible range for smooth irreducible curves was extended to $\mathbb{P}^4$ and $\mathbb{P}^5$ by Rathmann [93], and to $\mathbb{P}^6$ by Ciliberto [74].

Moreover, as we have seen before, curves C in $\mathbb{P}^3$ whose genus is big with respect to the degree must lie on surfaces of small degree, so it is natural to refine the bound introducing the minimal degree s allowed for surfaces containing C. Halphen himself gave, in fact, a bound for the genus of space curves of degree n, not contained in surfaces of degree $< s$.

As pointed out in [73], Halphen's theory can be generalized to curves in $\mathbb{P}^r$ in several ways: one may ask for the maximal genus of curves $C \subset \mathbb{P}^r$ as a function of the degree n and either of the minimal degree allowed for hypersurfaces through C, or of the minimal degree s allowed for surfaces through C. The first point of view seems to be still widely open; for the second, results when s is not too big with respect to r are contained in [27]. The paper [73] pushes further Eisenbud-Harris' point of view by establishing the bound for the genus of irreducible, nondegenerate curves of degree n in $\mathbb{P}^r$, not contained on surfaces of degree $< s$, when n is large with respect to s.

Another interesting problem concerns the existence of a smooth and irreducible family parametrizing irreducible nonsingular curves of degree n and genus g in $\mathbb{P}^r$ having "the expected number of moduli", in a suitable sense to be clarified later. Relevant results in this direction are due to Gieseker [19] and Sernesi [61].

5

The approach by Segre and Castelnuovo relied on the study of linear series on the canonical curve.

It is easy to show that if C has genus $g \geq 3$ and is not hyperelliptic (i.e. it has not a g^1_2: indeed, every genus 2 curve has a g^1_2, the canonical series, while for $g \geq 3$ not all curves of genus g are hyperelliptic, but there exist hyperelliptic curves of genus g for every g), then the canonical series maps C isomorphically onto a curve of degree $2g-2$ in $\mathbb{P}^{g-1}$, a so-called *canonical curve*, the *canonical model* of C.

In the hyperelliptic case, the canonical series defines a $2:1$ map of C onto a rational curve of degree $g-1$ in $\mathbb{P}^{g-1}$.

The study of linear series on a hyperelliptic curve is not difficult. If instead C is not hyperelliptic, and we identify it with the canonical model, we see that an effective divisor is special if and only if it is contained in a hyperplane of $\mathbb{P}^{g-1}$. More precisely, the index of speciality of D is the number of linearly independent hyperplanes containing D. Hence if $|D|$ is a g^r_n with $n \leq 2g-2$, then by Riemann-Roch the linear subspace $< D >$ n-secant C generated by D in $\mathbb{P}^{g-1}$ has dimension

$$\dim(< D >) = g - 1 - i = n - r - 1.$$

This geometric interpretation of the Riemann-Roch Theorem enlightens a remarkable property of the canonical curve: if C has an n-secant $\mathbb{P}^{n-r-1}$ then it has ∞^r of them.

For instance, if C has a trisecant line then it has ∞^1 of them, cutting out on C the divisors of a complete g^1_3.

More generally, every property of special linear series on C mirrors a projective property of the canonical curve, and vice versa, hence the interest for canonical curves. Noether had inaugurated their investigation by showing that they are projectively normal (recall that a curve $C \subset \mathbb{P}^r$ is said to be *projectively normal* if it is nonsingular and for every $d \geq 0$ hypersurfaces of degree d cut out on C a complete linear series).

Enriques proved in 1919, in a short note [12], that $C \subset \mathbb{P}^{g-1}$ is the intersection of the quadrics containing it with only two exceptions: either C has a g^1_3 (it is "trigonal"), or C has genus 6 and it is isomorphic to a nonsingular plane quintic. The proof by Enriques, very elegant and concise but incomplete, was resumed and completed by Babbage in 1939 [3].

The same subject was also addressed by Petri in a paper [50] dating back to 1922 but rediscovered only in the seventies. Petri proved that the quadrics containing the canonical curve generate its ideal except in the two cases described by Enriques, where the ideal is generated by quadrics and cubics. Petri explicitly describes, in a very tricky way, a basis of the vector space of quadrics containing the canonical curve C and expresses in terms of them every polynomial in the ideal. His proof has been written and clarified in modern language by Saint Donat [55].

6

A subsequent paper [51] was devoted by Petri to extending the previous result to projective curves more general than canonical curves. The general problem addressed by him is to find a procedure to explicitly describe the equations defining a curve in $\mathbb{P}^r$ (i.e. its ideal), or at least to deduce informations on the equations from informations on the curve, such as the genus, the degree and several properties of the hyperplane linear series and its multiples.

A related example is the theorem (due to G. Gherardelli [18]) stating that a nonsingular curve in $\mathbb{P}^3$ is a complete intersection if and only if it is projectively normal and its canonical linear series coincides with the series $|dH|$ for some $d \geq 0$, where H is the divisor of a plane section (this last property is usually referred to as C is "sub-canonical").

A class of curves wider than complete intersections, which are easily described, are the projectively normal ones in $\mathbb{P}^3$, investigated by Apery, Gaeta and Dubreil, and subsequently by Peskine and Szpiro [49]. These curves are characterized by the fact that their ideal is generated by the maximal minors of a suitable matrix M of dimension $m \times m+1$ with entries homogeneous polynomials: this is equivalent to their geometric property of being "of finite residual" ($C \subset \mathbb{P}^3$ is of finite residual means that there exists a sequence $C = C_1, \ldots, C_k$ of curves such that $C_i \cup C_{i+1}$ is a complete intersection for $i = 2, \ldots, k-1$ and C_k is a complete intersection). An important property of these curves is that they are all described by letting the corresponding matrix M vary generically (i.e. without imposing any closed condition to the coefficients of the polynomials defining it). Unluckily such a simple description is compensated by the fact that projectively normal curves in $\mathbb{P}^3$ are almost all very special among those of their genus.

Explicitly describing curves which are sufficiently general among those of fixed genus, in a suitable sense to be clarified later, becomes harder and harder as g grows. Petri was able to give only partial results in this direction, but some of his ideas have been subsequently resumed and extended giving rise to a very interesting research line.

In 1960 Mumford proved an analogue of the theorem by Enriques-Babbage-Petri, stating that if D is a divisor of sufficiently high degree n on a nonsingular curve C of genus g, the complete g^{n-g}_n $|D|$ embeds C in $\mathbb{P}^{r-g}$ and the image is projectively normal and its ideal is generated by quadrics [42]. Mumford's estimate is $n \geq 2g+1$ for projective normality and $n \geq 3g+1$ for quadratic generation of the ideal. This last one was later improved to $n \geq 2g+2$ by Saint Donat [55].

When n is low with respect to the genus it is much more difficult to provide even qualitative informations about the ideal of a projective curve of degree n. Several partial generalizations of the above results are now available, but there is still a lot of work to do in this direction.

In [21] M. Green conjectured that the various numerical characters of a minimal free resolution of the ideal of a canonical curve $C \subset \mathbb{P}^{g-1}$ closely follow geometric properties of C, thus generalizing the fact that the degrees of the generators of the ideal of C depend on geometric properties of C. About this problem we have only partial results.

Update. More precisely, the Clifford index of a line bundle L of degree d with k sections on a curve C of genus g is defined as $\mathrm{Cliff}(L) = d - 2(k-1)$. The Clifford index of the curve is defined as the minimum of the Clifford indices of its line bundles of degree at most $g-1$.

On the other hand, a canonical curve $C \subset \mathbb{P}^{g-1}$ is said to satisfy condition N_0 if it is projectively normal, property N_1 if its ideal is generated by quadrics and property N_p if it satisfies property N_{p-1} and the p-th syzygies are generated by linear relations among the $(p-1)$-st syzygies.

Green's conjecture states that a curve satisfies property N_p if and only if $\mathrm{Cliff}(C) > p$. This is classically known for $p = 0$ and 1.

The"only if" part was proven by Green and R. Lazarsfeld in the appendix of [21], and the conjecture was established for $g \leq 8$ and for $p = 2$ by Schreyer in [94] and [95]. More recently, Voisin was able to prove Green's conjecture for the generic curve of even genus in [99] and odd genus in [100] by considering curves contained in suitable $K3$ surfaces and studying syzygies on the surface and their restriction to the curve.

