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Petri's Approach to the Study of the Ideal Associated to a Special Divisor

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Introduction

The ideal defining a canonical curve was classically studied by Max Noëther, Enriques, Petri and Babbage [B]. Recently the subject was again brought to light by Bernard Saint-Donat and David Mumford.

The fundamental result is that the ideal of a canonical, non-hyperelliptic, curve C, of genus $p>3$, is generated by quadratic forms, with the exception of two cases: when C lies on a non-singular ruled surface of degree $p-2$ in $\mathbb{P}^{p-1}$ (in which case C is trigonal), or else when $p=6$ and C is contained in the Veronese surface of $\mathbb{P}^5$ (in which case C has a g_5^2).

More generally it is natural to ask what can be said about the ideal I_*^D of an irreducible curve $C \subset \mathbb{P}^{l-1}$ of genus p, on which the hyperplanes cut a complete linear series $|D|$. When D is *non-special* and $\deg D \geqq 3p+1$ the answer is simple: I_*^D is generated by quadratic forms (see [M-2]). When D is *special* the situation is more complicated.

To this case is devoted the central part of this paper, which closely follows Petri's approach. The result is the following. Take a basis $x_1, \ldots, x_l$ of $H^0(D)$ and a basis $y_1, \ldots, y_{l'}$ of $H^0(K-D)$. The $x_i y_j$'s can be naturally viewed as elements of $H^0(K)$, spanning a sub-vector space of $H^0(K)$ whose codimension we call τ. Choosing $z_1, \ldots, z_\tau \in H^0(K)$ so that the $x_i y_j$'s and the z_ν's generate $H^0(K)$, we consider relations of the following type:

(1) $\sum a_{ij} x_i y_j = 0$,

(2) $\sum P_j^{(2)}(x) y_j + \sum P_\nu^{(1)}(x) z_\nu = 0$

(the $P^{(\rho)}(x)$'s are forms of degree ρ in the x's).

It is then possible to show that *every element of I_*^D can be obtained by eliminating the y's and the z's from the above relations.* There are few possible exceptions to that rule and they are completely analogous to the ones occurring in the classical case $D=K$.

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This theorem was stated by K. Petri in [P-2]. In its proof a central role is played by Petri's analysis of the natural map [P-1]

$$\bigoplus_{\rho \geqq 0} S^\rho H^0(D) \otimes H^0(K) \xrightarrow{u_*} \bigoplus_{\rho \geqq 0} H^0(\rho D + K).$$

The kernel $I_*^{K,D}$ of u_* provides, after natural manipulations, the basic relations of type (1) and (2). This kernel is a $\bigoplus_{\rho \geqq 0} S^\rho H^0(D)$-module in which two different kinds of data coexist. The intrinsic data coming from the ideal I_*^K, of the canonically embedded curve, and the extrinsic ones coming from the ideal I_*^D of the special curve $C \subset \mathbb{P}^{l-1}$.

The study of this "*semicanonical ideal*" $I_*^{K,D}$ is carried out in the first three sections. The fourth section deals with the ideal I_*^D, and there we analyse the particular case of a *Klein's canonical curve*, (i.e. one for which there exists a positive integer d with $|dD| = |K|$). These curves appear, in a natural way, as sections of canonically embedded varieties.

The study of Klein's canonical curves leads us in fact, in the last section, to another generalization of the Max Noether-Enriques-Petri theorem. The following. Given a d-dimensional variety $V \subset \mathbb{P}^N$, *canonically embedded* and *arithmetically Cohen Macaulay* the ideal $I_*(V)$ of V is generated by forms of degree $\leqq d+1$, with few possible exceptions. The exceptional cases can be easily described, and the entire result particularizes to the classical one when V is a canonically embedded curve.

The study of the exceptional cases turned out to be linked with Griffiths and Harris' recent work on extremal varieties [G-H]. For example we can easily show that exceptional Klein's canonical curves are Castelnuovo's curves, (Theor. (4.7)).

At the end of § 4 some parenthetical remarks are made about an old moduli problem concerning special divisors. It may be possible that Petri's machinery could bring some light to those questions.

We thank Phillip Griffiths and Joseph Harris for pointing out to us the very interesting relations between their extremal varieties and our exceptional cases. We also thank Maurizio Cornalba for a number of very useful comments on this paper.

§ 1. General Remarks

Let C be an irreducible complete non-singular algebraic curve of genus $p > 0$ defined over an algebraically closed field $\mathbb{K}$. Given a sheaf L, and a divisor D on C we shall briefly denote by $H^i(D \otimes L)$ the cohomology group $H^i(C, O(D) \otimes L)$ whose dimension will be denoted by $h^i(D \otimes L)$.

The so called "*base point free pencil trick*" states that, for any pair of invertible sheaves L and M, on C, and for any pair of sections s_1 and s_2 of L, having no common zeroes, the kernel of the map

$$H^0(M)s_1 \oplus H^0(M)s_2 \to H^0(M \otimes L)$$
$$(t_1, t_2) \rightsquigarrow t_1 s_1 + t_2 s_2$$

is isomorphic to $H^0(M \otimes L^{-1})$.

We shall make frequent use of this elementary fact whose proof is given in [S].

Throughout the entire paper a *canonical divisor* will be denoted by K and, as usual, the *complete* linear series containing a divisor D will be denoted by $|D|$. If $\rho K > D$, $\rho \geqq 1$, the elements of $H^0(\rho K - D)$ will be thought of as regular ρ-fold differentials having zeros at D.

Our first aim is to prove a sort of *generalized* Noëther's theorem in which the canonical divisor on C interplays with the extrinsic datum of an arbitrary positive divisor D.

(1.1) **Lemma.** *Let $|D|$ be a base point free pencil. Then, for every positive integer ρ, the image of the natural map*

$$u_\rho \colon S^\rho H^0(D) \otimes H^0(K) \to H^0(K + \rho D)$$

is of codimension $\rho - 1$ in $H^0(K + \rho D)$.

The case $\rho = 1$ is a straightforward consequence of the base point free pencil trick and Riemann-Roch theorem. Let now $\rho = 2$. We have a commutative diagram

$$\begin{tikzcd} & S^2 H^0(D) \otimes H^0(K) \arrow[dr, "u_2"] & \\ H^0(D) \otimes H^0(D) \otimes H^0(K) \arrow[ur, "\pi"] \arrow[dr, "w"'] & & H^0(K + 2D). \\ & H^0(D) \otimes H^0(K + D) \arrow[ur, "v"'] & \end{tikzcd}$$

The case $\rho = 1$ implies that w is surjective. Obviously π too is surjective. Therefore $\dim \operatorname{Im}(u_2) = \dim \operatorname{Im}(v)$ and the base point free pencil trick gives $\dim \ker v = p$. Thus $\dim \operatorname{Im} v = 2(p + n - 1) - p = h^0(K + 2D) - 1$ where $n = \deg D$. As a consequence of the case $\rho = 2$ we can say that $H^0(K + 2D)$ is generated by $\operatorname{Im} v$ and by an element $\eta \in H^0(K + 2D) \setminus \operatorname{Im} v$. Let s_1 and s_2 be generators for $H^0(D)$. Notice that, by Castelnuovo's lemma ([M-2]) applied to the map v: $H^0(D) \otimes H^0(K + (v-1)D) \to H^0(K + vD)$, every element in $H^0(K + vD)$, $v \geqq 3$, may be written in the form $\sum_{i=1}^{p} P_i^{(v)} \varphi_i + P^{(v-2)} \eta$, where $P^{(\mu)}$ denotes a form of degree μ in s_1 and s_2, and $\varphi_1, \dots, \varphi_p$ is a basis of $H^0(K)$. We now prove the lemma by induction on ρ. Assume it is true for the positive integers less than ρ. Consider the following diagram:

(1.2) $$\begin{tikzcd} & S^\rho H^0(D) \otimes H^0(K) \arrow[dr, "u_\rho"] & \\ H^0(D) \otimes S^{\rho-1} H^0(D) \otimes H^0(K) \arrow[ur, "\pi"] \arrow[dr, "w"'] & & H^0(K + \rho D). \\ & H^0(D) \otimes H^0(K + (\rho - 1)D) \arrow[ur, "v"'] & \end{tikzcd}$$

We have

(1.3) $\dim \operatorname{Im}(u_\rho) = \dim \operatorname{Im}(w) - \dim(\ker(v) \cap \operatorname{Im}(w))$.

By induction

(1.4) $\dim \operatorname{Im}(w) = 2[h^0(K + (\rho - 1)D) - \rho + 2] = 2(p + (\rho - 1)n - \rho + 1)$.

