Neutral Linear Series *

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Abstract

On a projective nonsingular curve $C$ of genus $g$ we study the variety $N^r_n(A)$ of linear series of degree $n$ and dimension $r$ to which a given effective divisor $A$ of degree $s$ does not impose independent conditions. Our main result is that, in the appropriate ranges of $g, r, n, s$, if $C$ is a Brill-Noether curve and $A$ is general then $N^r_n(A)$ has the expected dimension. The proof does not use any degeneration argument. As a consequence we give a simple proof of the Brill-Noether-Castelnuovo statement, also known as the Brill-Noether theorem.

Introduction

Let $C$ be a smooth projective irreducible curve of genus $g$, defined over the complex numbers and let $p, q \in C$. A linear series on $C$ is said to be neutral with respect (w.r.) to $p + q$ if the divisor $p + q$ imposes at most one condition to it. Such series arise naturally as pullbacks of linear series on the singular curve $X$ obtained from $C$ after identifying $p$ with $q$.

The notion and the terminology of neutral linear series are classical: for example a systematic investigation was undertaken by Severi in [10] in the framework of quasi-abelian functions.

In this paper we introduce the schemes $N^r_n(p + q)$ of neutral linear series w.r. to $p + q$ of degree $n$ and dimension $r$, and we study some of their basic properties, concentrating on the relation with the theory of special divisors on a general curve. More generally, given nonnegative

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integers $r, n$ and denoting by $G^r_n$ the scheme of linear series of degree $n$ and dimension $r$ on $C$, for every effective divisor $A = p_1 + \cdots + p_s$ of degree $s$ such that $2 \leq s \leq r + 1$ we define a closed subscheme $N^r_n(A)$ of $G^r_n$ which is supported on the series to which $A$ imposes at most $s - 1$ conditions. Recall that every irreducible component of $G^r_n$ has dimension at least $\rho(g, r, n) := g - (r + 1)(g - n + r)$. The curve $C$ is said to be a Brill-Noether curve (abbreviated “B.N. curve”) if $G^r_n$ has pure dimension $\rho(g, r, n)$ for every $n, r \geq 0$, in particular if $G^r_n = \emptyset$ when $\rho(g, r, n) < 0$. Our main result is the following:

**Theorem 0.1** Assume that $C$ is a B.N. curve of genus $g$ and that $A$ is an effective divisor of degree $s$, with $2 \leq s \leq r + 1$, on $C$.

(a) If $\rho(g, r, n) \geq r - s + 2$ then $N^r_n(A) \neq \emptyset$ and every component of $N^r_n(A)$ has codimension at most $r - s + 2$ in $G^r_n$.

(b) If $A$ is general then $N^r_n(A)$ has pure codimension $r - s + 2$ in $G^r_n$; in particular $N^r_n(A) = \emptyset$ if $\rho(g, r, n) < r - s + 2$.

In the particular case $s = 2$ we obtain that the scheme $N^r_n(p + q)$ of neutral linear series w.r. to $p + q$ has pure codimension $r$ for general $p, q \in C$.

Part (a) is a consequence of the definition of $N^r_n(A)$ as a determinantal scheme and of a general result of Fulton-Lazarsfeld. Part (b) is proved by descending induction on $s$: for a general choice of $p, \ldots, p_{r+1} \in C$ the schemes

$$G^r_n, \ N^r_n(p_1 + \cdots + p_{r+1}), \ldots, N^r_n(p_1 + p_2)$$

are shown to be each (except for the first one of course) of pure codimension one in the previous.

Results similar to theorem 0.1 have already been proved by Coppens [3] and by D. Schubert [9] using degeneration techniques and the theory of limit linear series. In particular, part (b) in the case $s = 2$ follows from the main result of [3]. We use a different approach, proving everything directly on the curve $C$ without any degeneration. Recent related results have also been obtained by E. Cotterill [4] and by G. Farkas [6].

In the final part of this paper the above theorem is applied to give a new proof of the Brill-Noether-Castelnuovo statement, first proved in [8].