More generally, if L is a spanned line bundle on a curve C of genus g, Green investigated in [21] certain L-valued Koszul cohomology groups $K_{p,q}(C,L)$ of C. In particular, if d is the gonality of the curve and L is of sufficiently large degree, Green and Lazarsfeld conjectured that

$$K_{h^0(C,L)-d,1}(C,L) = 0$$

for the generic curve: this was later proved in [66] by Aprodu and Voisin for even genus and in [65] by Aprodu in the odd genus case.

7

Let us fix an irreducible nonsingular curve C and denote by $\mathrm{Pic}(C)$ the *Picard group* of C, i.e. the group of isomorphism classes of invertible sheaves on C. We have

$$\mathrm{Pic}(C) = \oplus_n \mathrm{Pic}^n(C)$$

where $\mathrm{Pic}^n(C)$ is the subset of invertible sheaves of degree n. Notice that $\mathrm{Pic}^0(C)$ is a subgroup of $\mathrm{Pic}(C)$ and for fixed $M \in \mathrm{Pic}^n(C)$ the map

$$\begin{array}{ccc} \mathrm{Pic}^0(C) & \to & \mathrm{Pic}^n(C) \\ L & \mapsto & L \otimes M \end{array}$$

is a bijection.

The first geometrically relevant fact is that $\mathrm{Pic}^0(C)$, hence every $\mathrm{Pic}^n(C)$, has a natural structure of irreducible and nonsingular projective variety of dimension g equal to the genus of C: it is indeed an abelian variety, namely the commutative group structure is compatible with the structure of variety. It is called the *Jacobian* of C, also denoted by $J(C)$.

If $g = 0$, then $J(C)$ is just one point and $\mathrm{Pic}(C) \cong \mathbb{Z}$: this means that all invertible sheaves of given degree n are isomorphic, hence all divisors of degree n are linearly equivalent. It is indeed a simple case, due to the fact that every curve of genus zero is rational and that on $\mathbb{P}^1$ any two point are linearly equivalent.

If $g = 1$, then $J(C)$ is a curve. Once fixed a point $p_0 \in C$, we can define a map

$$\begin{array}{rcl} C & \to & \mathrm{Pic}^0(C) \\ p & \mapsto & \mathcal{O}(p - p_0). \end{array}$$

This map (which is a morphism) is not constant (otherwise C would be rational, carrying a g^1_1), hence it is surjective. It is also injective by the same reason, hence C is isomorphic to its Jacobian.

More generally, let us consider a curve of arbitrary genus g. For every $n \geq 1$ the set of effective divisors of degree n on C has a natural structure of nonsingular projective algebraic variety: it is obtained from the cartesian product $C \times C \times \ldots \times C$ (n times) by taking the quotient by the natural action of the symmetric group σ_n permuting factors: it is denoted by $C^{(n)}$ and it is called the *n-th symmetric power* of C. For every n we have a natural map

$$\begin{array}{rcl} \psi_n : C^{(n)} & \to & \mathrm{Pic}^n(C) \\ D & \mapsto & \mathcal{O}(D) \end{array}$$

having as fibers the linear systems of degree n. It is straightforward from Riemann-Roch that ψ_n is surjective if $n \geq g$. Fixed $p_0 \in C$, we get

$$\begin{array}{rcl} C^{(n)} & \to & \mathrm{Pic}^0(C) \\ D & \mapsto & \mathcal{O}(D - np_0) \end{array}$$

which is the composition of ψ_n and the isomorphism of varieties

$$\begin{array}{rcl} \mathrm{Pic}^n(C) & \to & \mathrm{Pic}^0(C) \\ L & \mapsto & L(-np_0). \end{array}$$

Notice that $C^{(1)} = C$ and for every n the fibers of ψ_n are connected because they are projective spaces. In particular, it turns out that ψ_g is a birational isomorphism, i.e. the Jacobian of C is birationally isomorphic to $C^{(g)}$, from which it is obtained by contracting to a point every special linear system, as it follows from Riemann-Roch.

For instance, if $g = 2$ the map

$$C^{(2)} \to \mathrm{Pic}^2(C)$$

is bijective with the exception of the fiber of ω, which is a $\mathbb{P}^1$, the canonical g^1_2. Hence $J(C)$ is obtained from $C^{(2)}$ by contracting a curve (which is exceptional of the first kind).

Assume $g = 3$ and C non-hyperelliptic, so that the canonical curve of C is a nonsingular plane quartic. Consider the map

$$\psi_3 : C^{(3)} \to \mathrm{Pic}^3(C).$$

The special divisors $D \in C^{(3)}$ are the ∞^2 triples of aligned points of $C \subset \mathbb{P}^2$. The set of the corresponding $\mathcal{O}(D)$ in $\mathrm{Pic}^3(C)$ can be identified with the set of fourth points of intersection of C with the corresponding lines. Hence the locus in $\mathrm{Pic}^3(C)$ where ψ_3 is not an isomorphism is a curve isomorphic to C and its inverse image is a surface in $C^{(3)}$, birationally isomorphic to $C \times \mathbb{P}^1$.

More generally, it is easy to show (for instance, at least in the non-hyperelliptic case, by using the geometric version of Riemann-Roch) that the image $\psi_{g-1}(C^{(g-1)})$ is a divisor in $\mathrm{Pic}^{g-1}(C)$, which is called *theta divisor* and denoted by Θ, consisting of all special invertible sheaves of degree $g-1$. It is an ample divisor on $\mathrm{Pic}^{g-1}(C)$ and defines a so-called principal polarization. A celebrated theorem by Torelli [64] states that the pair $(\mathrm{Pic}^{g-1}(C), \Theta)$ identifies C, namely, if C' is a curve such that there exists an isomorphism between $\mathrm{Pic}^{g-1}(C)$ and $\mathrm{Pic}^{g-1}(C')$ preserving theta divisors, then C' is birationally isomorphic to C.

For instance, in the case $g = 2$ the theta divisor is isomorphic to C (more generally, $\psi_1(C) = C$ for every g).

If $g = 3$ then the theta divisor is isomorphic to $C^{(2)}$ unless C is hyperelliptic, in which case the g^1_2 is contracted to a point in $\mathrm{Pic}^2(C)$, singular for the theta divisor.

Let us go back to the general case and introduce

$$W^r_n = \{L \in \mathrm{Pic}^n(C) : h^0(C, L) \geq r + 1\},$$

i.e., the set of all complete g^s_n, with $s \geq r$. Fix a point $p_0 \in C$ and via multiplication by $\mathcal{O}(-np_0)$ realize every W^r_n as a subset of $\mathrm{Pic}^0(C) = J(C)$. It can be shown that W^r_n has a natural structure of closed subscheme of $J(C)$.

For instance, we have $W^0_n = \psi_n(C^{(n)})$. If $n \leq g$ then $\dim(W^0_n) = n$, while if $n \geq g$ then $W^0_n = W^1_n = \ldots = W^{n-g}_n$. In this notation, the theta divisor is W^0_{g-1} and obviously $W^{r+1}_n \subseteq W^r_n$.

It is a fundamental problem to study the structure of the W^r_n's. First of all: when is $W^r_n \neq \emptyset$?

The answer depends on g, r, n, but also on C. For instance, $W^1_2 \neq \emptyset$ if and only if C is hyperelliptic.

Brill and Noether gave a criterion on g, r, n in order to have $W^r_n \neq \emptyset$ on every curve of genus g, based on the following easy argument.

Assume C nonhyperelliptic (the hyperelliptic case can be treated separately), canonically embedded into $\mathbb{P}^{g-1}$. We can assume that $g-n+r > 0$, i.e. that W^r_n consists of special sheaves, since the nonspecial case is trivial. It is easy to see that if $W^r_n \neq \emptyset$ then $W^{r+1}_n \subsetneq W^r_n$, hence $W^r_n \neq \emptyset$ is equivalent to the existence of $L \in W^r_n \setminus W^{r+1}_n$. By arguing on the canonical curve, the existence of such an L is equivalent to that of a $\mathbb{P}^{n-r-1}$ n-secant C.

