Some remarks on symmetric correspondences

H. Lange
Mathematisches Institut
Universitat Erlangen-Nurnberg
Bismarckstr. $1\frac{1}{2}$
D-91054 Erlangen (Germany)

E. Sernesi
Dipartimento di Matematica
Università Roma Tre
Largo S.L. Murialdo 1
00146 Roma (italy)

Introduction

A (d_1, d_2)-correspondence on a projective connected nonsingular curve C can be defined as a curve $D \subset C \times C$ of type (d_1, d_2). The i-th projection restricts to a morphism of degree d_i

$$p_i : D \to C$$

for $i = 1, 2$. These morphisms induce a correspondence in the usual sense, associating to a point $x \in C$ the divisor $p_2(p_1^{-1}(x))$ of degree d_1. On any curve C there exist plenty of correspondences, but if the curve is general they are all correspondences with valency: namely there is an integer N such that the linear equivalence class of the divisor

$$p_2(p_1^{-1}(x)) + Nx$$

is independent of x. It is more difficult to find correspondences without valency.

A (d,d)-correspondence is called symmetric if it is mapped to itself by the involution on $C \times C$ which interchanges the factors. Hence for a symmetric correspondence D, the involution on $C \times C$ induces an involution on the curve D. If we assume that the correspondence is without fixed points, i.e. it does not meet the diagonal $\Delta \subset C \times C$, the involution on D is fixed-point free. This is the specific case that we will study in this paper. The interest of such correspondences comes from their relation with the theory of Prym-Tyurin varieties. We refer the reader to [1] for details on this theory and on how it is related with correspondences.

We consider a few geometrical configurations arising from a nonsingular symmetric fixed-point free (d,d)-correspondence and we study their deformation theory. This gives informations on the deformation theory of Prym-Tyurin varieties, a subject on which very little is known.

We are able to compute several invariants and to test them on a few known examples.

1 The second symmetric product

Let C be a complex projective nonsingular connected curve of genus $g \geq 2$. Denote by C^n and C_n the n-th cartesian, resp. symmetric, product of C, for any $n > 0$. We will be interested in the case $n = 2$.

In this section we want to compute the cohomology of T_{C_2}, the tangent bundle of C_2. For this purpose consider the universal divisor

$$\Delta_2 = \{(x, E) : E - x > 0\} \subset C \times C_2$$

and let

$$\begin{array}{ccc} C \times C_2 & & \\ \downarrow p & \searrow q & \\ C & & C_2 \end{array}$$

be the projections. We have the well known formula

$$T_{C_2} = q_*[\mathcal{O}_{\Delta_2}(\Delta_2)] \tag{1}$$

(see [4]). We have an obvious isomorphism

$$\begin{array}{ccc} C \times C & \stackrel{\epsilon}{\longrightarrow} & \Delta_2 \\ (x,y) & \mapsto & (x, x+y) \end{array}$$

and we can identify the natural map $\sigma : C \times C \to C_2$ with the composition

$$C \times C \quad \stackrel{\epsilon}{\longrightarrow} \quad \Delta_2 \quad \stackrel{q}{\longrightarrow} C_2$$

Moreover the composition $p\epsilon : C \times C \to C$ gets identified with the first projection which we will keep calling p. Now, using the identity

$$\epsilon^* \mathcal{O}_{\Delta_2}(\Delta_2) = p^* \omega_C^{-1} \otimes \mathcal{O}(\Delta)$$