As it is well known, the original proof of the Brill-Noether-Castelnuovo statement, which asserts the existence of a B.N. curve of genus $g$ for
all $g \geq 0$, has subsequently been simplified by Eisenbud and Harris
by means of the powerful theory of limit linear series they created.
Our approach is different, being based on the geometrically simplest
possible degeneration of a smooth curve of genus $g$ to a $1$-nodal curve
of geometric genus $g - 1$, rather than to a $g$-cuspidal rational curve.
We thus obtain a proof by induction on $g$, instead of directly reducing
to the genus 0 case.
The paper is organized as follows. In §1 we introduce the basic def-
initions and we state theorem 1.5. In §2 we prove Theorem 1.5. In
§3 the proof of the Brill-Noether-Castelnuovo statement is outlined,
referring to well known arguments which can be easily adapted to our
case.
We work over the field of complex numbers. Therefore all schemes
will be implicitly defined over $\mathbb{C}$.
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1 The schemes of neutral linear series

Let $C$ be a smooth, connected, projective curve of genus $g$. We will
denote by $C^{(n)}$ the $n$-th symmetric product of $C$. For any divisor $D$
and sheaf $\mathcal{F}$ on $C$ we will often write $H^i(D)$ and $H^i(\mathcal{F})$ instead of
$H^i(C, \mathcal{O}_C(D))$ and $H^i(C, \mathcal{F})$ respectively, and $h^i(\mathcal{L})$ for $\dim[H^i(\mathcal{L})]$.
For an invertible sheaf $\mathcal{L}$ on $C$ and a vector subspace $V \subset H^0(\mathcal{L})$
we denote by $|V|$ the linear series on $C$ defined by $V$, and by $|\mathcal{L}|$ the
(complete) linear series defined by $H^0(\mathcal{L})$. As usual, if $D$ is a divisor
on $C$ we denote by $|D|$ the complete linear series defined by $D$. The
dual of a vector space $V$ or of a locally free sheaf $\mathcal{F}$ will be denoted by $V^\vee$
respectively by $\mathcal{F}^\vee$. We shall use the expression “a $g^r_d$” as a
synonymous of “a linear series of degree $d$ and dimension $r$”.

Definition 1.1 Let $p + q \in C^{(2)}$ be an effective divisor of degree two,
$L$ a line bundle on $C$ and $V \subset H^0(L)$. We say that $V$ is a neutral
linear series w.r. to $p + q$ if $p + q$ imposes at most one condition to $V$
namely if
$$\dim[V(-p - q)] \geq \dim(V) - 1$$
where

\[ V(-p - q) := V \cap H^0(L(-p - q)) \]

(the intersection takes place in \( H^0(L) \)).

If \( p_1 + q_1, \ldots, p_\delta + q_\delta \) are effective divisors of degree two with pairwise disjoint supports we say that \( |V| \) is a neutral linear series w.r. to \( p_1 + q_1, \ldots, p_\delta + q_\delta \) if \( |V| \) is neutral w.r. to each of the divisors \( p_i + q_i \).

Each of the pairs \((p_i, q_i)\) is called a neutral pair for \( |V| \).

If \( |V| \) has a fixed point \( p \), then it is neutral w.r. to \( p + q \) for all \( q \in C \).

Let \( X \) be the integral curve obtained from \( C \) after identifying \( p_1 \) with \( q_1 \), \( p_2 \) with \( q_2 \), etc., and denote by \( \pi : C \to X \) the natural morphism. \( X \) has \( \delta \) double points \( x_1, \ldots, x_\delta \) such that \( \pi^{-1}(x_i) = \{p_i, q_i\} \), and no other singularity, has arithmetic genus \( g + \delta \), and \( \pi \) is the normalization map. If \( p_i \neq q_i \) (resp. \( p_i = q_i \)) then \( x_i \) is a node (resp. an ordinary cusp). Every linear series on \( X \) obviously pullback to a linear series on \( C \) which is neutral w.r. to \( p_1 + q_1, \ldots, p_\delta + q_\delta \).

Conversely, a neutral linear series on \( C \) is not necessarily the pullback of a linear series on \( X \). For example, if \( p_1, \ldots, p_\delta \) are fixed points of the series \( |V| \), but at least one of the \( q_i \)'s is not, then \( |V| \) is neutral w.r. to \( p_1 + q_1, \ldots, p_\delta + q_\delta \) but it is not a pullback.

Through this paper we will consider the case \( \delta = 1 \), i.e. neutral linear series w.r. to one divisor \( p + q \) of degree two, although many of the things we say generalize to any \( \delta \).

If \( \omega = \omega_C \) is the dualizing sheaf on \( C \), then \( |\omega(p + q)| \) is a neutral linear series w.r. to \( p + q \) called the canonical neutral linear series. It is the pullback of the complete canonical series on \( X = C/(p = q) \).