The $\mathbb{P}^{n-r-1}$'s containing at least one point of C form a nonempty irreducible subvariety of codimension $g-n+r-1$ of the Grassmannian $\mathbb{G}(n-r-1, g-1)$. Thus the set S of the $\mathbb{P}^{n-r-1}$'s n-secant C has codimension in $\mathbb{G}(n-r-1, g-1)$ not exceeding $n(g-n+r-1)$, *if it is not empty*. In this case S has dimension at least r, as it follows from the geometric version of Riemann-Roch. Hence we have

$$r \leq \dim(\mathbb{G}(n-r-1, g-1)) - n(g-n+r-1),$$

that is to say,

$$g - (r+1)(g-n+r) \geq 0.$$

The numerical quantity $\rho(g, r, n) := g - (r+1)(g-n+r)$ is called the *Brill-Noether number*. It was implicitly evident to Brill and Noether that

$S \neq \emptyset$, however the above argument definitely does not prove that $W^r_n \neq \emptyset$ if $\rho(g,r,n) \geq 0$. What it proves is that, if the above inequality is satisfied and $W^r_n \neq \emptyset$, then every irreducible component of W^r_n has dimension $\geq \rho(g,r,n)$.

A complete proof of the Brill-Noether criterion, i.e. of the fact that on every curve of genus g we have $W^r_n \neq \emptyset$ and $\dim(W^r_n) \geq \rho(g,r,n)$ if $\rho(g,r,n) \geq 0$ is due to Kleiman and Laksov [37] and, independently, to Kempf [35].

We may wonder whether and when the above estimate is sharp, namely, we have precisely $\dim(W^r_n) = \rho(g,r,n)$.

The answer was given once again by Brill and Noether, who claimed (but without proof) that $W^r_n = \emptyset$ if $\rho(g,r,n) < 0$ and $\dim(W^r_n) = \rho(g,r,n)$ if $\rho(g,r,n) \geq 0$ on every "sufficiently general" curve.

The precise meaning of the condition "C is sufficiently general" comes from the consideration of the moduli space, which we are going to address in a short time. By now, we content ourselves of its intuitive meaning.

Justifying the claim, or better the conjecture, of Brill and Noether, has been an open problem for many years. An attempt of proof is due to Severi [62]. In [36] Kleiman showed that Severi's argument could be reduced to a problem of enumerative geometry. Such a problem was then solved, hence the conjecture completely proven, by Griffith and Harris in [22] (in the case $r = 1$ the proof had already been given by Laksov in an appendix to [36]). From this result it follows that on a sufficiently general curve all schemes W^r_n are reduced.

Another interesting problem comes from the easy fact that W^{r+1}_n is always contained in the singular locus of W^r_n, for every curve C. Indeed, in the case $r = 0$, $n \leq g-1$, W^1_n coincides with the singular locus of W^0_n: more precisely, the s-uple points of W^0_n, $s \geq 2$, are the $L \in W^{s-1}_n \setminus W^s_n$, i.e. such that $h^0(C,L) = s$. This is the content of the so-called Riemann singularity theorem in the case $n = g-1$, generalized by Kempf in [34] to the other values $n \leq g-1$.

It is easy to give examples where W^{r+1}_n is different from $\mathrm{Sing}(W^r_n)$, but in all these examples the curve has very special properties that a more general curve of the same genus does not exhibit. It is hence natural to conjecture that if C is sufficiently general and $\rho(g,r,n) \geq 0$ then $W^{r+1}_n = \mathrm{Sing}(W^r_n)$. This conjecture, formulated by Mayer, is equivalent to a statement of cohomological nature made by Petri in [51]. For a thorough discussion of this equivalence we refer the interested reader to [1], where the conjecture has been proven in the case $r = 2$ (for $r = 1$ it was already known). In the general case it has been established by Gieseker [19].

In order to prove both Brill-Noether and Petri conjectures it is enough to show that they are true for just one curve C of given genus (this follows from rather elementary general facts). However, this does not trivializes the problem: indeed, it seems very difficult to find such a nonsingular curve of given genus. Paradoxically it happens that, even though on almost all curves the claims of Brill-Noether and Petri do hold, nevertheless on every nonsingular curve which can be explicitly found they turn out to be false. The method applied by Griffiths-Harris and Gieseker to bypass this obstacle is to look for particular *singular* curves on which it is possible to check the theorem (in a suitably generalized sense), and then extend it to nonsingular curves by deforming the curve.

The method of reducing problems about linear series on nonsingular curves to problems on singular curves has found more and more applications in the last decades. But singular curves are not yet well understood from the viewpoint of linear series. Progress in this direction is going to be of great importance.

In order to complete the picture of the geometric properties of W^r_n we mention a theorem by Fulton and Lazarsfeld [17] claiming that if C is *any* curve of genus g and $\rho(g,r,n) > 0$, then W^r_n is connected and if moreover C is sufficiently general then W^r_n is irreducible.

In the end, let us see what happens to the W^r_n's when the curve C is very special.

Assume $n \leq g-1$. By a rather elementary reason (namely, Clifford's Theorem) $W^r_n \neq \emptyset$ implies $n \geq 2r$, and if equality holds then C is hyperelliptic. It has been proven by H. H. Martens [39] that W^r_n has dimension $n-2r$ and if equality holds for at least one irreducible component then C is hyperelliptic.

Martens' Theorem has been refined by Mumford [43], who proved that if C is nonhyperelliptic then $\dim(W^r_n) \leq n-2r-1$, and if equality holds then C is either trigonal, or a nonsingular plane quintic, or a double covering of a curve of genus 1.

There exist further refinements of Martens' Theorem and various results of this kind, all going in the direction of explicitly characterizing those curves of fixed genus g for which a W^r_n has not the expected dimension computed by the Brill-Noether number $\rho(g,r,n)$.

Update. The main tool developed insofar to address degenerations of linear series on families of curves with singular fibers is the theory of limit linear series [78] by Eisenbud and Harris. Roughly speaking, a limit g^r_d on a curve of compact type (i.e., a union of smooth curves with dual graph a tree) is a collection of g^r_d's, one for each irreducible component, related by a compatibility condition on vanishing orders at the nodes. The main result of [78] is a smoothability criterion for a limit linear series on a special curve to be the limit of a honest linear series on the general fiber. A more functorial construction (working over fields of arbitrary characteristic) has been presented in [92].

8

A fundamental issue is the classification problem.

Let $\mathcal{M}_{g,n}$ be the set of isomorphism classes of birational isomorphism of curves of genus g with n marked points. The problem is to describe $\mathcal{M}_{g,n}$.

The solution is easy for the first values of g.

The space $\mathcal{M}_{0,3}$ has just one point.

Every curve C of genus one is isomorphic to a nonsingular plane cubic curve (indeed, every $D \in C^{(3)}$ defines a g^2_3 without base points embedding C in $\mathbb{P}^2$) and the cubic can be reduced to have affine equation $y^2 = x(x-1)(x-\lambda)$ for some $\lambda \neq 0, 1$.

It is a classical theorem (due to G. Salmon) that two such cubics $C(\lambda)$ and $C(\mu)$ are isomorphic if and only if there exists a projectivity of $\mathbb{P}^1$ sending the unordered quadruple $\{0, 1, \infty, \lambda\}$ to the unordered quadruple $\{0, 1, \infty, \mu\}$. For a given λ there exist six values of μ with this property (corresponding to the six projectivities permuting $0, 1, \infty$), namely,

$$\mu_1 = \lambda, \mu_2 = 1 - \lambda, \mu_3 = \frac{1}{\lambda}, \mu_4 = \frac{\lambda - 1}{\lambda}, \mu_5 = \frac{\lambda}{\lambda - 1}, \mu_6 = \frac{1}{1 - \lambda}.$$

The numerical expression

$$j(\lambda) = 256\frac{(\lambda^2 - \lambda - 1)^3}{\lambda^2(\lambda - 1)^2}$$

assumes the same value $j(\mu)$ precisely if μ is one among $\mu_1, \dots, \mu_6$, hence $j(\lambda)$ depends only on $C(\lambda) \in \mathcal{M}_{1,1}$. It is called the *$j$-invariant* of the curve $C(\lambda)$ and it assumes all values in $\mathbb{C}$. Hence $\mathcal{M}_{1,1}$ may be identified to $\mathbb{C}$ via j.