(see [4], lemma 1.1. Warning: Δ is the diagonal in $C \times C$ while Δ_2 is the diagonal in $C \times C_2$) we can reformulate (1) as follows:

$$T_{C_2} = \sigma_* [p^* \omega_C^{-1}(\Delta)] \tag{2}$$

Since σ is finite we have canonical identifications:

$$H^i(T_{C_2}) = H^i(p^* \omega_C^{-1}(\Delta)), \quad i = 0, 1, 2 \tag{3}$$

Lemma 1.1

$$\begin{aligned} h^0(p^* \omega_C^{-1}) &= 0 \\ h^1(p^* \omega_C^{-1}) &= 3g - 3 \\ h^2(p^* \omega_C^{-1}) &= g(3g - 3) \end{aligned}$$

and therefore

$$\chi(p^* \omega_C^{-1}) = 3(g-1)^2$$

Proof. Follows easily from the Kunneth formula or Leray spectral sequence. •

Lemma 1.2 *If $g \geq 3$ then*

$$h^0(R^1 p_* \mathcal{O}(\Delta) \otimes \omega_C^{-1}) = 0$$

and

$$h^1(R^1 p_* \mathcal{O}(\Delta) \otimes \omega_C^{-1}) = 3g^2 - 8g + 5$$

If $g = 2$ then $R^1 p_* \mathcal{O}(\Delta) \cong \omega_C$.

Proof. Assume $g \geq 3$. By pushing down by p the exact sequence

$$0 \to \mathcal{O} \to \mathcal{O}(\Delta) \to \omega_{\Delta}^{-1} \to 0$$

we obtain the exact sequence on C:

$$0 \to \mathcal{O}_C \to p_*\mathcal{O}(\Delta) \to \omega_C^{-1} \to R^1p_*\mathcal{O}_C \to R^1p_*\mathcal{O}(\Delta) \to 0 \qquad (4)$$

We have $R^1p_*\mathcal{O}_C = H^1(\mathcal{O}_C) \otimes \mathcal{O}_C$ and by rank reasons $O_C \cong p_*\mathcal{O}(\Delta)$. We are left with the exact sequence

$$0 \to \omega_C^{-1} \to H^1(\mathcal{O}_C) \otimes \mathcal{O}_C \to R^1p_*\mathcal{O}(\Delta) \to 0 \qquad (5)$$

which is a twist of the Euler sequence of $\mathbb{P}^{g-1}$ restricted to C via the canonical morphism $C \to \mathbb{P}^{g-1}$; in particular

$$R^1p_*\mathcal{O}(\Delta) \cong T_{\mathbb{P}^{g-1}|C} \otimes \omega_C^{-1}$$

Let $\mathbf{a} = x_1 + \cdots + x_{g-2}$ be a general effective divisor of degree $g-2$ on C. Then there is an exact sequence (see [3], (2.3): the proof is valid in the hyperelliptic case as well, see [6], , Lemma 1.4.1)

$$0 \to \omega_C^{-1}(\mathbf{a}) \to R^1p_*\mathcal{O}(\Delta)^\vee \to \bigoplus \mathcal{O}_C(-x_i) \to 0$$

Dualizing and twisting by ω_C^{-1} we obtain:

$$0 \to \bigoplus \omega_C^{-1}(x_i) \to R^1p_*\mathcal{O}(\Delta) \otimes \omega_C^{-1} \to \mathcal{O}(-\mathbf{a}) \to 0$$

which implies $h^0(R^1p_*\mathcal{O}(\Delta) \otimes \omega_C^{-1}) = 0$. The other estimate follows also from this sequence.

The case $g = 2$ follows from the exact sequence (5). •

The following result is well known (see [2]):

Proposition 1.3 *If $g \geq 3$ then*

$$\begin{aligned} h^0(T_{C_2}) &= 0 \\ h^1(T_{C_2}) &= 3g-3 \\ h^2(T_{C_2}) &= 3g^2-8g+5 \end{aligned}$$

and therefore

$$\chi(T_{C_2}) = 3g^2 - 11g + 8$$

If $g = 2$ then $h^0(T_{C_2}) = 0$, $h^1(T_{C_2}) = 4$ and $h^2(T_{C_2}) = 2$.

Proof. The equality $h^0(T_{C_2}) = 0$ is an immediate consequence of (3) for all $g \geq 2$.