A neutral linear series w.r. to \( p + q \) is called a complete neutral linear series if it is not contained in a larger neutral linear series w.r. to \( p + q \). Of course in this definition the completeness of a neutral linear series is referred to a given neutral pair, and a complete neutral linear series is not in general complete as an ordinary linear series on \( C \).

**Lemma 1.2** Let \( p + q \in C^{(2)} \) and \( V \subset H^0(D) \) such that \( |V| \) is a complete neutral linear series w.r. to \( p + q \). Then

\[ h^0(D) - 1 \leq \dim(V) \leq h^0(D) \]

**Proof.** If \( V \neq H^0(D) \) then \( |D| \) is not neutral w.r. to \( p + q \). Hence

\[ h^0(D - p - q) = h^0(D) - 2 \]
Since
\[ \dim[V \cap H^0(D - p - q)] \geq \dim(V) - 1 \]
there is a hyperplane \( H \subset H^0(D) \) containing both \( V \) and \( H^0(D - p - q) \).

Since
\[ \dim[H \cap H^0(D - p - q)] = \dim(H) - 1 \]
we see that \(|H|\) is a neutral linear series containing \(|V|\); hence \( V = H \).

\[ \square \]

**Example 1.3** Consider a smooth embedding of \( C \subset \mathbb{P}^r, r \geq 3 \), by a complete \( g^r_n[L] \), let \( p, q \in C \) be distinct points, and let \( \langle p, q \rangle \) be the line joining \( p \) and \( q \). For every point \( O \in \langle p, q \rangle \) the linear series \( \Sigma_O \) spanned on \( C \) by the hyperplanes through \( O \) is a \( g^{r-1} \), neutral w.r. to \( p + q \) (if \( O = p \) or \( O = q \) then \( \Sigma_O \) has \( O \) as a fixed point).

Since \( |L| \) is not neutral \( \Sigma_O \) is complete as a neutral linear series. If \( O \neq O' \) then \( \Delta := \Sigma_O \cap \Sigma_{O'} \) is the \( g^{r-2} \) cut on \( C \) by the hyperplanes through \( \langle p, q \rangle \) having both \( p \) and \( q \) as fixed points. \( \Delta \) is neutral w.r. to \( p + q \) and it is contained in all the complete neutral linear series \( \Sigma_O, O \in \langle p, q \rangle \). Therefore in general a neutral linear series is not contained in a unique complete neutral linear series.

For a given integer \( n \geq 0 \) denote by \( \text{Pic}_n(C) \) the subvariety of \( \text{Pic}(C) \) parametrizing isomorphism classes of invertible sheaves of degree \( n \), and by \( W^r_n \) the closed subscheme of \( \text{Pic}_n(C) \) defined by the condition \( h^0(L) \geq r + 1 \). Let moreover \( G^r_n = G^r_n(C) \) be the scheme of linear series of degree \( n \) and dimension \( r \) on \( C \) (see [2]). We have a commutative diagram:

\[ C \times G^r_n \longrightarrow C \times W^r_n \longrightarrow C \times \text{Pic}_n(C) \]

\[ \downarrow p \quad \downarrow \text{id} \quad \downarrow \]

\[ C \quad G^r_n \quad W^r_n \quad \text{Pic}_n(C) \]

Let \( \mathcal{P} \) be the pullback on \( C \times G^r_n \) of a Poincaré line bundle on \( C \times \text{Pic}_n(C) \), and let \( E^r_n \subset p_* \mathcal{P} \) be the tautological locally free sheaf of rank \( r + 1 \) on \( G^r_n \). Fix \( A = p_1 + p_2 + \cdots + p_s \), an effective divisor of degree \( s, \ 2 \leq s \leq r + 1 \). The sheaf \( \mathcal{F}_A := p_*[\mathcal{P} \otimes \alpha^*(\mathcal{O}_A)] \) is locally free of rank \( s \) on \( G^r_n \) and we have a natural restriction morphism:

\[ \sigma_A : E^r_n \longrightarrow \mathcal{F}_A \]
Definition 1.4 $N^r_n(A)$ is the closed subscheme of $G^r_n$ defined by the condition $\text{rk}(\sigma_A) \leq s - 1$. In case $A = p + q$ has degree 2 we call $N^r_n(A)$ the scheme of linear series of degree $n$ and dimension $r$ neutral w.r. to $p + q$.