The description of $\mathcal{M}_2$ is rather more involved, and it is due to Igusa [33]. The idea is analogous to the previous case. Indeed, every $C \in \mathcal{M}_2$ can be realized as a plane curve of equation $y^2 = g(x)$, where $g(x)$ is a polynomial of degree six with distinct roots. Two such curves, say $y^2 = g_1(x)$ and $y^2 = g_2(x)$, are birationally equivalent if and only if there exists a projectivity of $\mathbb{P}^1$ taking the six roots of $g_1(x)$ into the six roots of $g_2(x)$. This point of view leads to describe $\mathcal{M}_2$ as an irreducible affine variety of dimension three embedded in $\mathbb{C}^8$.

An explicit description of $\mathcal{M}_g$ for $g \geq 3$ is not known. Nevertheless, $\mathcal{M}_g$ has been extensively investigated and many of its properties are known. The most important one is certainly that it carries a natural structure of quasi-projective algebraic variety, as in the cases of $g = 0, 1, 2$. This structure exists due to the fact that curves vary in families.

A *family of curves* is defined as a flat morphism of algebraic varieties $f : \mathcal{C} \to S$ such that for every $s \in S$ the fiber $f^{-1}(s)$ is a curve. The base S is then called the variety parametrizing the family f. Flatness is a technical property which is very weak but strong enough to guarantee that the arithmetic genus $p_a(C) = 1 - h^0(\mathcal{O}_C) + h^1(\mathcal{O}_C)$ and other numerical characters of the fibers are locally constant as a function of $s \in S$. Hence if the fibers of f are nonsingular and S is connected, then all curves in the family have the same genus. In a similar way, one can define families of projective varieties.

For instance, plane curves of fixed degree n form a family since they are parameterized by the points of an algebraic variety, namely, the projective space $\mathbb{P}^N$, $N = \binom{n+2}{2} - 1$, having as homogeneous coordinates the coefficients of a homogeneous polynomial of degree n. In this case, $\mathcal{C}$ is the subvariety of $\mathbb{P}^2 \times \mathbb{P}^N$ defined by the equation $P(x_0, x_1, x_2) = 0$ where P is the homogeneous polynomial of degree n in x_0, x_1, x_2 with varying coefficients, $S = \mathbb{P}^N$ and $f : \mathcal{C} \to S$ the morphism defined by the projection of $\mathbb{P}^2 \times \mathbb{P}^N$ onto the second factor.

More generally, a family $f : \mathcal{C} \to S$ is said to be a family of curves in $\mathbb{P}^r$ if $\mathcal{C}$ is a closed subvariety in $\mathbb{P}^r \times S$ and f is the morphism induced by the projection of $\mathbb{P}^r \times S \to S$. In this case, flatness is equivalent to the fact that all fibers have the same Hilbert polynomial.

If $f : \mathcal{C} \to S$ is a family of nonsingular curves of genus g, then there is a natural map $S \to \mathcal{M}_g$ sending $s \mapsto f^{-1}(s)$.

The structure of algebraic variety on $\mathcal{M}_g$ is defined by requiring that all maps obtained this way from families of curves are indeed morphisms of algebraic varieties. Endowed with this structure, $\mathcal{M}_g$ is called the *moduli space of curves of genus g*.

Proving its existence is rather difficult: in order to do that, the most sophisticated tools from algebraic geometry are needed (see Mumford's proof in [42] and [44]). It would be interesting to see in detail how deeply the problem of constructing the moduli space of curves and other algebraic varieties has influenced the development of algebraic geometry in the last decades. Unluckily, brevity reasons suggest to drop this subject at all.

Having established that $\mathcal{M}_g$ is a variety, the condition "C is sufficiently general among curves of genus g" (or equivalently "C has general moduli") means that C can be chosen in a Zariski open subset of $\mathcal{M}_g$, i.e. that no closed condition is imposed on C. Otherwise, C is said to have "special moduli".

The fact that curves, and more generally algebraic varieties, are naturally distributed into families is a crucial phenomenon and it is essentially the reason why $\mathcal{M}_g$ exists (as an algebraic variety) and why its existence has always been taken for granted by geometers, even when a proof was out of reach. The word "moduli" dates back to Riemann, who denoted this way the continuous parameters on which a curve of genus g depends locally in $\mathcal{M}_g$. He found that its number (namely, the dimension of $\mathcal{M}_g$) is $3g-3$ for every $g \geq 2$.

A heuristic computation can be easily performed as follows.

Assume $g \geq 2$ and fix $n \geq 2g+1$. A classical theorem (namely, Riemann's existence theorem) states that for any choice of $\delta = 2(g+n-1)$ distinct points $p_1, \ldots, p_\delta$ of $\mathbb{P}^1$ it is always possible to construct in a finite number of ways a curve C of genus g and a base-point free g^1_n on C defining a morphism $q : C \to \mathbb{P}^1$ of degree n ramified precisely over $p_1, \ldots, p_\delta$. The number of parameters governing this construction is $\delta - 3 = 2g + 2n - 5$, since every $\delta - uple$ is transformed by the projectivities of $\mathbb{P}^1$ into ∞^3 others giving rise to the same curve with the same g^1_n. Every curve of genus g is obtained this way: indeed, for n big enough, every $C \in \mathcal{M}_g$ has ∞^g complete base-point free g^{n-g}_n, each containing $\infty^{2(n-g-1)}$ g^1_n defining a map of C onto $\mathbb{P}^1$ of degree n with δ distinct ramification points. It follows that the birationally distinct curves obtained via Riemann's construction depend on

$$2n + 2g - 5 - (g + 2n - 2g - 2) = 3g - 3$$

parameters.

A simple topological argument combined with the above construction shows that $\mathcal{M}_g$ is irreducible. This was done by Klein in [38] by using a canonical way of representing an n-fold covering of $\mathbb{P}^1$ due to Lüroth and Clebsch. For a very readable version of this approach we refer to [13].

Klein's method was later extended by Hurwitz [32], who investigated for every n and g the varieties of moduli of n-fold coverings of genus g of $\mathbb{P}^1$,

$H_{n,g}$, whose elements are the pairs (C, g^1_n), computed their dimension and proved their irreducibility. In particular, Hurwitz proved that the locus $\mathcal{M}^1_{g,n} \subseteq \mathcal{M}_g$ consisting of all curves carrying at least one g^1_n (i.e. the image of the natural map $H_{n,g} \to \mathcal{M}_g$) is an irreducible closed subset of dimension

$$\min(3g-3, 3g-3+\rho(g,1,n)) = \min(3g-3, 2g+2n-5)$$

(a gap in Hurwitz proof was pointed out by Severi and filled in by B. Segre in [58]). For instance, the locus $\mathcal{M}^1_{g,2}$ has dimension $2g-1 < 3g-3$ for $g \geq 3$, hence hyperelliptic curves of genus $g \geq 3$ have special moduli. Analogously, trigonal curves of genus $g \geq 5$ have special moduli because $\dim(\mathcal{M}^1_{g,3}) = 2g+1 < 3g-3$ for $g \geq 5$, and so on.

It was still an open problem to give a purely algebro-geometric proof (in the style of Clebsch-Noether) of the irreducibility of $\mathcal{M}_g$. Enriques in 1912 touched only marginally this question, while Severi addressed it in a systematic way. Both of them thought to be able to reduce the problem to a proof of the irreducibility of the family of plane curves of given genus g and degree n having only nodes as singularities. This family is parametrized by a locally closed subset $V_{n,g}$ of the $\mathbb{P}^N$, $N = \binom{n+2}{2} - 1$, parameterizing all plane curves of degree n. Every irreducible component of $V_{n,g}$ has dimension $N - \delta = 3n + g - 1$, where $\delta = \binom{n+1}{2} - g$ is the number of nodes of any curve in the family. Since for $n >> 0$ every curve of genus g is birationally isomorphic to an irreducible plane curve of degree n with δ nodes, from the irreducibility of $V_{n,g}$ it would follow that of $\mathcal{M}_g$, because by definition of $\mathcal{M}_g$ there is a surjective morphism $V_{n,g} \to \mathcal{M}_g$.

Unluckily, both the proofs by Enriques and Severi ([62], Anhang F) are incomplete as they stand. Both of them are of inductive nature and rely on the possibility of letting any irreducible plane curve with δ nodes degenerate to one with $\delta + 1$ nodes (which was not justified by neither Enriques nor Severi).