Assume $g \geq 3$. By the Leray spectral sequence for $p : C \times C \to C$ and equation (3) we have an exact sequence

$$0 \to H^1(\omega_C^{-1} \otimes p_*\mathcal{O}(\Delta)) \to H^1(T_{C_2}) \to H^0(\omega_C^{-1} \otimes R^1p_*\mathcal{O}(\Delta)) \to 0 \qquad (6)$$

The exact sequence (4) implies that $p_*\mathcal{O}(\Delta) = \mathcal{O}_C$ so that we have

$$H^1(T_{C_2}) = H^1(\omega_C^{-1}) \cong \mathbf{C}^{3g-3}$$

because $H^0(\omega_C^{-1} \otimes R^1p_*\mathcal{O}(\Delta)) = 0$ by Lemma 1.2. Thethe Leray spectral sequence for $p : C \times C \to C$ and equation (3) again we have

$$H^2(T_{C_2}) \cong H^1(R^1p_*\mathcal{O}(\Delta) \otimes \omega_C^{-1}) \qquad (7)$$

and again the conclusion follows from Lemma 1.2.

If $g = 2$ the exact sequence (6) becomes:

$$0 \to H^1(\omega_C^{-1}) \to H^1(T_{C_2}) \to H^0(\mathcal{O}_C) \to 0$$

by Lemma 1.2. Therefore $h^1(T_{C_2}) = 4$. Similarly (7) in this case gives

$$H^2(T_{C_2}) \cong H^1(\mathcal{O}_C) \cong \mathbf{C}^2$$

•

2 The set-up

Let $D \subset C \times C$ be a symmetric irreducible and nonsingular curve of type (d, d). Assume that $D \cdot \Delta = 0$, where $\Delta \subset C \times C$ is the diagonal. In particular D is a symmetric fixed-point free (d, d)-correspondence on C.

We have the following situation:

$$\begin{array}{ccc} C \times C & \xrightarrow{\sigma} & C_2 \\ \cup & & \cup \\ D & \to & \bar{D} \\ \downarrow p & & \\ C & & \end{array} \qquad (8)$$

Here p is induced by the projection, σ is the canonical quotient map, $\bar{D} := \sigma(D)$ is irreducible and nonsingular and the lower horizontal map is an etale double cover.

Let $g(D)$ and $\bar{g}$ be the genera of D and of $\bar{D}$ respectively. By the adjunction formula on $C \times C$ we have:

$$g(D) - 1 = 2d(g-1) + \frac{D^2}{2} = 2d(g-1) + \bar{D}^2$$

On the other hand by Hurwitz formula applied to p we have

$$2(g(D) - 1) = 2d(g-1) + |R|$$

where $|R|$ is the degree of the ramification divisor R of $p : D \to C$. By comparing the two formulas we therefore get:

$$\bar{D}^2 = -g(D) + 1 + |R| \tag{9}$$

But since $g(D) - 1 = 2\bar{g} - 2$ we also have:

$$\bar{D}^2 = 2\bar{g} - 2 - 2d(g-1) = 2 - 2\bar{g} + |R| \tag{10}$$

From these expressions we get the following Riemann-Roch formula for the normal sheaf $N_{\bar{D}}$ of $\bar{D} \subset C_2$:

$$\chi(N_{\bar{D}}) = 3 - 3\bar{g} + |R| \tag{11}$$

equivalently:

$$\chi(N_{\bar{D}}) = \bar{g} - 1 - 2d(g-1) \tag{12}$$

Consider the exact sequence on $C \times C$:

$$0 \to p^*\omega_C^{-1}(\Delta - D) \to p^*\omega_C^{-1}(\Delta) \to p^*\omega_C^{-1} \otimes \mathcal{O}_D \to 0 \tag{13}$$

(recall that $D \cdot \Delta = 0$). Taking σ_* we obtain the following exact sequence on C_2:

$$0 \to \sigma_*[p^*\omega_C^{-1}(\Delta - D)] \to T_{C_2} \to \sigma_*[p^*\omega_C^{-1} \otimes \mathcal{O}_D] \to 0 \tag{14}$$