Also the elements of $H^0(D)\otimes H^0(K+(\rho-1)D)$ can be identified with pairs $(s_1\otimes\lambda_1, s_2\otimes\lambda_2)$ where $\lambda_i=\sum_j P_{ij}^{(\rho-1)}\varphi_j+P_i^{(\rho-3)}\eta$, $i=1,2$. Clearly the elements of $\operatorname{Im} w$ are of type $(s_1\otimes\sum_j P_{1j}^{(\rho-1)}\varphi_j, s_2\otimes\sum_j P_{2j}^{(\rho-1)}\varphi_j)$. On the other hand the base point free pencil trick gives

$$\operatorname{Ker}(v)=\{(s_1\otimes s_2 h, -s_2\otimes s_1 h)\colon h\in H^0(K+(\rho-2)D)\};$$

we now prove that

$$\operatorname{Ker}(v)\cap\operatorname{Im}(w)=\{(s_1\otimes s_2 h', -s_2\otimes s_1 h')\colon h'\in\operatorname{Im}(u_{\rho-2})\}.$$

First observe that among the elements of type $Q^{(\rho-3)}\eta$ there are exactly $\rho-2$ linearly independent ones. Also, as we noticed, those elements generate $H^0(K+(\rho-1)D)$ modulo $\operatorname{Im}(u_{\rho-1})$. It follows then by the induction that non-zero elements of type $Q^{(\rho-3)}\eta$ do not belong to $\operatorname{Im}(u_{\rho-1})$. Therefore writing $h=\sum Q_j^{(\rho-2)}\varphi_j+Q^{(\rho-4)}\eta$ and imposing the condition $s_2 h\in\operatorname{Im}(u_{\rho-1})$ we get $h=h'\in\operatorname{Im}(u_{\rho-2})$. Therefore $\dim\operatorname{Ker}(v)\cap\operatorname{Im}(w)=\dim\operatorname{Im}u_{\rho-2}$. The lemma follows by using (1.2), (1.3) and the induction hypothesis.

We shall need the following classical result.

(1.5) **Lemma.** *Let C be an irreducible curve in $\mathbb{P}^r$, which is not contained in any hyperplane. Then, for every $s\leq r-2$, $s+1$ generic points $P_1,\dots,P_{s+1}$ on C span a $\mathbb{P}^s$ such that $\mathbb{P}^s\cdot C=P_1+\dots+P_{s+1}$.*

The assertion is obviously true when $r=3$ and $s=1$, since, as is well known, a non-planar curve in $\mathbb{P}^3$ does not possess ∞^2 trisecants. The case $s=1$, $r\geq 3$ can be reduced to the preceding by projecting C from a generic $\mathbb{P}^{r-4}$ of $\mathbb{P}^r$, into $\mathbb{P}^3$. The general case is then proved by induction on s by projecting C on a hyperplane, from a generic point of C.

(1.6) **Theorem.** *Let $|D|$ be a complete linear series on C free from base points and such that $h^0(D)=l\geq 3$. Furthermore assume that $|D|$ defines a birational morphism*

$$\pi_D\colon C\to\pi_D(C)\subset\mathbb{P}^{l-1}.$$

Then the natural map

$$\oplus u_\rho\colon \bigoplus_{\rho\geq 0} S^\rho H^0(D)\otimes H^0(K)\to\bigoplus_{\rho\geq 0} H^0(K+\rho D)$$

is surjective.

Let $n=\deg D$. Let $\mathfrak{a}=P_3+\dots+P_l$ be a generic positive divisor on C of degree $l-2$. Then our hypothesis on D, together with Lemma (1.5), imply that $|D-\mathfrak{a}|$ is a base point free pencil and that $h^1(D)=h^1(D-\mathfrak{a})$. Let $D^*=O(D)\otimes O_{\mathfrak{a}}$.

The exact sequence

$$0\to O(D-\mathfrak{a})\to O(D)\to D^*\to 0$$

induces an exact commutative diagram ($\rho \geqq 1$)

$$\begin{tikzcd}
0 \arrow[r] & H^0(D-\mathfrak{a})\otimes H^0(K+(\rho-1)D) \arrow[r] \arrow[d, "\sigma"] & H^0(D)\otimes H^0(K+(\rho-1)D) \arrow[r] \arrow[d, "v"] & H^0(D^*)\otimes H^0(K+(\rho-1)D) \arrow[r] \arrow[d, "v^*"] & {} \\
0 \arrow[r] & H^0(K+\rho D-\mathfrak{a}) \arrow[r] \arrow[d] & H^0(K+\rho D) \arrow[r] \arrow[d] & H^0(D^*\otimes(K+(\rho-1)D) \arrow[r] \arrow[d] & {} \\
& s(D-\mathfrak{a}, K+(\rho-1)D) \arrow[r] \arrow[d] & s(D, K+(\rho-1)D) \arrow[r] \arrow[d] & s(D^*, K+(\rho-1)D) \arrow[d] & \\
& 0 & 0 & 0 &
\end{tikzcd}$$

where $s(A, B) = \operatorname{coker}[H^0(A)\otimes H^0(B) \to H^0(A\otimes B)]$. We first consider the case $\rho = 1$. The base point free pencil trick gives

$$\dim \ker(\sigma) = h^0(K-D+\mathfrak{a}) = h^0(K-D) = h^1(D)$$

so that

$$\dim s(D-\mathfrak{a}, K) = h^0(K+D-\mathfrak{a}) - 2p + h^1(D) = 0.$$

By Castelnuovo's lemma ([M-2]) $s(D^*, K) = 0$, so that $s(D, K) = 0$, proving the surjectivity of u_1. When $\rho \geqq 2$ we have a commutative diagram of type (1.2). Therefore it is sufficient, by induction, to show that v is surjective. For this we proceed exactly as in the case $\rho = 1$.

§ 2. The "Semicanonical Ideal" of a Special Curve

From now on we shall assume that C is non-hyperelliptic. Let us fix once and for all a *special* linear series $|D|$ of degree n on C *free from base points* and defining a *birational morphism*

$$\pi_D\colon\ C \to \pi_D(C) \subset \mathbb{P}^{l-1}$$

where $l = h^\circ(D)$.

We set $i(D) = h^1(D)$. We fix a divisor $D' \in |K-D|$. We also fix, once and for all, l points $P_1, \ldots, P_l$ on C in generic position and a basis $\varphi_1, \ldots, \varphi_p$ of $H^0(K)$ in such a way that the first l elements $\varphi_1, \ldots, \varphi_l$ form a basis of $H^\circ(K-D') \cong H^0(D)$ having the property that

$$\varphi_i(P_j) = \delta_{ij}, \qquad i, j = 1, \ldots, l.$$

Furthermore we can assume that the divisors $(\varphi_1) - D'$ and $(\varphi_2) - D'$ consist, each, of distinct points. So that if we set $\mathfrak{a} = P_3 + \cdots + P_l$, φ_1 and φ_2 vanish of order 1 at the points of $\mathfrak{a}$.

(2.1) *Notation.* Let Δ be a special divisor. Consider the natural map $S^\rho H^0(K-\Delta)\otimes H^0(K)\to S^{\rho+1}(K)$. We shall denote by $S^\rho H^0(K-\Delta)\cdot H^0(K)$ the image of this map.

We have the following diagram of degree 0 morphisms of graded $\bigoplus_{\rho\geqq 0} S^\rho H^0(K-D')$-modules

$$\begin{array}{ccc}
\bigoplus_{\rho\geqq 0} S^{\rho+1}H^0(K) & \xrightarrow{i_*} & \bigoplus_{\rho\geqq 0} H^0((\rho+1)K), \\
\cup & & \cup \\
\bigoplus_{\rho\geqq 0} S^\rho H^0(K-D')\cdot H^0(K) & \xrightarrow{u_*} & \bigoplus_{\rho\geqq 0} H^0((\rho+1)K-\rho D'), \\
\cup & & \cup \\
\bigoplus_{\rho\geqq 0} S^{\rho+1}H^0(K-D') & \xrightarrow{d_*} & \bigoplus_{\rho\geqq 0} H^0((\rho+1)(K-D')).
\end{array}$$

Here i_* is the natural homomorphism while u_* and d_* are restrictions of i_*. From Theorem (1.6) it follows that i_* and u_* are surjective. As usual one says that $\pi_D(C)$ is *projectively normal* if d_* is surjective. We set

$$I_*^K=\bigoplus_{\rho\geqq 0} I_{\rho+1}^K=\operatorname{Ker} i_*,$$
$$I_*^{K,D}=\bigoplus_{\rho\geqq 0} I_{\rho+1}^{K,D}=\operatorname{Ker} u_*,$$
$$I_*^D=\bigoplus_{\rho\geqq 0} I_{\rho+1}^D=\operatorname{Ker} d_*.$$

I_*^K is the ideal of the canonically embedded curve while I_*^D is the ideal of $\pi_D(C)\subset \mathbb{P}^{l-1}$.