The problem of deciding whether $V_{n,g}$ is irreducible for every n, g or not is called "the Severi problem". A partial result by Arbarello and Cornalba [2] states that $V_{n,g}$ is irreducible for all n, g such that the Brill-Noether number $\rho(g,2,n) = g - 3(g-n+2)$ is positive, i.e. in almost all cases in which the morphism $V_{n,g} \to \mathcal{M}_g$ has dense image (except the case $\rho = 0$). However, their proof uses the irreducibility of $\mathcal{M}_g$, hence their result cannot be used to prove it.

In 1969 Deligne and Mumford [75] proved the irreducibility of $\mathcal{M}_g$ for curves defined over an algebraically closed field of arbitrary characteristic, by using the classical result topologically proven. A purely algebro-geometric proof of the irreducibility of $\mathcal{M}_g$ was discovered only in 1982 by Fulton [15]: it makes essential use of the compactification $\overline{H}_{n,g}$, constructed by Harris and Mumford, of the moduli space $H_{n,g}$ of n-fold coverings of $\mathbb{P}^1$ of genus g. The paper [16] by Fulton is devoted to the Severi problem.

Severi raised in [63] an important question about the moduli space of curves, by conjecturing that $\mathcal{M}_g$ is rational, or at least unirational.

Recall that the first condition means that there exists a birational isomorphism between $\mathcal{M}_g$ and $\mathbb{P}^{3g-3}$, while unirational means that for some N there exists a dominant rational map $\mathbb{P}^N \to \mathcal{M}_g$. Hence the unirationality of $\mathcal{M}_g$ would essentially correspond to the existence of a family of curves of genus g parametrized by the points of an affine or projective space, varying freely without any closed condition. Rationality, on the other hand, is of course a much stronger property.

Severi's conjecture relied on the obvious rationality of $\mathcal{M}_{1,1} \cong \mathbb{C}$ and on the unirationality of $\mathcal{M}_g$ for $g \leq 10$, of which Severi gave a very elementary proof, exploiting families of irreducible plane curves with nodes.

Here is his argument. Let us suppose (as indeed Severi did) that the varieties $V_{n,g}$ are irreducible. We know that if $\rho(g,2,n) \geq 0$, or equivalently if $n \geq 2g/3 + 2$, then $V_{n,g}$ parameterizes a family of curves with general moduli.

The idea is to show that if for every $g \leq 10$ we take the minimum n satisfying the above inequality, then the δ nodes of the varying curve vary generically in $\mathbb{P}^2$, i.e. they describe a dense subset of $(\mathbb{P}^2)^{(\delta)}$, the δ-fold symmetric power of $\mathbb{P}^2$.

This is easily checked to be equivalent to the fact that, chosen generically δ points in $\mathbb{P}^2$, there exist curves of degree n singular in those points: indeed, since every point imposes three conditions on the curves of given degree having it as a singular point, the above claim immediately follows from the inequality $3\delta < \binom{n+2}{2}$ holding in all those cases.

From the genericity of nodes and the rationality of $(\mathbb{P}^2)^{(\delta)}$ it immediately follows that $V_{n,g}$ is rational, since the map

$$V_{n,g} \to (\mathbb{P}^2)^{(\delta)}$$

sending p to the δ-uple of nodes of the curve parameterized by p has dense image and linear systems (i.e., projective spaces) as fibers. Since $V_{n,g}$ parameterizes curves with general moduli, the natural map $V_{n,g} \to \mathcal{M}_g$ has dense image, hence $\mathcal{M}_g$ is unirational. We point out that the irreducibility assumption is easily checked in all considered cases: for instance, the result by Arbarello and Cornalba quoted above works out all cases with $\rho(g,2,n) > 0$, hence all cases except $g = 3, 6, 9$ where it can be proven directly (indeed, for $g = 3, 6$ is trivially true).

When $g \geq 11$ and $n \geq 2g/3 + 2$, the nodes of the curves parameterized by $V_{n,g}$ do not vary generically in $\mathbb{P}^2$, but they describe a locally closed subset of $(\mathbb{P}^2)^{(\delta)}$ whose geometry is unknown. For an interesting discussion of this topic see [57].

For a long time this conjecture by Severi has remained widely open, except for Igusa's construction of $\mathcal{M}_2$, which implies its rationality.

In 1982 Harris and Mumford in [28] proved that for infinitely many values of g (namely, all odd $g \geq 23$) $\mathcal{M}_g$ is not unirational (we refer to [9] for an exposition of the work by Harris and Mumford). Their result has been later extended to all $g \geq 23$ by Harris [26] and Eisenbud-Harris [10] with similar methods. Indeed, at least for $g \geq 24$ much more is true: as a function of h, the dimension $\dim(H^0(\mathcal{M}_g, K^{\otimes h})$ increases as a polynomial of degree $3g-3$, thus meaning that $\mathcal{M}_g$ is a variety "of general type", in a certain sense the opposite of a unirational variety. For small values of g ($g = 11, 12, 13$) it has instead been proven that $\mathcal{M}_g$ is unirational: see [8] and [60].

We also mention the seminal paper [45].

Update. The Severi problem was finally solved by Harris [86] in the affirmative.

As far as the birational geometry of $\mathcal{M}_g$ is concerned, we report that the unirationality of $\mathcal{M}_{14}$ was established by Verra in [98] and a proof that $\mathcal{M}_{22}$ is of general type was announced by Farkas [82].

The pioneering paper [45], computing the Chow ring of $\mathcal{M}_2$, opened the road to further research by Faber (who addressed the cases $g = 3, 4$ in [80], [81]) and by Izadi (who worked out the case $g = 5$ in [90], see also [84] for a simpler proof).

In [85] Harer proved that the degree k rational cohomology group of $\mathcal{M}_g$ is independent on g as soon as $g >> k$. These stable cohomology groups form a graded commutative algebra, which Mumford [45] conjectured to be generated by certain tautological classes κ_i of (real) dimension $2i$. Such a statement was finally proven by Madsen and Weiss [91] via algebraic topology, but a purely geometric argument seems to be still elusive.

We point out that the moduli spaces $\mathcal{M}_{g,n}$ admit a natural compactification $\overline{\mathcal{M}}_{g,n}$ parameterizing isomorphism classes of pointed Deligne-Mumford stable curves, which have at most simple nodes as singularities and a finite automorphism group. The computation of the rational cohomology of $\overline{\mathcal{M}}_{g,n}$ is still a widely open problem: a beautifully simple inductive approach, which turns out to be effective at least in low degree, was proposed by Arbarello and Cornalba in [67].

9

Finally we turn to families of embedded projective curves. Classical geometers deeply investigated curves in $\mathbb{P}^3$. Both Noether and Halphen had discovered that the curves of given degree n in $\mathbb{P}^3$ vary in different irreducible families each of which has dimension $\geq 4n$: the components of dimension $4n$ were called regular, and the others irregular. The classification of these families and the investigation of their properties immediately turned out to be a difficult problem: Halphen tried in vain to determine numerical invariants discriminating the various irreducible families.

Some attempt was also tried by Severi, who was looking for standard types of singular curves (polygonal curves, i.e. union of lines) to which any nonsingular curve of $\mathbb{P}^3$ and more generally of $\mathbb{P}^r$, could degenerate, thus reducing to them the classification of families. This idea was very suggestive but the claims made in in [62] and [63] were not corroborated by rigorous arguments. The most important among the results stated by Severi claims that nonsingular curves of degree n and genus g in $\mathbb{P}^r$ form one irreducible family for $\rho(g, r, n) \geq 0$.

Indeed, the Vorlesungen by Severi represent the last systematic attempt to address the most relevant questions in the theory of algebraic curves with the methods of classical geometry. We can say that they are the crowning achievement of the program started by Clebsch and Noether, or at least of its first phase. From hence on, geometers had to realize the inadequacy of the geometric language of Severi and his predecessors. The main efforts were thus focused for a long time to the foundations of algebraic geometry and several classical problems were almost neglected. This is the reason why the investigation of families of projective curves was essentially stagnant until the beginning of the sixties.