Now, since the sheaf on the right is locally free of rank two and it is supported on $\bar{D}$, the last map factors as the restriction $T_{C_2} \to T_{C_2|\bar{D}}$ composed with a surjection of locally free rank two sheaves

$$T_{C_2|\bar{D}} \to \sigma_*[p^*\omega_C^{-1} \otimes \mathcal{O}_D] \tag{15}$$

which must therefore be an isomorphism. It also follows that

$$\sigma_*[p^*\omega_C^{-1}(\Delta - D)] \cong T_{C_2}(-\bar{D})$$

Using the fact that σ is finite we can state the following:

Lemma 2.1 *For all i we have:*

$$H^i(T_{C_2}(-\bar{D})) \cong H^i(p^*\omega_C^{-1}(\Delta - D))$$

$$H^i(T_{C_2|\bar{D}}) \cong H^i(p^*\omega_C^{-1} \otimes \mathcal{O}_D)$$

and the cohomology sequence of (13) is isomorphic to the cohomology sequence of

$$0 \to T_{C_2}(-\bar{D}) \to T_{C_2} \to T_{C_2|\bar{D}} \to 0$$

We will need the following exact and commutative diagram:

$$\begin{array}{cccccccc}
 & 0 & & 0 & & & \\
 & \downarrow & & \downarrow & & & \\
0 \to & p^*\omega_C^{-1}(-D) & \to & p^*\omega_C^{-1} & \to & p^*\omega_C^{-1} \otimes \mathcal{O}_D & \to 0 \\
 & \downarrow & & \downarrow & & \| & \\
0 \to & p^*\omega_C^{-1}(\Delta - D) & \to & p^*\omega_C^{-1}(\Delta) & \to & p^*\omega_C^{-1} \otimes \mathcal{O}_D & \to 0 \\
 & \downarrow & & \downarrow & & & \\
0 \to & \omega_\Delta^{-2} & = & \omega_\Delta^{-2} & & & \\
 & \downarrow & & \downarrow & & & \\
 & 0 & & 0 & & &
\end{array} \tag{16}$$

Lemma 2.2 *If $\bar{g} \geq 2$ then*

$$h^i(p^*\omega_C^{-1}(-D)) \quad = 0, \quad i = 0, 1$$

$$h^2(p^*\omega_C^{-1}(-D)) \quad = (g-1)[3(g-1) + 4d] + \bar{D}^2$$

Proof. Set $E = p^*\omega_C(D)$. Then

$$\begin{aligned}
E \cdot D \quad &= p^*\omega_C \cdot D + 2\bar{D}^2 = \\
&= 2d(g-1) + 2[2\bar{g} - 2 - 2d(g-1)] \quad = 4(\bar{g}-1) - 2d(g-1) \\
&= \bar{D}^2 + 2\bar{g} - 2 = |R| \geq 0
\end{aligned}$$

Moreover

$$\begin{array}{rlr} E^2 & = D^2 + 2d(2g-2) = 2\bar{D}^2 + 4d(g-1) & = \\ & = 2[2\bar{g} - 2 - 2d(g-1)] + 4d(g-1) & = 4(\bar{g}-1) > 0 \end{array}$$

Therefore, since E is effective, it is big and nef. Therefore the first part follows from the Kawamata-Viehweg vanishing theorem.

Since $\deg(p^*\omega_C^{-1} \otimes \mathcal{O}_D) = 2d(1-g)$, we have

$$\chi(p^*\omega_C^{-1} \otimes \mathcal{O}_D) = 2d(1-g) + 1 - g(D) = 4d(1-g) - \bar{D}^2$$

Using the exact sequence

$$0 \to p^*\omega_C^{-1}(-D) \to p^*\omega_C^{-1} \to p^*\omega_C^{-1} \otimes \mathcal{O}_D \to 0$$

and recalling Lemma 1.1 we obtain:

$$\begin{array}{rlr} & h^2(p^*\omega_C^{-1}(-D)) = \chi(p^*\omega_C^{-1}(-D)) & = \\ = & \chi(p^*\omega_C^{-1}) - \chi(p^*\omega_C^{-1} \otimes \mathcal{O}_D) & = (g-1)[3(g-1)+4d] + \bar{D}^2 \end{array}$$