It follows from the definition that the support of $N^r_n(A)$ consists of the linear series $|V| \in G^r_n$ to which the divisor $A$ imposes at most $s - 1$ conditions, i.e. such that

$$\dim[V(-A)] \geq \dim(V) - s + 1$$

where $V \subset H^0(L)$ and $V(-A) = V \cap H^0(L(-A))$. If $|V|$ is base-point free and defines a birational morphism of $C$ onto its image this condition means that the linear span $\langle A \rangle$ has dimension at most $s - 2$. In particular $N^r_n(p + q)$ is supported on the set of all $g^r_n$’s neutral w.r. to $p + q$. Our main result on the schemes $N^r_n(A)$ on a B.N. curve is the following:

Theorem 1.5 Assume that $C$ is a B.N. curve of genus $g$ and that $A$ is an effective divisor of degree $s$, with $2 \leq s \leq r + 1$, on $C$.

(a) If $\rho(g, r, n) \geq r - s + 2$ then $N^r_n(A) \neq \emptyset$ and every component of $N^r_n(A)$ has codimension at most $r - s + 2$ in $G^r_n$.

(b) If $A$ is general then $N^r_n(A)$ has pure codimension $r - s + 2$ in $G^r_n$; in particular $N^r_n(A) = \emptyset$ if $\rho(g, r, n) < r - s + 2$.

The proof of this theorem will be given in §2. Note that in the particular case $s = 2$ we obtain that the scheme $N^r_n(p + q)$ of neutral linear series w.r. to $p + q$ has pure codimension $r$ if $p, q \in C$ are general points.

In the course of the proof we will consider on the given curve $C$ the family of all schemes $N^r_n(A)$ as $A$ varies in $C^{(s)}$. It can be defined as follows. Fix $2 \leq s \leq r + 1$ and consider the following diagram:
where the arrows are all projections. Let $D \subset C \times C^{(s)}$ be the tautological effective Cartier divisor; it has relative degree $s$ w.r. to the projection $\eta_2$. Consider on $C \times C^{(s)} \times G_{\mathfrak{m}}^{r}$ the natural restriction

$\beta_{13}^{*}P \rightarrow \beta_{13}^{*}P \otimes \beta_{12}^{*}O_{D}$

and, on $C^{(s)} \times G_{\mathfrak{m}}^{r}$, the induced:

$q_{*}\beta_{13}^{*}P \rightarrow q_{*}[\beta_{13}^{*}P \otimes \beta_{12}^{*}O_{D}] \quad (1)$

We have an inclusion:

$\gamma_{2}^{*}E_{\mathfrak{m}}^{r} \rightarrow q_{*}\beta_{13}^{*}P \quad (2)$

and composing (2) with (1) we obtain:

$\sigma_{s} : \gamma_{2}^{*}E_{\mathfrak{m}}^{r} \rightarrow q_{*}[\beta_{13}^{*}P \otimes \beta_{12}^{*}O_{D}]$

This is a homomorphism of locally free sheaves of ranks $r + 1$ and $s$ respectively.

**Definition 1.6** $N_{\mathfrak{m}}^{r}(s)$ is the closed subscheme of $C^{(s)} \times G_{\mathfrak{m}}^{r}$ defined by the condition $rk(\sigma_{s}) \leq s - 1$, and

$\gamma : N_{\mathfrak{m}}^{r}(s) \rightarrow C^{(s)}$

is the projection.

From the definition it follows that for each $A \in C^{(s)}$ the fibre $\gamma^{-1}(A)$ is isomorphic to $N_{\mathfrak{m}}^{r}(A)$.

## 2 Proof of Theorem 1.5

A preliminary observation which follows immediately from the definition of the schemes $N_{\mathfrak{m}}^{r}(A)$ is the following.

Let $p_{1} + \cdots + p_{t} \in C^{(t)}$ for some $2 \leq t \leq r$, and assume that $|V| \in N_{\mathfrak{m}}^{r}(p_{1} + \cdots + p_{t})$. Then $|V| \in N_{\mathfrak{m}}^{r}(p_{1} + \cdots + p_{t} + p)$ for each $p \in C$. Therefore we have a set-theoretic inclusion

$N_{\mathfrak{m}}^{r}(p_{1} + \cdots + p_{t}) \subset N_{\mathfrak{m}}^{r}(p_{1} + \cdots + p_{t} + p)$

It is easy to check that this is also a scheme-theoretic inclusion, but we will not need this fact.
Let’s prove (a). Since the scheme $N^r_n(A)$ is defined by a determinantal condition, if it is not empty then each of its components has codimension at most $r - s + 2$ in $G^r_n$.