In this section we shall study the graded $\bigoplus_{\rho\geqq 0} S^\rho H^0(K-D')$-module $I_*^{K,D}$, which, by abuse of language, we may call the *semicanonical ideal of the special curve* $\pi_D(C)$. We set $I_*^{K,D}=I_*$. We shall see that I_* is generated by I_2 and I_3, if $l\geqq 4$, and it is generated by I_2 and I_4 if $l=3$. Moreover we shall see that, if $l\geqq 4$, I_* is generated by I_2 with the possible exception of two cases: the case in which $\pi_D(C)$ lies on a ruled surface of degree $l-2$ in $\mathbb{P}^{l-1}$, and the case in which $l=6$ and $\pi_D(C)$ is contained in the Veronese surface.

This result is essentially due to Petri and the case $D=K$ is exactly the Max Noëther-Enriques-Petri theorem proved by Bernard Saint-Donat in [S].

Our treatment will follow very closely his paper.

2.2) *Notation.* Elements in $\bigoplus_{\rho\geqq 0} S^\rho H^0(K)$ will be written as polynomial expressions in the *indeterminates* $\varphi_1,\dots,\varphi_p$. One such polynomial $P(\varphi)$ is called a *relation* if $P(\varphi)\in\operatorname{Ker} i_*$. In this case we write $P(\varphi)\underset{C}{=}0$. As usual $P^{(\rho)}(\varphi)$, $Q^{(\rho)}(\varphi)$, ... denote forms of degree ρ in the φ's.

Consider the graded $\bigoplus_{\rho\geqq 0} S^\rho H^0(K-D'-\mathfrak{a})$-submodule $\bigoplus_{\rho\geqq 0} S^\rho H^0(K-D'-\mathfrak{a})\cdot H^0(K)$ of $\bigoplus_{\rho\geqq 0} S^{\rho+1}H^0(K)$ and the natural map

$$\alpha_*=\bigoplus\alpha_\rho\colon \bigoplus_{\rho\geqq 0} S^\rho H^0(K-D'-\mathfrak{a})\cdot H^\circ(K)\to \bigoplus_{\rho\geqq 0} H^0((\rho+1)K-\rho D'-\rho\,\mathfrak{a}).$$

(2.3) **Lemma.** $\operatorname{Ker}(\alpha_\rho)$ *is a* $\mathbb{K}$*-vector space of dimension* $\rho(i(D)-1)$. *Moreover* $\ker\alpha_*$ *is generated, as* $\bigoplus_{\rho\geqq 0} S^\rho H^0(K-D'-\mathfrak{a})$*-module, by* $i(D)-1$ *linearly independent relations* $g_1, \ldots, g_{i(D)-1}$ *of the type*

$$(2.4)\quad g_j=\sum_{i=1}^{p} P_{ji}^{(1)}\varphi_i$$

where $P_{ji}^{(1)}$ *are linear forms in* φ_1 *and* φ_2.

Clearly $\dim S^\rho H^0(K-D'-\mathfrak{a})\cdot H^0(K)=(\rho+1)(p-2)+\rho+2$ and $h^\circ((\rho+1)K-\rho D'-\rho\mathfrak{a})=p+\rho(n-l+2)$ so that the first assertion is a consequence of Lemma (1.1). Let us now choose $i(D)-1$ linearly independent relations of the form (2.4). The second assertion will be proved by induction on ρ. Consider $(\rho-1)(i(D)-1)$ linearly independent relations of the form $\sum_{i=1}^{p} P_i^{(\rho-1)}\varphi_i$, where the $P_i^{(\rho-1)}$'s are forms of degree $\rho-1$ in φ_1 and φ_2. We then get $2(\rho-1)(i(D)-1)$ elements in $\operatorname{Ker}\alpha_\rho$ of type

$$(2.5)\quad \varphi_1\sum_{i=1}^{p} P_i^{(\rho-1)}\varphi_i, \qquad \varphi_2\sum_{i=1}^{p} P_i^{(\rho-1)}\varphi_i.$$

Since, by induction, we can express these elements in terms of the g_j's, it suffices to show that they generate $\operatorname{Ker}\alpha_\rho$. Suppose we have a linear relation

$$\lambda_1\varphi_1\Big(\sum_i P_i^{(\rho-1)}\varphi_i\Big)+\lambda_2\varphi_2\Big(\sum_i Q_i^{(\rho-1)}\varphi_i\Big)=0,$$

$\lambda_0, \lambda_1\in\mathbb{K}$. Then $\sum_i P_i^{(\rho-1)}\varphi_i$ is divisible by φ_2 and $\sum Q_i^{(\rho-1)}\varphi_i$ is divisible by φ_1. We therefore get a relation

$$\lambda_1\sum_i \tilde{P}_i^{(\rho-2)}\varphi_i+\lambda_2\sum_i \tilde{Q}_i^{(\rho-2)}\varphi_i=0.$$

By the first part the number of linearly independent relations of this type is $(\rho-2)(i(D)-1)$. Thus among the elements (2.5) at least $\rho(i(D)-1)$ are linearly independent; but we already know that there can't be more.

Let us consider the following elements in $H^\circ(K-D')\cdot H^0(K)$.

$$(2)\quad \begin{cases} \varphi_3^2, \ldots, \varphi_l^2 \\ (1)\ \begin{cases}\varphi_1\varphi_i,\ \varphi_2\varphi_i, \quad i=3, \ldots, p\\ \varphi_1^2, \varphi_1\varphi_2, \varphi_2^2.\end{cases}\end{cases}$$

(2.6) **Lemma.** *The elements* (2), *viewed in* $H^\circ(2K-D')$, *generate* $H^\circ(2K-D')$.

Since $\varphi_i(P_j)=\delta_{ij}$, $i,j=3, \ldots, l$, the elements $\varphi_3^2, \ldots, \varphi_l^2$ are linearly independent modulo $H^\circ(2K-D'-\mathfrak{a})$. On the other hand, by Lemma (1.1), the elements (1) generate $H^0(2K-D'-\mathfrak{a})$. The assertion follows now by counting dimensions.

Let us fix i and k such that $3\leqq i\leqq p$, $3\leqq k\leqq l$, $i\neq k$. Then $\varphi_i\varphi_k$, viewed in $H^\circ(2K-D')$, vanishes at P_1 and P_2. Therefore we have relations

$$\varphi_i \varphi_k \underset{C}{=} \sum_{s=3}^{p} a_{sik} \varphi_s + b_{ik} \varphi_1 \varphi_2$$

where $a_{sik} = \lambda_{sik} \varphi_1 + \mu_{sik} \varphi_2$, $\lambda_{sik}, \mu_{sik}, b_{ik} \in \mathbb{K}$. If we set

$$f_{ik} = \varphi_i \varphi_k - \sum a_{sik} \varphi_s - b_{ik} \varphi_1 \varphi_2$$

we get elements in I_2. It is easy to see that among the f_{ik}'s there are

$$\binom{p-1}{2} - \binom{p-l+1}{2} - (l-2)$$

which are linearly independent modulo the g_j's.

An easy count of dimensions, based on Theorem (1.6), gives the following.

(2.7) **Lemma.** *I_2 is generated by the relations f_{ik}'s and g_j's.*

Let now W be the subspace of $H^\circ(3K-2D'-2\mathfrak{a})$ generated by the elements

$$\begin{aligned}&\varphi_1^2 \varphi_i, \varphi_1 \varphi_2 \varphi_i, \varphi_2^2 \varphi_i, \qquad i=3, \dots, p,\\ &\varphi_1^3, \varphi_1^2 \varphi_2, \varphi_1 \varphi_2^2, \varphi_2^3.\end{aligned}$$

From Lemma (1.1) we deduce that W is of codimension 1 in $H^0(3K-2D'-2\mathfrak{a})$.

Let η be an element of $S^2 H^0(K-D') \cdot H^\circ(K)$ whose image in $H^\circ(3K-2D'-2\mathfrak{a})$ does not lie in W.