•

Proposition 2.3 *If* $\bar{g} \geq 2$ *then*

$$\begin{array}{rrl} & h^0(T_{C_2}(-\bar{D})) & = 0 \\ & h^1(T_{C_2}(-\bar{D})) & = 0 \\ \chi(T_{C_2}(-\bar{D})) = & h^2(T_{C_2}(-\bar{D})) & = (g-1)[3(g-1)+4d-5] + \bar{D}^2 \end{array}$$

Proof. The equality $h^0(T_{C_2}(-\bar{D})) = 0$ is clear. By Lemma 2.1 it suffices to show that $H^1(p^*\omega_C^{-1}(\Delta - D)) = 0$ in order to prove the second equality. We will use diagram (16). Note that $H^0(\mathcal{F}) = 0$ for all sheaves $\mathcal{F}$ in the diagram. Moreover, since $H^1(p^*\omega_C^{-1}(-D)) = 0$ (Lemma 2.2), the first column of (16) shows that it suffices to prove that

$$H^1(\omega_\Delta^{-2}) \to H^2(p^*\omega_C^{-1}(-D))$$

is injective. This amounts to show that the coboundary map

$$H^1(\omega_\Delta^{-2}) \to H^2(p^*\omega_C^{-1})$$

coming from the second column of (16) is injective. But the cohomology sequence of the second column of (16) is

$$\begin{array}{ccccc} 0 \to & H^1(p^*\omega_C^{-1}) & \to & H^1(p^*\omega_C^{-1}(\Delta)) & \to H^1(\omega_\Delta^{-2}) \to H^2(p^*\omega_C^{-1}) \to \cdots \\ & \| & & \| & \\ & \mathbf{C}^{3g-3} & & \mathbf{C}^{3g-3} & \end{array}$$

where we used Lemma 1.1, equation (3) and Proposition 1.3; so this proves that $h^1(T_{C_2}(-\bar{D})) = 0$. The identity for $h^2(T_{C_2}(-\bar{D}))$ follows from Lemma 2.1 and the first part and from the cohomology sequence of the first column of diagram (16). •

We now prove a result which is useful for the computation of the cohomology of $T_{C_2|\bar{D}}$.

Proposition 2.4 *We have an identity*

$$T_{C_2|\bar{D}} = \omega_{\bar{D}}^{-1} \otimes \sigma_* \mathcal{O}(R) \tag{17}$$

and an exact sequence:

$$0 \to \omega_{\bar{D}}^{-1} \oplus \omega_{\bar{D}}^{-1}\eta \to T_{C_2|\bar{D}} \to \mathbf{t}' \to 0 \tag{18}$$

where $\mathbf{t}'$ *is a torsion sheaf with* $h^0(\mathbf{t}') = |R|$.

Proof. We have the identifications

$$p^*\omega_C^{-1} \otimes \mathcal{O}_D = \omega_D^{-1}(R) = \sigma^*\omega_{\bar{D}}^{-1}(R)$$

The second one being because $D \to \bar{D}$ is unramified. Then by applying the projection formula we have

$$\sigma_*[p^*\omega_C^{-1} \otimes \mathcal{O}_D] = \omega_{\bar{D}}^{-1} \otimes \sigma_*\mathcal{O}(R)$$

and therefore, after recalling the isomorphism (15), we have the identity (17).

Let $\eta \in \mathrm{Pic}_2(\bar{D})$ be the 2-torsion point defining the double cover $D \to \bar{D}$. Then we have $\sigma_*\mathcal{O}_D = \mathcal{O}_{\bar{D}} \oplus \eta$ and the exact sequence on $\bar{D}$

$$0 \to \mathcal{O}_{\bar{D}} \oplus \eta \to \sigma_*\mathcal{O}(R) \to \mathbf{t} \to 0$$

where $\mathbf{t}$ is a torsion sheaf with $h^0(\mathbf{t}) = |R|$ obtained by pushing down the obvious sequence on D:

$$0 \to \mathcal{O}_D \to \mathcal{O}_D(R) \to \mathcal{O}_R(R) \to 0$$

Tensoring with $\omega_{\bar{D}}^{-1}$ and using (17) we obtain (18). •

Example 2.5 (The classical Prym varieties) Let C have a fixed-point free involution ι and let $D \subset C \times C$ be the graph of ι. In this case $d = 1$, $R = 0$, $D = C$, and $g = g(D) = 2\bar{g} - 1$. Assume $\bar{g} \geq 2$, i.e. $g \geq 3$. Using (10) we get $\deg(N_{\bar{D}}) = -g + 1 < 0$ and, by (11):

$$h^0(N_{\bar{D}}) = 0, \quad h^1(N_{\bar{D}}) = -\chi(N_{\bar{D}}) = 3\bar{g} - 3$$

Moreover by (18) we have

$$h^1(T_{C_2|\bar{D}}) = h^0(\omega_{\bar{D}}^{\otimes 2}) + h^0(\eta\omega_{\bar{D}}^{\otimes 2}) = 6\bar{g} - 6$$

On the other hand

$$h^1(T_{\bar{D}}) = 3\bar{g} - 3$$

and, by Proposition 1.3:

$$h^1(T_{C_2}) = h^1(T_C) = 3(2\bar{g} - 1) - 3 = 6\bar{g} - 6$$

Therefore, since $h^1(T_{C_2}(-\bar{D})) = 0$ (Proposition 2.3), we see that the restriction map

$$H^1(T_{C_2}) \to H^1(T_{C_2|\bar{D}})$$

is an isomorphism, and that:

$$H^1(T_{C_2}) \cong H^1(T_{C_2|\bar{D}}) \cong H^1(T_{\bar{D}}) \oplus H^1(N_{\bar{D}}) \tag{19}$$

3 Deformations of the correspondence

Let the notation be as in the last section. Let

$$T_{C_2}\langle \bar{D} \rangle \subset T_{C_2}$$

be the inverse image of $T_{\bar{D}} \subset T_{C_2|\bar{D}}$ under the natural restriction homomorphism:

$$T_{C_2} \to T_{C_2|\bar{D}}$$

The deformation theory of the pair $(\bar{D}, C_2)$ is controlled by the sheaf $T_{C_2}\langle \bar{D} \rangle$. Precisely the tangent and the obstruction space of the deformation functor of the pair are $H^1(C_2, T_{C_2}\langle \bar{D} \rangle)$ and $H^2(C_2, T_{C_2}\langle \bar{D} \rangle)$ respectively (see [7], §3.4.4).

For the computation of these two vector spaces we consider the following exact sequences on C_2:

$$0 \to T_{C_2}(-\bar{D}) \to T_{C_2}\langle \bar{D}\rangle \to T_{\bar{D}} \to 0 \tag{20}$$

and

$$0 \to T_{C_2}\langle \bar{D}\rangle \to T_{C_2} \to N_{\bar{D}} \to 0 \tag{21}$$

which fit into the commutative diagram with exact rows and columns:

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
 & & T_{C_2}(-\bar{D}) & = & T_{C_2}(-\bar{D}) & & \\
 & & \downarrow & & \downarrow & & \\
0 \to & & T_{C_2}\langle \bar{D}\rangle & \to & T_{C_2} & \to & N_{\bar{D}} \ \to 0 \\
 & & \downarrow & & \downarrow & & \| \\
0 \to & & T_{\bar{D}} & \to & T_{C_2|\bar{D}} & \to & N_{\bar{D}} \ \to 0 \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & &
\end{array} \tag{22}$$