In 1961 Grothendieck proposed a new approach, exploiting the language of schemes and functorial methods [23]. For every fixed numerical polynomial $p(X) \in \mathbb{Q}[X]$ (i.e., such that $p(d) \in \mathbb{Z}$ for every $d \in \mathbb{Z}$), he defined in a functorial way a projective scheme $\underline{Hilb}^r_{p(X)}$, the so-called *Hilbert scheme*, parametrizing the family of all closed subschemes of $\mathbb{P}^r$ having $p(X)$ as Hilbert polynomial (the *universal family*). These schemes are generalizations of Grassmannians, to which they reduce for suitable choices of $p(X)$ (namely, $p(X) = \binom{X+k}{k}$ for the Grassmannian $\mathbb{G}(k,r)$ of subspaces of projective dimension k of $\mathbb{P}^r$).

The local properties of $\underline{Hilb}^r_{p(X)}$ (which for brevity sake will be denoted by $\underline{Hilb}$ whenever r and $p(X)$ are inessential) at a closed point z correspond to properties of the scheme $X(z) \subseteq \mathbb{P}^r$ parametrized by z. The Zariski tangent space to $\underline{Hilb}$ at z is canonically isomorphic to $H^0(X(z), N)$, where N is the normal sheaf to $X(z)$ in $\mathbb{P}^r$, defined as $N = \underline{Hom}(I/I^2, \mathcal{O}_{X(z)})$, where $I \subseteq \mathcal{O}_{\mathbb{P}^r}$ is the ideal sheaf of $X(z)$ in $\mathbb{P}^r$. For instance, if X is a line in $\mathbb{P}^r$, then its normal sheaf is isomorphic to $\mathcal{O}(1)^{r-1}$ and $H^0(N)$ has dimension $2(r-1)$, according to the fact that $\mathbb{G}(1,r)$ is nonsingular of dimension $2(r-1)$.

The Hilbert scheme can be singular and even nonreduced, even at points parameterizing irreducible and nonsingular curves or varieties: several pathological examples are known, the first one due to Mumford [40]. A sufficient criterion in order that $\underline{Hilb}$ is nonsingular in z is that $H^1(X(z), N) = 0$, but this condition is not necessary at all, as very simple examples show (e.g., complete intersections of type (a, b) in $\mathbb{P}^3$, with $a + b \geq 6$).

In general, $H^1(X(z), N)$ contains the "obstructions" to the nonsingularity of $\underline{Hilb}$ at z (in a technical sense which is possible to make precise), but not all of its elements are necessarily obstructions: $X(z)$ is said to be *obstructed* if $H^1(X(z), N)$ contains nonzero obstructions, i.e. if $\underline{Hilb}$ is singular at z.

Both the local structure of $\underline{Hilb}$ and its global properties remain rather mysterious. For example, conditions of local reducedness or analytic irreducibility are not known. Our ignorance about the local properties of $\underline{Hilb}$ makes the study of the global ones even more difficult.

A very general result by Hartshorne [29] states that, for every r and $p(X)$, $\underline{Hilb}^r_{p(X)}$ is connected. However, an explicit description of its irreducible components turns out to be difficult from the very beginning (for instance, [52] is devoted to a detailed study of $\underline{Hilb}^3_{3X+1}$).

Even the following simple question has no answer yet: do there exist irreducible and nonsingular curves in $\mathbb{P}^3$ which are specializations of complete intersections without being complete intersections?

The most interesting Hilbert schemes from the point of view of curves are those corresponding to $\mathbb{P}^r$ and to polynomials $p(X) = nX + 1 - g$ such that $n \geq (r/r+1)g + r$, i.e. parameterizing curves of degree n and arithmetic genus g with $\rho(g, r, n) \geq 0$. A refinement of the theorem by Kleiman-Laksov [37] and Kempf [35] guarantees that on every sufficiently general curve $C \in \mathcal{M}_g$ there exists an $L \in W^r_d$ defining an embedding in $\mathbb{P}^r$ if $\rho(g, r, n) \geq 0$ (see [11] and [61]). Hence it easily follows that in this case there exists an irreducible component $I^r_{n,g}$ of $\underline{Hilb}^r_{p(X)}$ parametrizing a family of curves generically nonsingular and with general moduli. The question left unsolved by Severi translates into the problem of checking if there exists another irreducible component of $\underline{Hilb}^r_{p(X)}$, besides $I^r_{n,g}$, generically parameterizing nonsingular and irreducible curves. It is known that $I^r_{n,g}$ is the unique irreducible component consisting of curves with general moduli. In the case $\rho(g,r,n) > 0$ this follows from a result of Fulton and Lazarsfeld [17] combined with one by Gieseker [19], and in the case $\rho(g,r,n) = 0$ it has been shown by Eisenbud and Harris.

It is clear that another component cannot exist if n is big enough with respect to g, precisely if $n \geq 2g-1$ (this easily follows from the irreducibility of $\mathcal{M}_g$ and from the fact that the hyperplane section of every curve in the family is necessarily nonspecial). This condition has been improved by Harris [27] to $n > (2r-1)g/(r+1)+1$. Even in these cases, when the irreducibility is known, the problem of proving it directly, without using the irreducibility of $\mathcal{M}_g$, remains. If instead $\rho(g,r,n) < 0$ there are many examples showing that $\underline{Hilb}^r_{p(X)}$, $p(X) = nX + 1 - g$, can have several irreducible components of different dimensions, generically parametrizing irreducible and nonsingular curves. No estimates, even conjectural, are known about the number and the dimension of such components. We only know that, due to general reasons, each of them has dimension $\geq n(r+1) - (g-1)(r-3)$.

The other question addressed by Severi, if any irreducible and nonsingular curve in $\mathbb{P}^r$ can be degenerated to a polygonal curve, is equivalent to establishing if any irreducible component of $\underline{Hilb}$ generically parametrizing nonsingular curves contains points corresponding to unions of lines.

In general, it is easier to decide if a singular curve in $\mathbb{P}^r$ can deform in $\mathbb{P}^r$ becoming nonsingular (i.e., if it is "smoothable"): for smoothability criteria of projective curves we refer to [30] and [61].

A very nice problem concerning the component $I^r_{n,g}$ of $\underline{Hilb}^r_{p(X)}$ in the case $\rho(g,r,n) \geq 0$ is the *maximal rank conjecture* (essentially due to Noether). It claims that for all curves $C(z)$ parametrized by a sufficiently general $z \in I^r_{n,g}$ and for every d the restriction map

$$H^0(\mathbb{P}^r, \mathcal{O}(d)) \to H^0(C(z), \mathcal{O}(d))$$

has maximal rank (namely, it is either injective or surjective). In other words, this means that if a curve contained in a hypersurface of degree d has incomplete the linear series cut out by the hypersurfaces of degree d then it is a special curve in the family. For instance, a curve of type $(2,4)$ on a quadric (both nonsingular) is not of maximal degree because quadrics cut out an incomplete linear series, but by moving it in $\underline{Hilb}$ we can deform it outside the quadric and make it to be of maximal rank. This conjecture can be stated for every component of $\underline{Hilb}$, not necessarily with general moduli, but rather easy examples show that it does not hold in such a general form. Partial results about the conjecture have been proven in [31] and [4].

For an interesting discussion of problems concerning the Hilbert scheme see [27]. Many of the arguments mentioned in this notes are fully dealt with in the book [68] by Arbarello, Cornalba, Griffiths, and Harris.

Update. A self-contained thorough introduction to algebraic deformation theory providing the tools needed in the local study of Hilbert schemes is now available [96].

The local structure of Hilbert schemes can be arbitrarily bad: indeed, as shown in [97], every singularity of finite type over $\mathbb{Z}$ appears on the Hilbert scheme of curves in projective space.

As far as the fundamental connectedness property of the Hilbert scheme, a few remarks are in order (see [89]). First of all, as noted by Halphen and Weyr already in 1874, the Hilbert scheme of *smooth* curves of given degree and genus in $\mathbb{P}^3$ need not to be connected. Next, Hartshorne's connectedness theorem is rather unsatisfactory in that, even to connect one smooth curve to another smooth curve, one cannot avoid passing by way of nonreduced schemes with embedded points and isolated points. So the following question sounds natural: is the Hilbert scheme of locally Cohen-Macaulay curves (namely, one dimensional schemes with no embedded points or isolated points) of degree d and arithmetic genus g in $\mathbb{P}^3$ connected? For a survey of partial answers we refer to [89].

Unluckily, both statements of Severi discussed above turn out to be false. Indeed, examples by Ein and Harris ([76] and [77]) show that the Hilbert scheme of curves of degree d and genus g in $\mathbb{P}^r$ can be reducible for $d \geq g+r$ and exceptional components arise for positive values of the Brill-Noether number (see for instance [79]).

