An overview of classical deformation theory

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1. GENERALITIES - Deformation theory is closely related to the problem of classification in algebraic geometry.

If we have a class $\mathcal{M}$ of algebro-geometric objects, e.g.

$$\mathcal{M} = \{\text{projective nonsingular curves of genus } g\}/(\text{isomorphism})$$

$$\mathcal{M} = \{\text{closed subschemes of } \mathbb{P}^r \text{ with given Hilbert polynomial}\}$$

$$\mathcal{M} = \{\text{vector bundles of given rank and Chern classes on a smooth projective variety } X\}$$

the problem is: to describe $\mathcal{M}$.

The interest and the difficulty of this problem come from the existence of families. Roughly speaking, the existence of families of objects in $\mathcal{M}$ implies that $\mathcal{M}$ is not just a set but has some kind of "structure", hopefully will be a scheme, which will be the *moduli space* of the classification problem. In most cases $\mathcal{M}$ is not a scheme but has a weaker structure.

In order to make this statement more precise we have to specify the notion of family. This notion is different for every different class $\mathcal{M}$ but in each case it is related to the natural fact that all objects of algebraic geometry can be "deformed" by varying the coefficients of their defining equations.

If for example we want to consider a class $\mathcal{M}$ of algebraic varieties (curves, varieties of given dimension and numerical characters, etc.) a family will be a morphism:

$$\begin{array}{c} \mathcal{X} \\ \downarrow \pi \\ S \end{array}$$

whose fibres $\mathcal{X}(s) = \pi^{-1}(s)$, $s \in S$, are elements of $\mathcal{M}$ and with at least the extra technical condition of being *flat*; if the class $\mathcal{M}$ consists of projective and/or nonsingular varieties, then π will be also required to be proper and/or smooth. Here $\mathcal{X}$ and S are called the *total space* and the *parameter space* of the family. If S is connected then π is called a *family of deformations* of $\mathcal{X}(s_0)$ for any $s_0 \in S$.

If $\mathcal{X}$ and S are complex manifolds with S connected, and π is proper and smooth then all fibres $\mathcal{X}(s)$ are diffeomorphic and we are just considering a family of compact complex structures on a fixed differentiable manifold.

If instead we want to consider a class $\mathcal{M}$ of closed subschemes of a given scheme Y a family will be a commutative diagram:

$$\begin{array}{lcl} \mathcal{X} & \subset & S \times Y \\ \downarrow \pi & & \\ S & & \end{array}$$

where π is the restriction of the first projection, the inclusion is closed, and all fibres of π are in $\mathcal{M}$.

Typically, a family of hypersurfaces of degree d in $\mathbb{P}^r$ parametrized by an affine space $\mathbf{A}^n = \mathrm{Spec}(k[t_1, \ldots, t_n])$, k a field, will be a hypersurface $H \subset \mathbf{A}^n \times \mathbb{P}^r$ defined by a polynomial $P(\underline{t}, \underline{X}) \in k[t_1, \ldots, t_n, X_0, \ldots, X_r]$ homogeneous of degree d in $X_0, \ldots, X_r$.

A less ambitious goal is the study of local deformations of a given object $m \in \mathcal{M}$. This means to consider deformations of m parametrized by spectra of local rings so that m is the fibre over the closed point. This will lead to the understanding of the local structure of $\mathcal{M}$ at m. This was the point of view of Kodaira-Spencer who initiated modern deformation theory in a series of papers published in 1958 on Annals of Mathematics, where they studied local deformations of compact complex manifolds, i.e. local deformations of complex structures on a fixed compact differentiable manifold.