As a consequence of Proposition 1.3 we obtain

$$h^0(T_{C_2}\langle \bar{D}\rangle) = 0 \tag{23}$$

In view of Proposition 2.3 if $\bar{g} \geq 2$ then the cohomology of this diagram is the following:

$$\begin{array}{ccccc}
0 & & 0 & & \\
\downarrow & & \downarrow & & \\
H^1(T_{C_2}\langle \bar{D}\rangle) & \xrightarrow{\beta_1} & H^1(T_{C_2}) & \xrightarrow{\alpha} & H^1(N_{\bar{D}}) \\
\downarrow & & \downarrow & & \| \\
H^1(T_{\bar{D}}) & \xrightarrow{\gamma} & H^1(T_{C_2|\bar{D}}) & \to & H^1(N_{\bar{D}}) \ \to 0 \\
\downarrow & & \downarrow & & \\
H^2(T_{C_2}(-\bar{D})) & = & H^2(T_{C_2}(-\bar{D})) & & \\
\downarrow & & \downarrow & & \\
H^2(T_{C_2}\langle \bar{D}\rangle) & \xrightarrow{\beta_2} & H^2(T_{C_2}) & \to & 0 \\
\downarrow & & \downarrow & & \\
0 & & 0 & &
\end{array} \tag{24}$$

Note that

$$\ker(\beta_2) \quad = \operatorname{coker}(\alpha)$$

$$\ker(\beta_1) \quad = H^0(N_{\bar{D}}) \quad = \ker(\gamma)$$

In general we have

$$h^1(T_{C_2|\bar{D}}) \neq 0 \neq h^1(N_{\bar{D}})$$

so that $\bar{D}$ is neither stable nor costable in C_2 (see [7], §3.4.5, for the definition of stability and costability).

Lemma 3.1

$$\chi(T_{C_2}\langle\bar{D}\rangle) = 1 - \bar{g} + (g-1)(3g+2d-8)$$

Proof. From the exact sequence (20) we deduce the following identity:

$$\chi(T_{C_2}\langle\bar{D}\rangle) = \chi(T_{C_2}(-\bar{D})) + \chi(T_{\bar{D}})$$

Using Proposition 2.3 and (10) we obtain:

$$\chi(T_{C_2}\langle\bar{D}\rangle) = \bar{D}^2 + (g-1)(3g+4d-8) + 3 - 3\bar{g} = 1 - \bar{g} + (g-1)(3g+2d-8)$$

•

Remarks 3.2 If we denote by $\mu(\bar{D}, C_2)$ the number of moduli of the pair $(\bar{D}, C_2)$ (see [7]) then we have:

$$-\chi(T_{C_2}\langle\bar{D}\rangle) \leq \mu(\bar{D}, C_2) \leq h^1(T_{C_2}\langle\bar{D}\rangle)$$

The second inequality is an equality if and only if the pair $(\bar{D}, C_2)$ is unobstructed. Since $\chi(T_{C_2}\langle\bar{D}\rangle)$ tends to be positive, the lower bound is negative in general and therefore it does not give any useful information about the deformations of the pair.

Remarks 3.3 The sheaf $T_{C_2}(-\bar{D})$ controls the deformations of $(\bar{D}, C_2)$ which induce a trivial deformation of $\bar{D}$. The fact that $H^1(T_{C_2}(-\bar{D})) = 0$ (Proposition 2.3) means that there are no such infinitesimal deformations. This can be explained geometrically as follows.

To deform $(\bar{D}, C_2)$ without deforming $\bar{D}$ is the same as deforming (D, C^2) without deforming D (because D is etale over $\bar{D}$). This means to deform C without deforming its cover D and this is not possible because it would contradict the theorem of de Franchis for D.

4 Examples

We consider a few examples and we apply the previous calculations.

1. *The classical Prym varieties* - This is the example considered at the end of §2. By comparing the expressions given by Propositions 1.3 and 2.3 we get

$$h^2(T_{C_2}(-\bar{D})) = h^2(T_{C_2})$$

and from this it immediately follows that

$$H^1(T_{C_2}\langle\bar{D}\rangle) = H^1(T_{\bar{D}})$$

In particular the space of first order deformations of the pair $(\bar{D}, C_2)$, i.e. those which preserve the involution ι on C, has dimension $3\bar{g}-3 = 3\dim(P)$, where P is the Prym variety of the pair (C, ι), a very well known fact. The map β_2 of diagram (24) is bijective because α is surjective.