On the other hand, the problem of deciding if every family of space curves contains limit curves which are composed of lines is usually referred to as Zeuthen's problem and was actually proposed in 1901 as a prize problem by the Royal Danish Academy, which remained without a winner. Nodal curves whose irreducible components are lines are now called stick figures and a negative answer to Zeuthen's problem was provided by Hartshorne in [88] as an application of the already mentioned theory of generalized divisors.

Finally, the maximal rank conjecture in $\mathbb{P}^r$ was proved by Ballico and Ellia for any $d \geq g + r$ (see [69] for $r = 3$, [70] for $r = 4$, and [71] for $r \geq 5$). Then, in the subsequent paper [72], they find a component of the Hilbert scheme such that the generic member satisfies the maximal rank condition in a much wider range (d, g, n). Roughly speaking, they borrow from Hirschowitz the so-called *méthode d'Horace* to construct a suitable reducible curve X in $\mathbb{P}^r$ satisfying the conjecture. Next, they apply [61] to deform X to a nonsingular curve having the same degree and arithmetic genus. In the case of space curves, further results are obtained in [83] via a clever application of stick figures.

References

[1] E. Arbarello and M. Cornalba: Su una congettura di Petri. Comment. Math. Helv., 56 (1981), 1–38.

[2] E. Arbarello and M. Cornalba: A few remarks about the variety of irreducible plane curves of given degree and genus. Ann. E.N.S. 16 (1983), 467–488.

[3] D. W. Babbage: A note on the quadrics through a canonical curve. J. London Math. Soc. 14 (1939), 310–315.

[4] E. Ballico and Ph. Ellia: Generic curves of small genus in $\mathbb{P}^3$ are of maximal rank. Math. Ann. 264 (1983), 211-225.

[5] A. Brill and M. Noether: Ueber die algebraischen Functionen und ihre Anwendung in der Geometrie. Math. Ann. 7 (1873), 269–310.

[6] G. Castelnuovo: Ricerche di geometria sulle curve algebriche. Atti R. Acc. d. Sc. di Torino 24 (1889).

[7] G. Castelnuovo: Sui multipli di una serie lineare di gruppi di punti appartenente ad una curva algebrica. Rend. Circ. Matem. di Palermo 7 (1893).

[8] M. C. Chang and Z. Ran: Unirationality of the moduli spaces of curves of genus 11, 13 (and 12). Invent. Math. 76 (1984), 41–54.

[9] M. Cornalba: Systemes pluricanoniques sur l'espace des modules des courbes et diviseurs de courbes k-gonales (d'apres Harris at Mumford). Seminaire Bourbaki, exp. 615 (1983).

[10] D. Eisenbud and J. Harris: The Kodaira dimension of the moduli space of curves of genus ≥ 23. Invent. Math. 90 (1987), 359–387.

[11] D. Eisenbud and J. Harris: Divisors on general curves and cuspidal rational curves. Invent. Math. 74 (1983), 371–418.

[12] F. Enriques: Sulle curve canoniche di genere p sullo spazio a $p-1$ dimensioni. Rend. Acc. Sc. Ist. di Bologna 23 (1919), 80–82.

[13] F. Enriques and O. Chisini: Teoria geometrica delle equazioni e delle funzioni algebriche. Vol. 3. Zanichelli, Bologna (1924).

[14] G. Fano: Sopra le curve di dato ordine e dei massimi generi in uno spazio qualunque. Torino Mem. (2) XLIV (1894), 335–382.

[15] W. Fulton: On the irreducibility of the moduli space of curves. Appendix to the paper of Harris and Mumford Invent. Math. 67 (1982), 87–88.

[16] W. Fulton: On nodal curves. Algebraic geometry - Open Problems. Proceedings, Ravello 1982, Springer L. N. 997 (1982).

[17] W. Fulton and R. Lazarsfeld: On the connectedness of degeneracy loci and special divisors. Acta Math. 146 (1981), 271–283.

[18] G. Gherardelli: Sulle curve sghembe algebriche intersezioni complete di due superficie. Rendiconti Reale Acc. d'Italia, s. VII, Vol. 4 (1942), 128–132.

[19] D. Gieseker: Stable curves and special divisors: Petri's conjecture. Invent. Math. 66 (1982), 251–275.

[20] D. Gieseker: A construction of special space curves. Algebraic Geometry - Ann Arbor, Springer L. N. 1008 (1981).

[21] M. Green: Koszul cohomology and the geometry of projective varieties. J. Differential Geom. 19 (1984), 125–171.

[22] P. Griffiths and J. Harris: On the variety of special linear systems on a general algebraic curve. Duke Math. J. 47 (1980), 233–272.

[23] A. Grothendieck: Les schemas de Hilbert. Seminaire Bourbaki, exp. 221 (1961).

[24] L. Gruson and C. Peskine: Genre des curves de l'espace projectif. Springer L. N. 687 (1978).

[25] G. Halphen: Memoire sur la classification des curves gauches algebriques. J. Ec. Polyt. 52 (1882), 1–200.

[26] J. Harris: On the Kodaira dimension of the moduli space of curves. II. The even-genus case. Invent. Math. 75 (1984), 437–466.

[27] J. Harris: Curves in projective space. Univ. de Montreal (1982).

[28] J. Harris and D. Mumford: On the Kodaira dimension of the moduli space of curves. Invent. Math. 67 (1982), 23–86.

[29] R. Hartshorne: Connectedness of the Hilbert scheme. Publ. IHES 29 (1966), 261–304.

[30] R. Hartshorne and A. Hitschowitz: Smoothing algebraic space curves. Algebraic geometry, Sitges (Barcelona), 1983, 98–131, Springer L. N. 1124 (1985).

[31] A. Hirtschowitz: Sur la postulation generique des courbes rationelles. Acta Math. 146 (1981), 209–230.

[32] A. Hurwitz: Uber Riemann'sche Flachen mit gegebenen Verzweigungspunkten. Math. Ann. 39 (1891), 1–61.

[33] J. Igusa: Arithmetic varieties of moduli of genus two. Ann. of Math. 72 (1960), 612–649.

[34] G. Kempf: On the geometry of a theorem of Riemann. Ann. of Math. 98 (1973), 178–185.
[35] G. Kempf: Schubert methods with an application to algebraic curves. Public. of Math. Centrum, Amsterdam (1971).
[36] S. Kleiman: r-special subschemes and an argument of Severi. Adv. in Math. 22 (1976), 1–31.
[37] S. Kleiman and D. Laksov: On the existence of special divisors. Amer. J. of Math. 94 (1972), 431–436.
[38] F. Klein: Über Riemann's Theorie der algebraischen Functionen. Leipzig, Teubner (1882).
[39] H. H. Martens: On the varieties of special divisors on a curve. J. Reine Angew. Math. 227 (1967), 111–120.
[40] D. Mumford: Further pathologies in algebraic geometry. Amer. J. of Math. 84 (1962), 642–648.
[41] D. Mumford: Varieties defined by quadratic equations. Corso CIME 1969 (Questions on algebraic varieties). Ed. Cremonese, Roma (1970).
[42] D. Mumford: Geometric Invariant Theory. Springer, (1965).
[43] D. Mumford: Prym varieties I. Contributions to Analysis, a collection of papers dedicated to L. Bers (1974), 325–350.
[44] D. Mumford: Stability of projective varieties. Ens. Math. 23 (1977), 39–110.
[45] D. Mumford: Towards an enumerative geometry of the moduli space of curves. Shafarevich volume, Birkhauser (1983).
[46] M. Noether: Ueber einen Satz aus der Theorie der algebraischen Functionen. Math. Ann. 6 (1872), 351–365.
[47] M. Noether: Über die algebraischen Functionen, note II. Goettingen Nachr. 7 (1871).
[48] M. Noether: Zur Grundlegung der Theorie der algebraischen Raumkurven. Abhand der preussischen Ak. der Wiss. (1882), 1–120.
[49] C. Peskine and L. Szpiro: Liaison des varietés algebriques I. Invent. Math. 26 (1974), 271–302.
[50] K. Petri: Uber die invariante Darstellung der algebraischen Functionen einer Veranderliche. Math. Ann. 88 (1922), 242–289.
[51] K. Petri: Uber Spezialkurven I. Math. Ann. 93 (1924), 182–209.
[52] R. Piene: Degenerations of complete twisted cubics. Progr. in Math. 24, Birkhauser (1982).
[53] B. Riemann: Grundlagen für enine allgemeine Theorie der Funktionen einer veranderlichen complexen Grosse (1851).
[54] B. Riemann: Theorie der Abel'schen Functionen. J. Reine Angew. Math. 54 (1859).
[55] B. Saint Donat: On Petri's analysis of the linear system of quadrics through a canonical curve. Math. Ann. 206 (1973), 157–175.
[56] B. Saint Donat: Sur les equations definissant une curve algebrique. CRAS Paris 274 (1972), 324–327.
[57] B. Segre: Sui moduli delle curve algebriche. Ann. di Mat. Pura e Appl. (4) 7 (1930), 71–102.
[58] B. Segre: Sui moduli delle curve poligonali, e sopra un complemento al teorema di esistenza di Riemann. Math. Ann. 100 (1928), 537–551.
[59] C. Segre: Recherches generales sur les courbes et les surfaces reglées algébriques. Math. Ann. 30 (1887), 203–226.
[60] E. Sernesi: L'unirazionalità della varietà dei moduli delle curve di genere dodici. Ann. Sc. Norm. Super. Pisa (4), 8 (1981), 405–439.
[61] E. Sernesi: On the existence of certain families of curves. Invent. Math. 75 (1984), 25–57.
[62] F. Severi: Vorlesungen uber algebraische Geometrie. Leipzig, Teubner (1921).
[63] F. Severi: Sulla classificazione delle curve algebriche e sul teorema d'esistenza di Riemann. Rend. R. Acc. Naz. dei Lincei (5) (1915), 241.
[64] R. Torelli: Sulle varietà di Jacobi. Rend. R. Accad. Lincei (5), 22 (1913), 98–103.
[65] M. Aprodu: Green-Lazarsfeld gonality conjecture for a generic curve of odd genus. Int. Math. Res. Not. 2004, 3409–3416.