Assume first $\rho(g, r, n) \geq r + 1$. In this case $G^r_{n-1} \neq \emptyset$ because

$$\dim(G^r_{n-1}) = \rho(g, r, n - 1) = \rho(g, r, n) - (r + 1) \geq 0$$

and $C$ is a B.N. curve. Let $|V| \in G^r_{n-1}$. For every $p \in C$ the linear series $|V| + p$ belongs to $G^r_n$ and $|V| + p \in N^r_n(A)$ for every $A = p + p_2 + \cdots + p_s$. Therefore $N^r_n(A) \neq \emptyset$ for all $A \in C^{(s)}$.

Assume now $\rho(g, r, n) \leq r$. One possibility is that $g - n + r < 0$: this means that all the series $|V| \in G^r_n$ are incomplete. This implies that $G^r_{n-1} \neq \emptyset$ and we can conclude as before that $N^r_n(A) \neq \emptyset$ for all $A \in C^{(s)}$.

The second possibility is that $g - n + r \geq 0$: we then have

$$\rho(g, r + 1, n) = \rho(g, r, n) - (r + 1) - (g - n + r) - 1 < 0$$

so that $W^{r+1} = \emptyset$; this implies $G^r_n = W^r_n$. Fixing a point $b \in C$ and an integer $m \gg 0$ we may embed $\text{Pic}_m(C) \subset \text{Pic}_m(C)$ tensoring by $O((m - n)b)$. We obtain the image of $W^r_n$ as the determinantal scheme defined by the condition $\text{rk}(\tau) \leq m - g - r$, where

$$\tau : \pi_*P \rightarrow \pi_*[P \otimes \alpha^*O_{(m-n)b}]$$

is the restriction map coming from the diagram:

$$\begin{array}{ccc}
C \times \text{Pic}_m(C) & \alpha^* & \pi^* \\
\downarrow & & \downarrow \\
C & & \text{Pic}_m(C)
\end{array}$$

and $P$ is a Poincaré bundle on $C \times \text{Pic}_m(C)$. We have $E^r_n = \ker(\tau|_{W^r_n})$, which is a subsheaf of the locally free $\mathcal{F} := (\pi_*P)|_{W^r_n}$. Therefore the dual $E^{r^\vee}_n$ is a quotient of $\mathcal{F}^\vee$ which is ample because it is the restriction to $W^r_n$ of the ample sheaf $(\pi_*P)^\vee$ (compare [7], §2). It follows that $E^{r^\vee}_n$ is ample: from this fact one deduces as in [7] that $N^r_n(A) \neq \emptyset$ for all $A \in C^{(s)}$. This completes the proof of (a).

We now prove (b). Because of (a), in order to prove part (b) of Theorem 1.5 it suffices to show that if $A \in C^{(s)}$ is general and $N^r_n(A) \neq \emptyset$ then all its components have codimension at least $r - s + 2$ in $G^r_n$. The proof is by induction on $r - s + 2$. 

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Suppose that \( s = r + 1 \). In this case it is enough to show that each component of \( N^r_n(A) \) is properly contained in a component of \( G^r_n \) for a general choice of \( A \in C^{(r+1)} \). For this it clearly suffices to prove that given \( |V| \in G^r_n \) there is at least one \( A \in C^{(r+1)} \) such that \( |V| \not\in N^r_n(A) \). This is obvious because each \( A \) such that \( |V| \in N^r_n(A) \) satisfies \( V(-A) \not= (0) \), hence the set of all such \( A \)’s has the same dimension \( r \) as the set of divisors \( D \in |V| \), so it is contained in a proper closed subset of \( C^{(r+1)} \).