In each different case the notion of family has the fundamental property of being funtorial. Let's consider, to fix ideas, the case of a class $\mathcal{M}$ of isomorphism classes of projective varieties defined over a fixed algebraically closed field k, and families of objects in $\mathcal{M}$. For each scheme S we call two different such families over S:

$$\begin{array}{lcl} \mathcal{X} & & \mathcal{X}' \\ \downarrow \pi & \text{and} & \downarrow \pi' \\ S & & S \end{array}$$

isomorphic if there is an isomorphism $\varphi : \mathcal{X} \to \mathcal{X}'$ such that $\pi = \pi' \circ \varphi$. We can define a contravariant functor

$$F : (\text{Schemes}/k) \to (\text{Sets})$$

by

$$F(S) = \{\text{isomorphism classes of families of objects of } \mathcal{M} \text{ over } S\}$$

For each morphism $f : T \to S$ we have an induced

$$F(f) : F(S) \to F(T)$$

by pulling back families by f:

$$F(f)([\mathcal{X} \to S]) = [T \times_S \mathcal{X} \to T]$$

where $[-]$ denotes the isomorphism class of $-$ and

$$\begin{array}{ccc} T \times_S \mathcal{X} & \to & \mathcal{X} \\ \downarrow & & \downarrow \pi \\ T & \xrightarrow{f} & S \end{array}$$

is the induced pullback diagram.

This observation was the starting point of the development of deformation theory under the influence of Grothendieck. According to his point of view we may ask whether this functor is represented by a scheme M, namely if there is an isomorphism of functors:

$$\mu : \mathrm{Hom}(-, M) \to F$$

Such an isomorphism will be induced by pulling back a uniquely determined family $\xi : \mathcal{Y} \to M$, called the *universal family* (Infact $[\xi] = \mu(M)(id_M) \in F(M)$). If this is the case M will be a moduli space for $\mathcal{M}$ in the strongest sense. In particular its closed points will be in one-to-one correspondence with the element of $\mathcal{M}$ by the chain of bijections:

$$\mathcal{M} \leftrightarrow \{\text{families parametrized by } \mathrm{Spec}(k)\} \leftrightarrow \mathrm{Hom}(\mathrm{Spec}(k), M) \leftrightarrow \{\text{closed points of } M\}$$

Such a moduli space very seldom exists. Most of the time $\mathcal{M}$ will have a weaker structure corresponding to a property of the functor F weaker than representability. But let's suppose for a moment that M exists in our case. Then in principle all informations concerning its structure and all its properties are encoded in the functor F. In particular we can investigate its infinitesimal, local and formal properties around a point $m \in M$ by looking at various special families of deformations of the fibre $\mathcal{Y}(m)$ of the universal family. For example the tangent space $T_{M,m}$ can be recovered considering "first order deformations".

A *first order deformation* of a scheme X is a commutative diagram:

$$\begin{array}{ccc} X & \to & \mathcal{X} \\ \downarrow & & \downarrow \pi \\ \mathrm{Spec}(k) & \subset & \mathrm{Spec}(k[\epsilon]) \end{array}$$

where π is a flat morphism, $\mathrm{Spec}(k[\epsilon]) = \mathrm{Spec}(k[t]/(t^2))$, and such that the induced morphism

$$X \to \mathrm{Spec}(k) \times_{\mathrm{Spec}(k[\epsilon])} \mathcal{X}$$

is an isomorphism. First order deformations can be viewed as derivatives of $\mathcal{Y}(m)$ along a tangent vector of M at m. Infact we have the following chain of bijections:

$$T_{M,m} \leftrightarrow \mathrm{Hom}_m(\mathrm{Spec}(k[\epsilon]), M) \leftrightarrow \{\text{first order deformations of } \mathcal{Y}(m)\}/(\text{isomorphism})$$

where we have denoted by $\mathrm{Hom}_m(\mathrm{Spec}(k[\epsilon]), M)$ the set of morphisms $\mathrm{Spec}(k[\epsilon]) \to M$ mapping the unique closed point of $\mathrm{Spec}(k[\epsilon])$ to m, and where the last bijection is $\mu(\mathrm{Spec}(k[\epsilon]))$.

