Clarification of the proof of the main theorem of Bruno-Sernesi: A Note on the Petri Loci

Denote by G the union of the components of $\mathcal{W}^r_n(\mathcal{C}/B)$ that surject onto B (by known results there is only one such component, but we do not need this fact). The proof given in the paper, as it stands, shows that $q(\widetilde{P}^r_{g,n} \cap G)$ has pure codimension one in B because $\widetilde{P}^r_{g,n} \cap G$ is a degeneracy locus and we can apply Steffen's theorem to each component of G.

What we still have to show is that, if there is a component Z of $\mathcal{W}^r_n(\mathcal{C}/B)$ such that $q(Z)$ has codimension ≥ 2 then $q(Z) \subset q(\widetilde{P}^r_{g,n} \cap G)$. For this purpose it will suffice to show that for every $(C, L) \in Z$ there is $M \in W^r_n(C)$ such that $(C, M) \in \widetilde{P}^r_{g,n} \cap G$. Note that for each $(C, L) \in Z$ the Petri map is not injective, and therefore $Z \subset \widetilde{P}^r_{g,n}$.

Assume first that $\rho(g, r, n) > 0$. Then $\mathcal{W}^r_n(\mathcal{C}/B)$ is connected because $W^r_n(\mathcal{C}(b))$ is connected for each $b \in B$. Therefore $Z \cap G \neq \emptyset$. Let W be an irreducible component of $W^r_n(C)$ such that $W \cap G \neq \emptyset$. If $\dim(W) > \rho(g, r, n)$ then all the points of W are in $\widetilde{P}^r_{g,n}$, in particular the points of $W \cap G$ are in $\widetilde{P}^r_{g,n} \cap G$, and we are done. If $\dim(W) = \rho(g, r, n)$ then $W \subset G$. In this case we have two possibilities. The first possibility is that $W = W^r_n(C)$ and then $Z \cap G \neq \emptyset$ and we are done. The second possibility is that W meets other components of $W^r_n(C)$. In this case the intersection points of W with $\overline{W^r_n(C) \setminus W}$ are in $\widetilde{P}^r_{g,n} \cap G$, because they are singular points of $W^r_n(C)$ and therefore the Petri map cannot be injective there, and we are done again. This takes care of the case $\rho(g, r, n) > 0$.

Assume now that $\rho(g, r, n) = 0$. Then G is a generically finite cover of B of degree given by Castelnuovo's number $c(g, r, n)$ (see [1], p. 211 for its expression). Let $(C, L) \in Z$. We may assume that $(C, L) \notin G$, because otherwise we conclude as before. Then L belongs to a positive dimensional component W of $W^r_n(C)$, all whose members are therefore in $\widetilde{P}^r_{g,n}$. If $W \cap G \neq \emptyset$ we are done. If instead $W \cap G = \emptyset$ then we need to exclude that $G \cap W^r_n(C)$ consists of $c(g, r, n)$ distinct points all having injective (isomorphic) Petri map. But in this case, since $W^r_n(C) \subset J(C)$ is the vanishing scheme of a section σ of an ample vector bundle V of rank g such that $c_g(V) = c(g, r, n)$, it would follow (by applying Th. 12.2 of [2] to the intersection of $\mathrm{Im}(\sigma)$ with the zero section of V) that $W = \emptyset$, a contradiction.

References

[1] E. Arbarello, M. Cornalba, P. Griffiths, J. Harris: *Geometry of Algebraic Curves, vol. I*, Springer Grundlehren b. 267 (1985).

[2] W. Fulton: *Intersection Theory*, Springer Ergebnisse b. 2 (1984).