2. Let $f : X \to X'$ be an unramified double cover, $\iota : X \to X$ the involution, and assume given a g^1_5 $X' \to I\!\!P^1$ with simple ramifications. Then [1] in the induced map

$$f^{(5)} : X_5 \to X'_5$$

we have

$$f^{(5)-1}(g^1_5) = C \cup \tilde{C}$$

with C and $\tilde{C}$ nonsingular and $\iota(C) = \tilde{C}$. Define a correspondence $D \subset C \times C$ by

$$D = \{(x_1 + \cdots + x_5, \iota x_1 + \cdots + \iota x_4 + x_5) \in C \times C : f^{(5)}(x_1 + \cdots + x_5) \in g^1_5\}$$

It is a fixed-point free nonsingular correspondence with $d = 5$ of exponent $e = 4$ (see [1]). Moreover $P = \text{Prym}(X, \iota)$ so that $\dim(P) = g' - 1$ where $g' = g(X')$. Therefore

$$g(C) = g = d + e\ \dim(P) = 4g' + 1$$

By a computation one finds

$$|R| = 8|r| = 16(g' + 4)$$

where r is the ramification divisor of the g^1_5. Moreover

$$2g(D) - 2 = 5(2g-2) + 16(g'+4) = 40g' + 16(g'+4) = 56g' + 64$$

and it follows that

$$g(D) = 28g' + 33$$

and

$$\bar{g} = 14g' + 17$$

We have

$$\deg(N_{\bar{D}}) = -g(D) + 1 + |R| = -28g' - 33 + 1 + 16(g'+4) = -12g' + 32 < 0$$

if $g' \geq 3$ and

$$h^1(N_{\bar{D}}) = -\chi(N_{\bar{D}}) = 26g' - 16$$

Therefore we obtain

$$h^1(T_{C_2|\bar{D}}) = h^1(N_{\bar{D}}) + h^1(T_{\bar{D}}) = (26g'-16) + (42g'+48) = 68g'+32 = 6\bar{g} - 6 - |R|$$

which is compatible with the exact sequence (18).

3. Let X be hyperelliptic of genus $\gamma \geq 3$, $f : \tilde{X} \to X$ an etale $3:1$ morphism, and let $C = f^{(2)-1}(g^1_2)$ where $f^{(2)} : \tilde{X}_2 \to X_2$. It can be shown that for general choice of X and f the curve C is irreducible and nonsingular (see [5]). The induced map

$$f^{(2)} : C \to \mathbb{P}^1 = g^1_2$$

is of degree 9. If $x = y \in g^1_2$ then we have:

$$f^{(2)-1}(x+y) = \{P_{ij} := x_i + y_j : i, j = 1, 2, 3\}$$

where $\{x_1, x_2, x_3\} = f^{-1}(x)$ and $\{y_1, y_2, y_3\} = f^{-1}(y)$. Define a correspondence D on C by

$$D(P_{12}) = P_{11} + P_{13} + P_{22} + P_{32}$$

etc. Then one can show that this is a symmetric nonsingular and irreducible correspondence of degree $d = 4$, exponent $e = 3$, the corresponding Prym-Tyurin variety P is of dimension $\gamma - 3$ and $g = g(C) = 3\gamma - 5$. Moreover the number of moduli in this case can be computed to be $2\gamma - 1$ [5]. It is pretty clear that $R = 0$ so that

$$2g(D) - 2 = 4(2g - 2) = 24\gamma - 48$$

thus:

$$g(D) = 12\gamma - 23$$

$$\bar{g} = 6\gamma - 11$$

We have

$$\deg(N_{\bar{D}}) = -g(D) + 1 = -12\gamma + 24 < 0$$

so that

$$h^1(N_{\bar{D}}) = -\chi(N_{\bar{D}}) = 18\gamma - 36$$

and we get

$$h^1(T_{C_2|\bar{D}}) = h^1(N_{\bar{D}}) + 3\bar{g} - 3 = 2(18\gamma - 36)$$

which is compatible with the value given by the exact sequence (18).

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