More generally an *infinitesimal deformation* of a scheme X is a commutative diagram

$$\begin{array}{ccc} X & \to & \mathcal{X} \\ \downarrow & & \downarrow \pi \\ \mathrm{Spec}(k) & \subset & \mathrm{Spec}(A) \end{array} \qquad (1)$$

where π is a flat morphism, A is a local artinian k-algebra and the morphism $X \to \text{Spec}(k) \times_{\text{Spec}(A)} \mathcal{X}$ induced by the diagram is an isomorphism. Then, in the same vein as above, infinitesimal deformations of X give informations on the infinitesimal structure of M at the point $m = \mu(\text{Spec}(k))^{-1}([X])$ because we have bijections

$$\text{Hom}_m(\text{Spec}(A), M) \leftrightarrow \{\text{infinit. deform.s of } X \text{ parametrized by Spec}A)\}$$

An infinitesimal deformation (1) is called *trivial* if $\mathcal{X} = X \times \text{Spec}(A)$.

Deformation theory is the study of infinitesimal deformations as a tool to understand the local structure of the moduli space. The goal is to be able to describe the restriction of the universal family to a small neighborhood of $m \in \mathcal{M}$, or, more precisely, its restriction to the germ of M at m.

What is interesting here is that we can study first order and infinitesimal deformations even though the functor F is not representable or simply we don't yet know it is. This is the most frequent case. Such an investigation will reveal the infinitesimal properties at $[X]$ of a yet unknown global structure on $\mathcal{M}$ which will be hopefully understood at a subsequent stage of the investigation. In order words it turns out to be possibile and convenient to separate the *global moduli problem* from the *local moduli problem*, and deformation theory studies the latter, with the purpose of constructing a family of deformations of a given object parametrized by the spectrum of a local ring, and having properties as close as possible to a universal property.

2. FIRST ORDER DEFORMATIONS - The first consequence of the local point of view is that, whenever we want to study infinitesimal deformations of some object, we don't need to specify the global class $\mathcal{M}$, i.e. the global moduli problem, inside which we are going to move it: all we have to do is to define what we mean by an infinitesimal deformation of it. Of course our definition will often be tailored to some specific global problem, but not always.

Let's apply these ideas to the study of first order deformations. We will only consider algebraic k-schemes where k is an algebraically closed field. We will see that isomorphism classes of first order deformations are elements of a cohomology vector space. It is a technical easy fact to check that this vector space structure coincides with the structure of tangent space in the corresponding moduli problem (whatever this means).

a) Nonsingular affine varieties. Let $X = \text{Spec}(R)$ be a nonsingular affine variety. Then every first order deformation of X is trivial. Infact let

$$\begin{array}{ccc} X & \to & \mathcal{X} \\ \downarrow & & \downarrow \pi \\ \text{Spec}(k) & \subset & \text{Spec}(k[\epsilon]) \end{array}$$

be such a deformation. We have a commutative diagram:

$$\begin{array}{ccc} X & \subset & \mathcal{X} \\ \cap & & \downarrow \pi \\ X \times \text{Spec}(k[\epsilon]) & \to & \text{Spec}(k[\epsilon]) \end{array}$$

and the nonsingularity of X implies the existence of a morphism $\phi : X \times \mathrm{Spec}(k[\epsilon]) \to \mathcal{X}$ such that the diagram

$$\begin{array}{ccc} X & \subset & \mathcal{X} \\ \cap & \phi \nearrow & \downarrow \pi \\ X \times \mathrm{Spec}(k[\epsilon]) & \to & \mathrm{Spec}(k[\epsilon]) \end{array}$$

is still commutative. One easily checks that ϕ is an isomorphism, and this proves that the given deformation is trivial.

b) Nonsingular varieties.

LEMMA: Let $\pi : \mathcal{X} \to S$ be a morphism of schemes, $\phi : X \subset \mathcal{X}$ a closed embedding defined by a sheaf of ideals $J \subset \mathcal{O}_{\mathcal{X}}$ such that $J^2 = 0$. Then there is a canonical $1-1$ correspondence:

$$\{S\text{-automorphisms of } \mathcal{X} \text{ inducing the identity on } X\} \leftrightarrow \mathrm{Hom}_{\mathcal{O}_X}(\phi^*\Omega^1_{\mathcal{X}/S}, J)$$