Consider the following elements in $S^2 H^0(K-D') \cdot H^\circ(K)$

$$(3) \quad \begin{cases} \varphi_3^3, \dots, \varphi_l^3 \\ \varphi_1 \varphi_3^2, \dots, \varphi_1 \varphi_l^2 \\ \eta, \varphi_1^2 \varphi_i, \varphi_1 \varphi_2 \varphi_i, \varphi_2^2 \varphi_i, \qquad i=3, \dots, p \\ \varphi_1^3, \varphi_1^2 \varphi_2, \varphi_1 \varphi_2^2, \varphi_2^3. \end{cases}$$

(2.8) **Lemma.** *The elements* (3), *viewed in $H^0(3K-2D')$, generate $H^0(3K-2D')$.*

The proof is entirely similar to the one of Lemma (2.6). In fact one easily sees that $\varphi_3^3, \dots, \varphi_l^3$, viewed in $H^0(3K-2D')$, are linearly independent modulo $H^\circ(3K-2D'-\mathfrak{a})$. Also, since φ_1 and φ_2 vanish of order 1 at P_i, $i=3, \dots, l$, $\varphi_1 \varphi_3^2, \dots, \varphi_1 \varphi_l^2$, viewed in $H^0(3K-2D')$, are linearly independent modulo $H^0(3K-2D'-2\mathfrak{a})$. It is then clear that the elements in the first two rows are linearly independent modulo $H^\circ(3K-2D'-2\mathfrak{a})$. On the other hand we noticed that the elements of the last two rows generate $H^0(3K-2D'-2\mathfrak{a})$. The assertion follows now by counting dimensions.

We need the following lemma.

(2.9) **Lemma.** *Let $3 \leqq i \leqq l$. Assume that $\alpha = \lambda_1 \varphi_1 + \lambda_2 \varphi_2$, $\lambda_1, \lambda_2 \in \mathbb{K}$, has a zero of order 2 at P_i. Then the image of $\alpha \varphi_i^2$ in $H^0(3K-2D'-2\mathfrak{a})$ does not lie in W.*

Assume that $\alpha \varphi_i^2 \underset{C}{=} \sum_{j=1}^{p} (\mu_j \varphi_1^2 + \nu_j \varphi_1 \varphi_2 + \pi_j \varphi_2^2) \varphi_j$. Since φ_1 and φ_2 vanish of order 1 at P_i, $\lambda_1 \neq 0$ and $\lambda_2 \neq 0$, and we can write the above relation in the form

(2.10) $\alpha\varphi_i^2 \underset{C}{=} \alpha\varphi_1\varphi + \alpha\varphi_2\psi + \varphi_2^2\vartheta, \qquad \varphi, \psi, \vartheta \in H^0(K).$

By hypothesis the divisor (α) satisfies $(\alpha) \geqq \mathfrak{a} + D'$. Let $(\alpha) = \Delta + \mathfrak{a} + D'$ and $(\varphi_2) = \Delta' + \mathfrak{a} + D'$. By hypothesis $\operatorname{supp}\Delta \cap \operatorname{supp}\Delta' = \phi$. If $Q \in \operatorname{supp}\Delta \setminus \operatorname{supp}(\mathfrak{a} + D') \cap \operatorname{supp}\Delta$ then relation (2.10) implies that $\operatorname{mult}_Q(\vartheta) \geqq \operatorname{mult}_Q(\alpha)$. Let now $Q \in \operatorname{supp}(\mathfrak{a} + D')$. We let $s = \operatorname{mult}_Q(\varphi_1) = \operatorname{mult}_Q(\varphi_2) = \operatorname{mult}_Q(\mathfrak{a} + D')$, $r = \operatorname{mult}_Q(\alpha)$, so that $r - s = \operatorname{mult}_Q \Delta$. If $Q \neq P_i$, $\operatorname{mult}_Q(\varphi_i) \geqq s$ and relation (2.10) gives $\operatorname{mult}_Q(\vartheta) + 2s \geqq r + s$. If $Q = P_i$ $\operatorname{mult}_Q(\varphi_i) = 0$, $s = 1$ so that $\operatorname{mult}_Q(\vartheta) + 2 \geqq r$.

Therefore $\vartheta \in H^0(K - \Delta + P_i) = H^0(\mathfrak{a} + D' + P_i)$. Since the points $P_3, \ldots, P_l$ of $\mathfrak{a}$ are generic, we use Lemma (1.5) to conclude that $h^0(\mathfrak{a} + D' + P_i) = h^0(\mathfrak{a} + D')$. Therefore $\vartheta \in H^0(K - \Delta)$. But this is absurd as one sees by comparing the order of zero at P_i of both sides of (2.10).

Consider now the element η. From its definition and from the preceding Lemma it follows that, for each $i = 3, \ldots, l$ there exists a unique

(2.11) $\alpha_i = \lambda_{i1}\varphi_1 + \lambda_{i2}\varphi_2$

vanishing of order 2 at P_i and such that

$$\alpha_i\varphi_i^2 \underset{C}{=} \eta + \theta_i$$

where θ_i belongs to $S^2 H^0(K - D' - \mathfrak{a}) \cdot H^0(K)$.

If $3 \leqq h$, $k \leqq l$, $h \neq k$ we set

$$G_{hk} = \alpha_h\varphi_h^2 - \alpha_k\varphi_k^2 + \theta_k - \theta_h.$$

The G_{hk}'s obviously belong to I_3 and are not defined for $l = 3$. Also the following relations hold

$$G_{kh} + G_{hk} = 0,$$
$$G_{kh} + G_{hj} = G_{kj}.$$

We can now prove the main result of this section.

(2.12) **Theorem.** (i) *If $l \geqq 4$ $I_*^{K,D}$ is generated, as $\bigoplus_{\rho \geqq 0} S^\rho H^0(K - D')$-module, by the* g_j's, *the* f_{ik}'s *and the* G_{kr}'s.

(ii) *If $l = 3$, $I_*^{K,D}$ is generated by the* g_j's, *the* f_{i3}'s, *and by at most one degree* 4 *relation of the type $F = F^{(1)}\varphi_3^3 + F^{(2)}\varphi_3^2 + \sum_{i=1}^{p} F_i^{(3)}\varphi_i$, where $F^{(1)}$, $F^{(2)}$, $F_i^{(3)}$ are homogeneous polynomials in φ_1, φ_2 of degree* 1, 2, 3 *respectively.*

We shall prove the theorem for each homogeneous component I_ρ of $I_* = I_*^{K,D}$.

Lemma (2.7) settles the case $\rho = 2$. Let us prove the theorem for I_3. Consider an element of I_3:

$$R = \sum_{\substack{1 \leqq i \leqq p \\ 1 \leqq j, k \leqq l}} \gamma_{ijk}\varphi_i\varphi_j\varphi_k \underset{C}{=} 0, \qquad \gamma_{ijk} \in \mathbb{K}.$$

Let us perform the following substitutions:

(a) for every triple of indices i, j, k such that $i \neq j$, $i, j \geqq 3$, we use the equality

$$\varphi_i \varphi_j \varphi_k = \left(f_{ij} + \sum_{s=3}^{p} a_{sij} \varphi_s + b_{ij} \varphi_1 \varphi_2\right) \varphi_k.$$

Afterwords, if $k \geqq 3$, $s \neq k$ we use the equality

$$a_{sij} \varphi_s \varphi_k = a_{sij} \left(f_{sk} + \sum_{t=3}^{p} a_{tsk} \varphi_t + b_{sk} \varphi_1 \varphi_2\right).$$

(b) According to (2.11) we substitute the terms of type

$$v_i \varphi_2 \varphi_i^2, \quad 3 \leqq i \leqq l \text{ with } v_i \lambda_{i2}^{-1} (\alpha_i - \lambda_{i1} \varphi_1) \varphi_i^2.$$

After these substitutions R can be written in the form

$$R = \sum_{i=3}^{l} \gamma_{iii} \varphi_i^3 + \sum \gamma'_{ijk} \varphi_i f_{jk} + \sum_{i=3}^{l} \delta_i \varphi_1 \varphi_i^2 + \sum_{i=3}^{l} \mu_i \alpha_i \varphi_i^2 + w$$

where $w \in H^0(K - D' - \mathfrak{a}) \cdot H^0(K)$. The properties of the system of elements (3) imply that $\gamma_{iii} = \delta_i = 0$. If $l = 3$ it follows that also $\mu_3 = 0$, therefore w is a relation which can be expressed in terms of the g_j's (Lemma (2.3)). Assume $l \geqq 4$. For each $i = 4, \ldots, l$ we make the substitution

$$\mu_i \alpha_i \varphi_i^2 = \mu_i G_{i3} + \mu_i \alpha_3 \varphi_3^2 + \mu_i \theta_i - \mu_i \theta_3$$

and we obtain

$$R = \sum \gamma'_{ijk} \varphi_i f_{jk} + \sum_{i=4}^{l} \mu_i G_{i3} + \left(\sum_{i=3}^{l} \mu_i\right) \alpha_3 \varphi_3^2 + w'$$

where $w' \in H^0(K - D' - \mathfrak{a}) \cdot H^0(K)$. It follows that $\sum \mu_i = 0$, and therefore w' is a relation which, by Lemma (2.3), can be expressed in terms of the g_j's. This proves our assertion for I_3.