We now assume that (b) has been proved for an integer \( s+1 \) such that \( 3 \leq s+1 \leq r+1 \); we let \( A = p_1 + \cdots + p_{s+1} \) be general and \( \Sigma \) an irreducible component of \( N^r_n(A) \). By the inductive hypothesis \( \Sigma \) has codimension \( r-s+1 \) in \( G^r_n \). Let \( |V| \) be a general element of \( \Sigma \). We will show that \( |V| \not\in N^r_n(A-p_k) \) for some \( 1 \leq k \leq s+1 \), where

\[
A - p_k = p_1 + \cdots + \hat{p}_k + \cdots + p_{s+1}
\]

This will imply that each component of \( N^r_n(A-p_k) \) contained in \( \Sigma \) is different from \( \Sigma \), hence has codimension at least \( r-s+2 \) in \( G^r_n \), and part (b) of the theorem will follow.

The set of \( g_r^n \)’s with a fixed point has codimension \( r \) in \( G^r_n \). Since \( \Sigma \) has codimension \( r-s+1 \leq r-1 \), the series \( |V| \) is base-point free. If in addition \( |V| \) is not composed with an involution the proof can be obtained as follows.

Let \( \varphi : C \longrightarrow \mathbb{P}(V^\vee) \cong \mathbb{P}^r \) be the birational morphism defined by \( |V| \): the hypothesis \( |V| \in N^r_n(A) \) means that the linear span \( \langle A \rangle \) of \( A \) in \( \mathbb{P}^r \) satisfies

\[
\dim(\langle A \rangle) \leq s - 1
\]

Suppose that \( |V| \in N^r_n(A-p_k) \) for all \( k \). This means that

\[
\dim(\langle A - p_k \rangle) \leq s - 2 \quad 1 \leq k \leq s + 1 \quad (3)
\]

Consider the two possibilities:

(i) \( \dim(\langle A \rangle) = s - 1 \).

(ii) \( \dim(\langle A \rangle) \leq s - 2 \).

In case (i) we immediately deduce from (3) that the points \( p_1, \ldots, p_{s+1} \) are independent in \( \mathbb{P}^r \), and this contradicts (i). In case (ii) we must have

\[
\langle A \rangle = \langle A - p_k \rangle
\]
for some $k$. Since $\dim(\langle A \rangle) < r$ and $C$ is non-degenerate in $\mathbb{P}^r$, we have:

$$\dim((A - p_k + p)) = \dim(\langle A - p_k \rangle) + 1 \leq s - 1$$

for a general choice of $p \in C$. Therefore $|V| \in N^r_n(A - p_k + p)$; more precisely $|V|$ is a general element of a component $\Sigma'$ of $N^r_n(A - p_k + p)$ which specializes to $\Sigma$ as $p$ specializes to $p_k$. Clearly this process, repeated if necessary, produces a general effective divisor $B = q_1 + \cdots + q_{s+1}$ such that $|V|$ is a general element of a component $\Sigma^*$ of $N^r_n(B)$ which specializes to $\Sigma$ as $B$ specializes to $A$, and such that either we are in case (i) again, and we have a contradiction, or $|V| \notin N^r_n(B - q_j)$ for some $j$, and the desired conclusion holds for $\Sigma^*$, contradicting the generality of $A$.

If we do not assume $\varphi : C \dashrightarrow \mathbb{P}^r$ birational the proof is similar and will be left to the reader. This concludes the proof of (b) and of the theorem. $\square$

3 The Brill-Noether-Castelnuovo statement

The so-called Brill-Noether-Castelnuovo statement is the following theorem, first proved in [8]:

**Theorem 3.1** For every $g \geq 0$ a B.N. curve $C$ of genus $g$ exists.

Theorem 3.1 can be easily deduced from Theorem 1.5 along the lines of existing degeneration arguments. In this section we give an outline of the reduction of 3.1 to 1.5. The proof is by induction on $g$. For the first few values of $g$ (say $g \leq 3$) the theorem is easily checked directly. We will therefore prove the inductive step only.

As before, consider a smooth curve $C$ of genus $g$, let $p, q \in C$ be two distinct points, and let $X = C/(p = q)$. Denote by $z$ the node of $X$, and by $\pi : C \rightarrow X$ the natural morphism. Since $X$ is integral, as in the smooth case it is possible to define a scheme $G^r_n(X)$ which parametrizes linear series of degree $n$ and dimension $r$ on $X$. Denote by $G^r_n(X, z)$ the open subset of $G^r_n(X)$ corresponding to linear series for which $z$ is not a base point. The scheme $G^r_n(X)$ is not proper but has a natural compactification $\overline{G^r_n(X)}$ which parametrizes $(r + 1)$-dimensional spaces $V$ of sections of torsion-free rank-one sheaves on
The construction of $G_n^r(X)$ and of $\overline{G_n^r(X)}$ requires the existence of Poincaré sheaves for the jacobian of $X$ and for the compactified jacobian respectively, which exist because $X$ is integral [1] (compare [5], last paragraph of p. 390 for an outline). For every $(r+1)$-dimensional $V \subset H^0(X, L)$, where $L$ is an invertible sheaf of degree $n$, we denote by $\pi^*V$ the corresponding $(r+1)$-dimensional subspace of $H^0(C, \pi^*L)$.