[66] M. Aprodu and C. Voisin: Green-Lazarsfeld's conjecture for generic curves of large gonality. C. R. Math. Acad. Sci. Paris 336 (2003), 335–339.

[67] E. Arbarello, M. Cornalba: Calculating cohomology groups of moduli spaces of curves via algebraic geometry. Inst. Hautes Études Sci. Publ. Math. No. 88 (1998), 97–127 (1999).

[68] E. Arbarello, M. Cornalba, P. A. Griffiths, and J. Harris: Geometry of algebraic curves. Vol. I. Grundlehren der Mathematischen Wissenschaften, 267. Springer-Verlag, New York, 1985.

[69] E. Ballico and Ph. Ellia: The maximal rank conjecture for nonspecial curves in $\mathbb{P}^3$. Invent. Math. 79 (1985), 541–555.

[70] E. Ballico and Ph. Ellia: On postulation of curves in $\mathbb{P}^4$. Math. Z. 188 (1985), 215–223.

[71] E. Ballico and Ph. Ellia: The maximal rank conjecture for nonspecial curves in $\mathbb{P}^n$. Math. Z. 196 (1987), 355–367.

[72] E. Ballico and Ph. Ellia: On the existence of curves with maximal rank in $\mathbb{P}^n$. J. Reine Angew. Math. 397 (1989), 1–22.

[73] L. Chiantini, C. Ciliberto, and V. Di Gennaro: The genus of projective curves. Duke Math. J. 70 (1993), 229–245.

[74] C. Ciliberto: On the degree and genus of smooth curves in a projective space. Adv. Math. 81 (1990), 198–248.

[75] P. Deligne and D. Mumford: The irreducibility of the space of curves of given genus. Inst. Hautes Études Sci. Publ. Math. 36 (1969), 75–109.

[76] L. Ein: Hilbert scheme of smooth space curves. Ann. Sci. Ècole Norm. Sup. (4) 19 (1986), 469–478.

[77] L. Ein: The irreducibility of the Hilbert scheme of smooth space curves. Algebraic geometry, Bowdoin, 1985 (Brunswick, Maine, 1985), 83–87, Proc. Sympos. Pure Math., 46, Part 1, Amer. Math. Soc., Providence, RI, 1987.

[78] D. Eisenbud and J. Harris: Limit linear series: basic theory. Invent. Math. 85 (1986), 337–371.

[79] D. Eisenbud and J. Harris: Irreducibility of some families of linear series with Brill-Noether number -1. Ann. Sci. École Norm. Sup. (4) 22 (1989), 33–53.

[80] C. Faber: Chow rings of moduli spaces of curves. I. The Chow ring of $\overline{\mathcal{M}}_3$. Ann. of Math. (2) 132 (1990), 331–419.

[81] C. Faber: Chow rings of moduli spaces of curves. II. Some results on the Chow ring of $\overline{\mathcal{M}}_4$. Ann. of Math. (2) 132 (1990), 421–449.

[82] G. Farkas: The global geometry of the moduli space of curves. arXiv:math/0612251 (2006).

[83] G. Floystad: Construction of space curves with good properties. Math. Ann. 289 (1991), 33–54.

[84] C. Fontanari and E. Looijenga: A perfect stratification of $\mathcal{M}_g$ for g at most 5. Preprint arXiv:0708.3424 (2007).

[85] J. H. Harer: Stability of the homology of the mapping class groups of orientable surfaces. Ann. of Math. (2) 121 (1985), 215–249.

[86] J. Harris: On the Severi problem. Invent. Math. 84 (1986), 445–461.

[87] R. Hartshorne: Generalized divisors on Gorenstein curves and a theorem of Noether. J. Math. Kyoto Univ. 26 (1986), 375–386.

[88] R. Hartshorne: Families of curves in $\mathbb{P}^3$ and Zeuthen's problem. Mem. Amer. Math. Soc. 130 (1997), no. 617, viii+96 pp.

[89] R. Hartshorne: Questions of Connectedness of the Hilbert scheme of Curves in $\mathbb{P}^3$. Algebra, arithmetic and geometry with applications (West Lafayette, IN, 2000), 487–495, Springer, Berlin, 2004.

[90] E. Izadi: The Chow ring of the moduli space of curves of genus 5. The moduli space of curves (Texel Island, 1994), 267–304, Progr. Math., 129, Birkhäuser Boston, Boston, MA, 1995.

[91] I. Madsen and M. Weiss: The stable moduli space of Riemann surfaces: Mumford's conjecture. Ann. of Math. (2) 165 (2007), 843–941.

[92] B. Osserman: A limit linear series moduli scheme. Ann. Inst. Fourier (Grenoble) 56 (2006), 1165–1205.

[93] J. Rathmann: The genus of curves in $\mathbb{P}^4$ and $\mathbb{P}^5$. Math. Z. 202 (1989), 525–543.
[94] F.-O. Schreyer: Syzygies of canonical curves and special linear series. Math. Ann. 275 (1986), 105–137.
[95] F.-O. Schreyer: A standard basis approach to syzygies of canonical curves. J. Reine Angew. Math. 421 (1991), 83–123.
[96] E. Sernesi: Deformations of algebraic schemes. Grundlehren der Mathematischen Wissenschaften, 334. Springer-Verlag, Berlin, 2006.
[97] R. Vakil: Murphy's law in algebraic geometry: badly-behaved deformation spaces. Invent. Math. 164 (2006), 569–590.
[98] A. Verra: The unirationality of the moduli spaces of curves of genus 14 or lower. Compos. Math. 141 (2005), 1425–1444.
[99] C. Voisin: Green's generic syzygy conjecture for curves of even genus lying on a $K3$ surface. J. Eur. Math. Soc. (JEMS) 4 (2002), 363–404.
[100] C. Voisin: Green's canonical syzygy conjecture for generic curves of odd genus. Compos. Math. 141 (2005), 1163–1190.

E-mail address: `claudio.fontanari@polito.it`

Current address: Dipartimento di Matematica, Politecnico di Torino, Corso Duca degli Abruzzi 24, 10129 Torino, Italy.

E-mail address: `sernesi@mat.uniroma3.it`

Current address: Dipartimento di Matematica, Università Roma Tre, Largo S. L. Murialdo 1, 00146 Roma, Italy.