Consider now the elements

$$(5) \quad \left\{ \begin{array}{l} \varphi_3^\rho, \ldots, \varphi_l^\rho \\ \varphi_1 \varphi_3^{\rho-1}, \ldots, \varphi_1 \varphi_l^{\rho-1} \\ \cdots \\ \varphi_1^{\rho-2} \varphi_3^2, \ldots, \varphi_1^{\rho-2} \varphi_l^2 \\ (4) \quad \left\{ \begin{array}{ll} \varphi_1^{\rho-3} \eta, \varphi_1^{\rho-4} \varphi_2 \eta, \ldots, \varphi_2^{\rho-3} \eta & \\ \varphi_1^{\rho-1} \varphi_i, \varphi_1^{\rho-2} \varphi_2 \varphi_i, \ldots, \varphi_2^{\rho-1} \varphi_i, & i = 3, \ldots, p \\ \varphi_1^\rho, \varphi_1^{\rho-1} \varphi_2, \ldots, \varphi_2^\rho. & \end{array} \right. \end{array} \right.$$

(2.13) **Lemma.** *For each $\rho \geqq 4$, the elements* (5), *viewed in* $H^0(\rho K - (\rho - 1) D')$, *generate* $H^0(\rho K - (\rho - 1) D')$.

This lemma is a straightforward generalization of Lemma (2.8). We sketch its proof. One first shows that the elements $\varphi_1^j \varphi_3^{\rho-j}, \ldots, \varphi_1^j \varphi_l^{\rho-j}$, $j \leqq \rho - 2$, viewed in

$H^0(\rho K-(\rho-1)D')$, are linearly independent modulo $H^0(\rho K-(\rho-1)D'-(j+1)\mathfrak{a})$. Then, by induction, as in the proof of Lemma (1.1), one shows that the elements (4), viewed in $H^0(\rho K-(\rho-1)D')$, generate $H^0(\rho K-(\rho-1)(D'+\mathfrak{a}))$. The final step consists in counting dimensions.

We are now going to prove Theorem (2.12) for I_4. Let us first suppose $l \geqq 4$. We must show that I_4 is generated by the f_{ik}'s, the g_j's and the G_{kr}'s.

Let us consider a relation in I_4:

$$R=\sum_{\substack{1\leqq i\leqq p\\ 1\leqq j,k,r\leqq l}} \gamma_{ijkr}\varphi_i\varphi_j\varphi_k\varphi_r \underset{C}{=} 0.$$

We make the following substitutions.

(a) if $i\neq j$, $i,j\geqq 3$, in every term $\varphi_i\varphi_j\varphi_k\varphi_r$, we set

$$\varphi_i\varphi_j=f_{ij}+\sum_{s=3}^{p} a_{sij}\varphi_s+b_{ij}\varphi_1\varphi_2.$$

Then if $k\geqq 3$ and $s\neq k$, we set

$$a_{sij}\varphi_s\varphi_k\varphi_r=a_{sij}\left(f_{sk}+\sum_{t=3}^{p} a_{tsk}\varphi_t+b_{sk}\varphi_1\varphi_2\right)\varphi_r.$$

We can continue till all the terms containing $\varphi_h\varphi_k$, $h\neq k$, $h,k\geqq 3$ will disappear.

(b) In every term of type $\mu_i\varphi_1\varphi_2\varphi_i^2$, $v_i\varphi_2^2\varphi_i^2$, $\pi_i\varphi_2\varphi_i^3$, $i=3,\ldots,l$, we set $\varphi_2=\lambda_{i2}^{-1}(\alpha_i-\lambda_{i1}\varphi_1)$.

We then get:

$$(2.14)\quad R=\sum_{i=3}^{l}\gamma_{iiii}\varphi_i^4+\sum P_{kj}f_{kj}+\sum_{i=3}^{l}\delta_i\varphi_1\varphi_i^3+\sum_{i=3}^{l}\delta_i'\varphi_1^2\varphi_i^2$$
$$+\sum\varepsilon_i\alpha_i\varphi_i^3+\sum\varepsilon_i'\varphi_1\alpha_i\varphi_i^2+\sum\varepsilon_i''\alpha_i^2\varphi_i^2+w,$$

where P_{kj} is a form of degree 2 in $\varphi_1,\ldots,\varphi_l$ and $w\in S^3H^0(K-D'-\mathfrak{a})\cdot H^0(K)$. The properties of the elements (5), for $\rho=4$, give $\gamma_{iiii}=\delta_i=0$, $i=3,\ldots,l$.

(c) When $i=4,\ldots,l$, we substitute the terms of the form $\varepsilon_i\alpha_i\varphi_i^3$ with $\varepsilon_i\varphi_i(G_{i3}+\alpha_3\varphi_3^2-\theta_3+\theta_i)$, we then use (a) and (b), if necessary, and we get

$$R=\sum P_{kj}'f_{kj}+\sum L_{i3}G_{i3}+\sum(\beta_i\varphi_1+\beta_i'\alpha_i)\alpha_i\varphi_i^2+w'+\sum_{i=3}^{l}\delta_i''\varphi_1^2\varphi_i^2$$

where the L_{i3}'s are linear in $\varphi_1,\ldots,\varphi_l$ and w' is as before. The properties of the elements (5), for $\rho=4$, give $\delta_i''=0$. Substituting $\alpha_i\varphi_i^2$ with $G_{i3}+\alpha_3\varphi_3^2+\theta_i-\theta_3$ we get

$$R=\sum P_{kj}'f_{kj}+\sum L_{i3}'G_{i3}+\sum(\beta_i\varphi_1+\beta_i'\alpha_i)\alpha_3\varphi_3^2+w''.$$

Clearly $\sum(\beta_i\varphi_1+\beta_i'\alpha_i)=0$, and, according to Lemma (2.3), w'' can be expressed in terms of the g_j's.

Let us now consider the case $l=3$. Proceding as before we reduce R to the form (2.14) with $\gamma_{iiii}=0$. We then have a relation

$$(2.15)\quad P^{(1)}\varphi_3^3+P^{(2)}\varphi_3^2+\sum_{i=1}^{p} P_i^{(3)}\varphi_i$$

where $P^{(1)}$, $P^{(2)}$, $P_i^{(3)}$ are forms in φ_1, φ_2 of degree 1, 2, 3 respectively.

We now observe that there exist $3i(D)-2$ linearly independent relations of the form (2.15). In fact the $4p+3$ elements φ_3^4, $\varphi_1\varphi_3^3$, $\varphi_2\varphi_3^3$, $\varphi_1^2\varphi_3^2$, $\varphi_2^2\varphi_3^2$, $\varphi_1\varphi_2\varphi_3^2$, $\varphi_1^3\varphi_i$, $\varphi_1^2\varphi_2\varphi_i$, $\varphi_1\varphi_2^2\varphi_i$, $\varphi_2^3\varphi_i$, $i=1,\ldots,p$ generate $H^0(4K-3D')$ which is of dimension $p+3n-1$. Therefore among them there are $3i(D)-2$ linearly independent relations. But, by Lemma (2.3) there are only $3i(D)-3$ linearly independent relations of type (2.15) having $P^{(1)}=P^{(2)}=0$. This means that in order to generate I_4, we must add to the g_j's and the f_{ik}'s *at most one* additional relation.

$$F=F^{(1)}\varphi_3^3+F^{(2)}\varphi_3^2+\sum_{i=1}^{p} F_i^{(3)}\varphi_i.$$

This proves the theorem for I_4.

The general case ($\rho>4$) can be treated exactly with the same methods used for the case $\rho=4$. Any further detail would be a useless repetition.

§ 3. The Exceptional Cases

We keep the assumptions and the notation of the preceding sections and we let Γ denote the canonical image of C. The main result of this section is the following.

(3.1) **Theorem.** *If $l\geqq 4$ the "semicanonical ideal" $I_*^{K,D}$ is generated, over $\bigoplus_{\rho\geqq 0} S^\rho H^0(K-D')$, by $I_2^{K,D}$ with two possible exceptions: when $\pi_D(C)\subset\mathbb{P}^{l-1}$ lies on a ruled surface of degree $l-2$, or else when $l=6$ and $\pi_D(C)$ lies in the Veronese surface of $\mathbb{P}^5$. If $l=3$ $I_*^{K,D}$ is generated by $I_2^{K,D}$ and $I_4^{K,D}$.*

The assertion relative to the case $l=3$ is a restatement of Theorem (2.12). Let $l\geqq 4$. We shall prove the theorem by showing that, if C is not exceptional, then $I_*^{K,D}$ is generated by the f_{ik}'s and the g_j's.

Let us write the relations f_{ik}'s in the form

$$(3.2)\quad \sum_{\substack{s=3\\ s\neq k}}^{p} (\delta_{is}\varphi_k-a_{sik})\,\varphi_s=b_{ik}\varphi_1\varphi_2+a_{kik}\varphi_k,$$

where δ_{is} is the Kronecker symbol and $i=3,\ldots,p$, $k=3,\ldots,l$, $i\neq k$.