**Proposition 3.2** Associating to a $(V, L) \in G_n^r(X, z)$ the pair $(\pi^*V, \pi^*L)$ we obtain an open embedding

$$\Phi_0 : G_n^r(X, z) \subset N_r^+(p+q)$$

**Proof.** Let $N_r^+(p+q)_0$ be the open subset of $N_r^+(p+q)$ consisting of neutral linear series without base points at $p$ and $q$. We define a map:

$$\Psi_0 : N_r^+(p+q)_0 \longrightarrow G_n^r(X, z)$$

in the following way. Given $(W, M) \in N_r^+(p+q)_0$ we consider the torsion-free rank-one sheaf $\pi_*M$ on $X$ and we define a sub line bundle $L$ of $\pi_*M$ by setting $L_b = (\pi_*M)_b$ for every point $b \neq z$, and $L_z := W\mathcal{O}_z = \text{Im}[W \otimes \mathcal{O}_z \rightarrow (\pi_*M)_z]$.

Since $W \otimes m_z = W(-p-q) \otimes \mathcal{O}_z$ and $W$ is neutral w.r.t. $p+q$ and has no base points at $p$ and $q$, we deduce that $L_z$ is cyclic, thus it is free of rank one. Therefore $(W, L) \in G_n^r(X, z)$. Obviously $\Psi_0$ and $\Phi_0$ are inverse of each other. One easily checks that they are morphisms. □

From Proposition 3.2 and Theorem 1.5 we can deduce the following result:

**Theorem 3.3** If $C$ is a B.N. curve and $p, q \in C$ are general points then $G_n^r(X)$ has pure dimension $\rho(g+1, r, n)$.

**Proof.** In outline the proof goes as follows, along the lines of similar arguments for cuspidal curves as in [5].

Every irreducible component of $\overline{G_n^r(X)}$ has dimension at least $\rho(g+1, r, n)$ ([5], Lemma 4.6). From Corollary 4.4 of [5] we deduce that

$$\dim[\overline{G_n^r(X)} \setminus G_n^r(X)] = \dim[G_{n-1}^r(C)] = \rho(g, r, n-1) = \rho(g+1, r, n) - 1$$
because $C$ is a B.N. curve. It follows that $G^r_n(X)$ is dense in $\overline{G^r_n(X)}$. Now we apply Proposition 3.2 and Theorem 1.5 to deduce the conclusion.

The proof of Theorem 3.1 can be now completed as follows. By the inductive hypothesis we may assume that a B.N. curve of genus $g$ exists. Let $p, q \in C$ be general points and let $X = C/(p = q)$. Let

$$f : X \to S = \text{Spec}(\mathbb{C}[[t]])$$

be a flat family of projective curves such that, if $O$ and $\eta$ are the closed point and the generic point of $S$ respectively, the special fibre $X_O$ is isomorphic to $X$ and the geometric generic fibre $X_\eta$ is a smooth projective curve of genus $g + 1$.

Since $f$ has integral geometric fibres, the relative Picard scheme and the relative compactified Picard scheme exist, endowed with Poincaré sheaves [1]. Then the definition of the schemes $G^r_n$ and $\overline{G^r_n}$ for a fixed curve generalizes to relative constructions, so that in particular we can speak of the scheme $G^r_n(X/S)$ for the family $f$. It is a proper scheme over $S$ such that

$$\overline{G^r_n(X/S)_\eta} = \overline{G^r_n(X_\eta)} = G^r_n(X_\eta)$$

and

$$\overline{G^r_n(X/S)}_O = \overline{G^r_n(X)}$$

From these facts and from Theorem 3.3 it follows that every component of $G^r_n(X_\eta)$ has dimension at most $\rho(g + 1, r, n)$. Since moreover every component of $G^r_n(X_\eta)$ has dimension at least $\rho(g + 1, r, n)$, the conclusion follows.

References


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