We fix a particular k, $3\leqq k\leqq l$. Let $\Delta^{(k)}$ be the determinant of the $(p-3)\times(p-3)$ matrix

$$(\delta_{is}\varphi_k-a_{sik}),\qquad 3\leqq i\leqq p,\ 3\leqq s\leqq p,\ i\neq k,\ s\neq k.$$

Since $\Delta^{(k)}(P_k)\neq 0$, the hypersurface $\Delta^{(k)}=0$ in $\mathbb{P}^{p-1}$ does not contain the canonical image Γ of C. Let V_k be the algebraic variety defined by (3.2). By using

Cramer's rule for the system (3.2), it follows that *there exists a unique irreducible rational component* F_k of V_k containing Γ. By construction F_k is birationally equivalent to the plane having homogeneous coordinates $\varphi_1, \varphi_2, \varphi_k$.

(3.3) **Lemma.** *The* g_j*'s vanish on* F_k. *for every* k, $3 \leqq k \leqq l$.

Fix k. We can write $g_j = \varphi_1 C_j + \varphi_2 D_j$, $j = 1, \ldots, i(D)-1$, $f_{ik} = \varphi_i \varphi_k - \varphi_1 A_i - \varphi_2 B_i$, where C_j, D_j, A_i, B_i are linear forms in $\varphi_1, \ldots, \varphi_p$. Clearly, for each j, we can write

$$\varphi_k g_j = \sum A_{ji} f_{ik} + \varphi_1^2 \Phi_j + \varphi_1 \varphi_2 \Psi_j + \varphi_2^2 \chi_j,$$

where A_{ji} are linear in φ_1, φ_2 and Φ_j, Ψ_j, χ_j are linear in $\varphi_1, \ldots, \varphi_p$. From Lemma (2.3) it then follows that

$$\varphi_1^2 \Phi_j + \varphi_1 \varphi_2 \Psi_j + \varphi_2^2 \chi_j = \sum_{s=1}^{i(D)-1} B_{js} g_s$$

where the B_{js}'s are linear forms in φ_1 and φ_2. We can therefore write

$$\sum_s (\delta_{js} \varphi_k - B_{js}) g_s = \sum_{\substack{i=3 \\ i \neq k}}^{p} A_{ji} f_{ik}.$$

By using Cramer's rule we get

$$(3.4) \quad g_j H = \sum_{\substack{i=3 \\ i \neq k}}^{p} H_{ji} f_{ik}$$

where the H_{ij}'s are forms and $H = \det(\delta_{js} \varphi_k - B_{js})$. Since H does not vanish at P_k, H does not contain F_k. But the f_{ik}'s do. Thus the g_j's vanish on F_k.

Exactly in the same manner as in [S] (except that for the presence of the g_j's, for which one uses the preceding Lemma) one can now prove the following results.

(3.5) **Proposition.** *Let* $l \geqq 4$.

A) *Let* i, k, r *be distinct indices such that,* $3 \leqq i \leqq p$, $3 \leqq k, r \leqq l$. *Then there exist linear forms in* φ_1 *and* φ_2 β_{kirj}, $j = 1, \ldots, i(D)-1$ *such that*

$$f_{ik} \varphi_r - f_{ir} \varphi_k = \sum_{s=3}^{p} (a_{sir} f_{sk} - a_{sik} f_{sr}) + \rho_{kir} G_{kr} + \sum_{j=1}^{i(D)-1} \beta_{kirj} g_j.$$

where $f_{kk} = f_{ll} = 0$ *and* $a_{kir} = \rho_{kir} \alpha_k$, k, i, r *distinct. Moreover* $\rho_{kir} = \rho_{rik}$.

B) *Fix* k *and* r, $k \neq r$, $3 \leqq k, r \leqq l$. *Then* $\rho_{kir} = 0$ *for every* $i = 3, \ldots, p$, $i \neq k, r$, *if and only if* $F_k = F_r$.

C) *If there exists one* ρ_{kir} *different from zero then* $I_* = \operatorname{Ker} u_*$ *is generated by the quadratic relations* g_j*'s and* f_{ik}*'s.*

In order to complete the proof of Theorem (3.1) we must interpret geometrically the vanishing of the ρ_{kir}'s. If all the ρ_{kir}'s vanish, then, in particular, we have

$$f_{ik} = \varphi_i \varphi_k - a_{kik} \varphi_k - a_{iik} \varphi_i - b_{ik} \varphi_1 \varphi_2.$$

For $3 \leqq i$, $k \leqq l$, $i < k$, the equations $f_{ik} = 0$ represent $\frac{(l-2)(l-3)}{2}$ linearly independent quadrics in $\mathbb{P}^{l-1}$ *containing a surface* F, which in turn contains $\pi_D(C)$. Keeping this in mind the theorem can now be proved by repeating, word by word, the argument given in [S], pag. 173, where the "g" should be replaced by a "l".

§ 4. The Ideal of a Special Curve

We keep the assumptions of Sections 2 and 3 and we make the following change in notation. We fix a basis

$$x_1, \ldots, x_l, \quad l = l(D)$$

of $H^0(K-D')$, and a basis

$$y_1, \ldots, y_{l'}, \quad l' = i(D)$$

of $H^0(D')$. The elements of $H^0(D')$ will be thought of as rational functions f such that $(f) + D' \geqq 0$. We assume that $y_1 = 1$.

We consider the natural homomorphism

$$v \colon H^0(K-D') \otimes H^0(D') \to H^0(K)$$
$$x_i \otimes y_j \rightsquigarrow x_i y_j.$$

Let $\tau = \operatorname{codim}(\operatorname{Im} v)$. If $\tau > 0$ we complete the set $\{x_i y_j\}$ to a system of generators of $H^0(K)$ by adding to it τ regular 1-forms

$$z_1, \ldots, z_\tau \in H^0(K)$$

which are linearly independent modulo $\operatorname{Im} v$.

Letting V be the subspace of $H^0(K)$ spanned by $z_1, \ldots, z_\tau$ we have a natural surjective map, which we also call v,

$$v \colon H^0(K-D') \otimes H^0(D') \oplus V \to H^0(K).$$

We now consider indeterminates $X_1, \ldots, X_l$, $Y_1, \ldots, Y_{l'}$, $Z_1, \ldots, Z_\tau$ and we identify $\mathbb{K}[X] = \mathbb{K}[X_1, \ldots, X_l]$ with $\bigoplus_{\rho \geqq 0} S^\rho H^0(K-D')$ in the obvious way. We also define a graded $\mathbb{K}[X]$-module $M = \bigoplus_{\rho \geqq 1} M_\rho$ by setting

$$M_\rho = \left\{ \sum_{j=1}^{l'} P_j^{(\rho)}(X) Y_j + \sum_{\nu=1}^{\tau} P_\nu^{(\rho-1)}(X) Z_\nu \right\}$$

where, as usual, $P_j^{(\rho)}(X)$, and $P_\nu^{(\rho-1)}(X)$ denote homogeneous polynomials of degree ρ and $\rho - 1$ respectively in $X_1, \ldots, X_l$.

We have a natural degree zero $\mathbb{K}[X]$-homomorphism

$$\Phi\colon M \to \bigoplus_{\rho \geqq 1} H^0(\rho K)$$
$$P(X, Y, Z) \rightsquigarrow P(x, y, z).$$

Let Λ be the degree one part of $\operatorname{Ker}\Phi$; Λ consists of all polynomials of the form

$$\sum \alpha_{ij} X_i Y_j, \quad \alpha_{ij} \in \mathbb{K}$$

such that $\sum \alpha_{ij} x_i y_j = 0$. We denote by (Λ) the graded $\mathbb{K}[X]$-submodule of M generated by Λ. Consider the following commutative diagram of degree zero maps of graded $\mathbb{K}[X]$-modules:

$$\begin{tikzcd}
 & M \arrow[dl] \arrow[dr, "\Phi"] & \\
M/(\Lambda) \arrow[rr, "\bar{\Phi}"'] & & \bigoplus_{\rho \geqq 1} H^0(\rho K) \\
 & \bigoplus_{\rho \geqq 1} S^{\rho-1} H^0(K-D') \cdot H^0(K) \arrow[ul, "\sigma"] \arrow[ur, "u_*"'] &
\end{tikzcd}$$

where $\bar{\Phi}$ is the obvious map and σ is defined as follows. Being v surjective each element of $S^{\rho-1} H^0(K-D') \cdot H^0(K)$ can be expressed in this way

$$P^{(\rho)}(x) + \sum_{j \geqq 2} P_{ij}^{(\rho-1)}(x) x_i y_i + \sum P_v^{(\rho-1)}(x) z_v$$

then we set

$$\sigma\left(P^{(\rho)}(x) + \sum_{j \geqq 2} P_{ij}^{(\rho-1)}(x) x_i y_j + \sum P_v^{(\rho-1)}(x) z_v\right)$$
$$= P^{(\rho)}(X) Y_1 + \sum_{j \geqq 2} P_{ij}^{(\rho-1)}(X) X_i Y_j + \sum P_v^{(\rho-1)}(X) Z_v.$$

Let us briefly say that C is *exceptional* if $\pi_D(C)$ lies on a ruled surface of degree $l-2$ in $\mathbb{P}^{l-1}$ $(l \geqq 4)$ or else if $l=6$ and $\pi_D(C)$ lies on the Veronese surface of $\mathbb{P}^5$.

(4.1) **Proposition.** *If* $l \geqq 4$ $\operatorname{Ker}\Phi$ *is generated, as a* $\mathbb{K}[X]$*-module by its elements of degree* 1, 2, 3. *If* C *is not exceptional* $\operatorname{Ker}\Phi$ *is generated by its elements of degree* 1 *and* 2. *If* $l=3$ $\operatorname{Ker}\Phi$ *is generated by its elements of degree* 1, 2 *and* 4.

This follows immediately from Theorems (2.12) and (3.1) by chasing the above diagram.

We shall express the fact that $P(X, Y, Z) \in \operatorname{Ker}\Phi$ by writing $P(X, Y, Z) \underset{C}{=} 0$ or simply $P(x, y, z) = 0$.

By the preceding proposition we can say that, when $l \geqq 4$, $\operatorname{Ker}\Phi$ is generated by relations of the following types:

(1) $\sum \alpha_{ij} X_i Y_j \underset{C}{=} 0,$

(2) $\sum_{j=1}^{l'} P_j^{(2)}(X) Y_j + \sum_{v=1}^{\tau} P_v^{(1)}(X) Z_v \underset{C}{=} 0,$

(3) $\sum_{j=1}^{l'} P_j^{(3)}(X) Y_j + \sum_{v=1}^{\tau} P_v^{(2)}(X) Z_v \underset{C}{=} 0.$

If C is not exceptional the relations of type (3) are not needed. When $l=3$ one must add to the relations (1) and (2) at most one fourth degree relation of the type

(4) $\sum_{j=1}^{l'} P_j^{(4)}(X) Y_j + \sum_{v=1}^{\tau} P_v^{(3)}(X) Z_v \underset{C}{=} 0.$

If we consider the injection $\mathbb{K}[X] \overset{i}{\hookrightarrow} M$, given by $i(P(X)) = P(X) Y_1$ it follows, from the definition of Φ, that $\operatorname{Ker}(\Phi \circ i)$ is the ideal I_*^D of the curve $\pi_D(C) \subset \mathbb{P}^{l-1}$. As a consequence of Proposition (4.1) we may therefore conclude that:

(4.2) **Theorem.** *A system of generators for I_*^D can be obtained, from the relations of type* (1), (2), (3) *and* (4) *by eliminating the Y's and the Z's. When $l \geqq 4$ and C is not exceptional it suffices to use relations of type* (1) *and* (2).

We now apply this result to the so called "*Klein's canonical curves*".

The curve C is a *Klein's canonical curve of type d* if there exists a positive integer d such that

$$|dD| = |K|.$$

For example a non-singular curve in $\mathbb{P}^r$ which is the complete intersection of $r-1$ hypersurfaces of degree $n_1, \ldots, n_{r-1}$, respectively, is a Klein's canonical curve of type $d = n_1 + \cdots + n_{r-1} - r - 1$.

If C is a Klein's canonical curve of type d, for each $\rho \geqq 1$ we set $l_\rho = l(\rho D)$, $l_1 = l$, and we denote by π_ρ: $C \to \mathbb{P}^{l_\rho - 1}$ the rational map induced by $|\rho D|$, $(\pi_1 = \pi_D)$.

(4.3) **Theorem.** *Let C be a Klein's canonical curve of type d. Assume that for each $\rho > 0$, the natural map $S^\rho H^0(D) \to H^0(\rho D)$ is surjective. Then if $l \geqq 4$, the ideal I_*^D of $\pi_D(C)$ in $\mathbb{P}^{l-1}$ is generated by forms of degree $\leqq d+2$. If C is not exceptional, I_*^D is generated by forms of degree $\leqq d+1$. If $l=3$ $\pi_D(C)$ is a plane non-singular curve of degree $n = d+3$.*

First observe that for $\rho \gg 0$, π_ρ: $C \to \pi_\rho(C) \subset \mathbb{P}^{l_\rho - 1}$ is an embedding. On the other hand being $S^\rho H^0(D) \to H^0(\rho D)$ surjective we can write $\pi_\rho = \lambda \circ \pi_D$ where λ is a Veronese map. Therefore π_D is a birational isomorphism of C onto $\pi_D(C)$.

The case $l=3$ is trivial. Suppose $l \geqq 4$. Let us use the notation introduced in the beginning of this section. Set $D' = (d-1)D \sim K - D$. Consider the basis $x_1, \ldots, x_l$ of $H^0(K-D')$. We may assume that $(x_1) = dD = D' + D$. We then have a isomorphism

$$H^0((d-1)(K-D')) \to H^0(D')$$

$$\Psi \rightsquigarrow \frac{\Psi}{x_1^{(d-1)}}.$$

By hypothesis we have surjective maps

(a) $S^{d-1}H^0(D) \cong S^{d-1}H^0(K-D') \to H^0((d-1)(K-D')) \cong H^0(D')$,

(b) $S^d H^0(D) \to H^0(K)$.

From (a) it follows that there exist l' forms, of degree $d-1$, $f_1(X), \ldots, f_{l'}(X)$ such that

(4.4) $x_1^{d-1} y_j = f_j(x_1, \ldots, x_l), \quad j=1, \ldots, l'$.

From (b) it follows that $H^0(K-D') \otimes H^0(D') \to H^0(K)$ is surjective so that $\tau = 0$.

Recalling (1), (2), (3) we get, in I_*^D, elements of the following types

(1)′ $\sum \alpha_{ij} X_i f_j(X)$,

(2)′ $\sum P_j^{(2)}(X) f_j(X)$,

(3)′ $\sum P_j^{(3)}(X) f_j(X)$.

These are forms of degree d, $d+1$, $d+2$ respectively. Let $F(X) \in I_\rho^D$, $\rho \geqq d$. With an obvious multiindex notation we write

$$F(X) = \sum_{|J|=d-1} F_J(X) X^J.$$

For every multiindex J we have, by virtue of (a),

(4.5) $x^J = x_1^{d-1} L_J(y)$

where L_J is a linear form in $y_1, \ldots, y_{l'}$. Let

$$G(X, Y) = \sum_{|J|=d-1} F_J(X) L_J(Y).$$

Clearly $G(X, Y) \in \operatorname{Ker} \Phi$. Setting $f(X) = (f_1(X), \ldots, f_{l'}(X))$, it then follows from Proposition (4.1), that $G(X, f(X))$ can be expressed as a combination of forms of types (1)′, (2)′ and (3)′ (the forms of type (3)′ are not necessary when C is not exceptional). On the other hand

$$F(X) - G(X, f(X)) = \sum_{|J|=d-1} F_J(X)(X^J - L_J(f(X))).$$

From (4.4) and (4.5) it follows that $X^J - L_J(f(X))$ is an element of degree $d-1$ in I_*^D, proving the assertion.

(4.6) *Remark*. In the preceding theorem the hypothesis that $\pi_D(C)$ is projectively normal is certainly redundant; we only used the surjectivity of the maps (a) and (b). On the other hand the surjectivity of the latter is necessary as the following example shows.

Let Q be a non-singular quadric in $\mathbb{P}^3$, let l and m be two non-equivalent lines on Q. Let Γ be an irreducible curve in $|4l+5m|$ having, as its only singularities, two ordinary double points on the line m. The normalization C of Γ, together with the pull-back D of a plane section is a Klein's canonical curve of type $d=2$ for which the map (b) is not surjective. On the other hand the ideal of $\Gamma=\pi_D(C)$ can *not* be generated by forms of degree less or equal than $d+2$.

(4.7) **Theorem.** *Let C be a Klein's canonical curve of type $d\geqq 2$ and genus p. Assume $\pi_D(C)\subset\mathbb{P}^{l-1}$, $l\geqq 4$. Then*

$$(*)\quad p\geqq 1+(l-1)d+(l-2)\binom{d}{2}.$$

Equivalently, for the degree $n=(2p-2)/d$ of $\pi_D(C)$ we have

$$(**)\quad n\geqq 2(l-1)+(l-2)(d-1).$$

If equalities hold C is a "Castelnuovo's curve" (i.e. a curve in $\mathbb{P}^{l-1}$ of maximal genus with respect to its degree); in particular C is exceptional. On the other hand if C is exceptional and projectively normal then C is a Castelnuovo's curve.

In case $\pi_D(C)\subset\mathbb{P}^2$ then $n\geqq d+3$ and the equality holds if and only if C is non-singular.

The statement about plane curves is obvious. So let $l\geqq 4$. By Castelnuovo's bound (see [H]), if h is the lowest integer such that hD is non-special, then $h\leqq(n-2)/(l-2)$, so that

$$(n-2)/(l-2)\geqq d+1.$$

This inequality is equivalent to $(*)$ and $(**)$. Suppose we have equality in $(*)$; then one easily sees that

$$p=\sum_{\rho\geqq 0}\max(0,n-(\rho+1)(l-1)+\rho).$$

Hence $\pi_D(C)$ is a Castelnuovo's curve (see [G-H]); since $d\geqq 2$ $\pi_D(C)$ is exceptional (see [C]).

On the other hand assume C is exceptional. Then $\pi_D(C)$ is contained in a surface F of degree $l-2$ which is either a non-singular ruled surface, or a cone over a rational normal curve in $\mathbb{P}^{l-2}$, or else the Veronese surface in $\mathbb{P}^5$. In all cases the Hilbert polynomial of F is

$$\chi(v)=1+(l-1)v+(l-2)\binom{v}{2}.$$

Since C is projectively normal we have a *surjective* map

$$H^0(F,O_F(d))\to H^0(C,K_C)$$

so that

$$p\leqq\chi(d)=1+(l-1)d+(l-2)\binom{d}{2}.$$

Therefore, be the first part, we get equality, proving that C is a Castelnuovo's curve.

We end this section by making some parenthetical remarks about an old conjecture of Brill and M. Noëther.

We first remind the geometrical significance of the number

$$\tau=\operatorname{codim}\operatorname{Im}(v\colon H^0(K-D')\otimes H^0(D')\to H^0(K)).$$

For this we let $V_{l,n}=\{\tilde{D}\in C^{(n)}\colon l(\tilde{D})\geqq l\}$. It is then well known that

$$\dim(\text{Zariski tang. space to } V_{l,n} \text{ at } D)=\tau+l-1.$$

Let now $\lambda_1=\#$ (linearly independent relations of type (1)) we have

$$\lambda_1=l i-p+\tau.$$

The conjecture is that if $\lambda_1>\tau$ then C is *special in the sense of moduli.* (Classically the conjecture was stated by saying that C is special in the sense of moduli if there exists, on C, a g_n^r with $(r+1)(n-r)-rp<0$). Petri in [P-2] states, as a fact, that C is special in the sense of moduli if $\lambda_1>0$. This too is a natural conjecture.

One natural way of looking at this problem is to consider the λ_1 relations as a basis of $H^0(\Omega_C^1\otimes\Omega_{\mathbb{P}^{l-1}}^1|_C)$ and see if the natural map

$$H^0(\Omega_C^1\otimes\Omega_{\mathbb{P}^{l-1}}^1|_C)\to H^0(2K_C)$$

is not zero.

§ 5. The Ideal of a Certain Class of Projective Varieties

In what follows $V\subset\mathbb{P}^N$, $N\geqq 2$, will denote a normal projective (irreducible) algebraic variety and $L=O_{\mathbb{P}^N}(1)\otimes O_V$ the hyperplane bundle. We assume that $N+1=h^0(V,L)$.

As is well known V is said to be *arithmetically Cohen-Macaulay* if the following conditions hold

(a) $H^q(V,L^\rho)=0$, $\rho\in\mathbb{Z}$, $1\leqq q\leqq\dim V-1$,

(b) for every $\rho\geqq 0$, the map $H^0(\mathbb{P}^N,O(\rho))\cong S^\rho H^0(V,L)\to H^0(V,L^\rho)$ is surjective.

When only condition (b) holds then V is said to be *projectively normal.*

The following lemmas are standard. They follow at once by examining the cohomology of the following exact commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & I_V(\rho-1) \arrow[r] \arrow[d] & I_V(\rho) \arrow[r] \arrow[d] & I_{V\cap H}(\rho) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & O_{\mathbb{P}^N}(\rho-1) \arrow[r] \arrow[d] & O_{\mathbb{P}^N}(\rho) \arrow[r] \arrow[d] & O_H(\rho) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & L^{\rho-1} \arrow[r] \arrow[d] & L^\rho \arrow[r] \arrow[d] & L^\rho_{V\cap H} \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

(5.1) **Lemma.** *Let V be projectively normal. Let H be a hyperplane in* $\mathbb{P}^N$. *Suppose that the graded ideal $I_*(V \cap H)$ of $V \cap H$ is generated by its homogeneous components of degree $\leqq v$. Then the ideal $I_*(V)$ is also generated by its homogeneous components of degree $\leqq v$.*

(5.2) **Lemma.** *Let V be arithmetically Cohen-Macaulay,* $\dim V \geqq 2$. *Let H be a hyperplane in $\mathbb{P}^N$, then $V \cap H$ is arithmetically Cohen-Macaulay.*

We now prove the following generalized version of the Max Noëther-Enriques-Petri theorem.

(5.3) **Theorem.** *Let $V \subset \mathbb{P}^N$ be a non-singular, irreducible, arithmetically Cohen Macaulay, variety of dimension d. Let ω_V be the canonical bundle on V. Let L be the hyperplane bundle. Assume that there exists an integer e such that $L^e \cong \omega_V$. Then the ideal $I_*(V)$ of V in $\bigoplus S^\rho H^0(V, L)$ is generated by elements of degree $\leqq d+e$, with the exception of the following cases*

I) V is a hypersurface of degree $n \geqq 2$, so that $n = e+d+2$,

II) V is a hypersurface on an irreducible variety $W \subset \mathbb{P}^N$ which has the minimal degree: $N-d=\operatorname{codim} W+1$.

When $e+d \geqq 3$ this is the same as saying that the intersection of V with a generic $\mathbb{P}^{N-d+1}$ is a Castelnuovo's curve.

In this case $I_*(V)$ is generated by elements of degree $\leqq e+d+1$.

III) $e+d=1$ in which case $I_*(V)$ is generated by quadrics, (unless we are in case I) with $n=3$.

Let $\mathbb{P}$ be a generic $(N-d+1)$-dimensional linear subspace of $\mathbb{P}^N$. Consider the non-singular curve $C = V \cap \mathbb{P}$. By adjunction

$$\omega_C = (L|_C)^{e+d-1}.$$

If $N-d+1=2$ we are in case *I*). Let's assume $N-d+1 \geqq 3$. After a repeated application of Lemma (5.2) it follows that C is arithmetically Cohen Macaulay. Suppose $e+d-1 \geqq 1$. Then C is a projectively normal Klein's canonical curve of type $e+d-1$ in $\mathbb{P}$. We now apply Theorem (4.3). If C is not exceptional the ideal $I(C)$ in $\bigoplus_{\rho \geqq 0} S^\rho H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(1))$ is generated by elements of degree $\leqq d+e$ and the theorem follows by a repeated application of Lemmas (5.1) and (5.2). If C is exceptional it is contained in a surface F of minimal degree $N-d$ in $\mathbb{P}$, which is the intersection of $h = \binom{N-d}{2}$ linearly independent quadrics $Q_1, \ldots, Q_h$ in $\mathbb{P}$. Chasing the diagram, in the beginning of the paragraph, one finds h linearly independent quadrics $\tilde{Q}_1, \ldots, \tilde{Q}_h$ in $\mathbb{P}^N$ containing V and such that $\tilde{Q}_i \cap \mathbb{P} = Q_i$. These quadrics define a variety W of dimension $d+1$ and degree $N-d$, containing V, and so we are in case II). Of course Theorem (4.7) gives the equivalent version of case II). From Theorem (4.3) and Lemmas (5.1) and (5.2) it follows that, in this case, the ideal $I_*(V)$ is generated by elements of degree $\leqq d+e+1$.

Let now $e+d-1=0$, so that C is an elliptic curve. Since we are assuming that C is not a plane cubic, it follows from [M-2] that we are in case III).

Finally it is easy to see that the case $e+d \leqq 0$ can occur only if V is a quadric hypersurface, and we are in case I).

Remarks. The above theorem gives that the ideal of a regular projectively normal canonical surface, is generated by quadrics and cubics; the only exceptions being the quintic surface in $\mathbb{P}^3$ and the canonical surfaces for which c_1^2 is minimal (i.e. $c_1^2 = 3p_g - 7$).

Examples for which $e+d-1=1$ are canonical curves, $K=3$ surfaces, Fano threefolds and fourfolds.

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