Proof

The question is local. Therefore we may assume that everything is affine and we have a commutative diagram:

$$\begin{array}{ccc} B & \to & B/J \\ \uparrow \tilde{\pi} & & \uparrow \\ A & \xrightarrow{\tilde{\pi}} & B \end{array}$$

Every A-automorphism ψ of B inducing the identity on B/J is of the form $\psi = 1_B + D$, where $D : B \to J$ is A-linear and satisfies:

$$D(b_1b_2) = (\psi - 1_B)(b_1b_2) = \psi(b_1b_2) - \psi(b_1)b_2 + \psi(b_1)b_2 - b_1b_2 =$$

$$= \psi(b_1)(\psi(b_2) - b_2) + (\psi(b_1) - b_1)b_2 = \psi(b_1)D(b_2) + D(b_1)b_2 = b_1D(b_2) + D(b_1)b_2$$

In other words D is an A-derivation of B in J. Therefore the set of A-automorphisms of B inducing the identity on B/J is in $1-1$ correspondence with

$$\mathrm{Der}_A(B, J) = \mathrm{Hom}_B(\Omega_{B/A}, J) = \mathrm{Hom}_{B/J}(\Omega_{B/A} \otimes_B B/J, J)$$

q.e.d.

Consider now a nonsingular variety X and a first order deformation of X:

(2)
$$\begin{array}{ccc} X & \to & \mathcal{X} \\ \downarrow & & \downarrow \pi \\ \mathrm{Spec}(k) & \subset & \mathrm{Spec}(k[\epsilon]) \end{array}$$

Let $\{U_\alpha\}$ be an affine open cover of X. Then by the previous case there are $\mathrm{Spec}(k[\epsilon])$-isomorphisms:

$$\theta_\alpha : \mathcal{X}_{|U_\alpha} \cong U_\alpha \times \mathrm{Spec}(k[\epsilon])$$

inducing the identity on the central fibre $U_\alpha = U_\alpha \times \mathrm{Spec}(k)$. Therefore by the lemma:

$$\theta_\beta\theta_\alpha^{-1} \in \Gamma(U_{\alpha\beta}, Hom(\Omega^1_{\mathcal{X}/\mathrm{Spec}(k[\epsilon])} \otimes \mathcal{O}_X, \mathcal{O}_X)) = \Gamma(U_{\alpha\beta}, \Theta_X)$$

where we denote, as usual, $U_{\alpha\beta} = U_\alpha \cap U_\beta$. It follows that the system $\{\theta_{\alpha\beta} = \theta_\beta\theta_\alpha^{-1}\}$ defines a Cech 1-cocycle in Θ_X and this defines an element of $H^1(X, \Theta_X)$. One easily checks that this element is independent of the chosen affine cover. Therefore we have defined a map

$$T_{\mathcal{M},[X]} \to H^1(X, \Theta_X)$$

which is easily seen to be a bijection.

Another equivalent way to define this map is the following. To a first order deformation (2) we can associate the exact sequence:

$$0 \to \pi^*\Omega^1_{\mathrm{Spec}(k[\epsilon])} \to \Omega^1_{\mathcal{X}} \to \Omega^1_{\mathcal{X}/S} \to 0$$

which tensored by $\mathcal{O}_X$ (i.e. restricted to X) gives the exact sequence:

$$0 \to \mathcal{O}_X \to \Omega^1_{\mathcal{X}} \otimes \mathcal{O}_X \to \Omega^1_X \to 0$$

This is an element of $\mathrm{Ext}^1(\Omega^1_X, \mathcal{O}_X) = H^1(X, \Theta_X)$ which can be checked to be the same as the one defined above.

If $\pi : \mathcal{X} \to S$ is an infinitesimal deformation of $X = \pi^{-1}(0)$ then the differential at 0 of the functorial morphism $S \to \mathcal{M}$ is a linear map

$$KS : T_{S,0} \to H^1(X, \Theta_X)$$

called the *Kodaira-Spencer map* of π, and $KS(v) \in H^1(X, \Theta_X)$ is the *Kodaira-Spencer class* of $v \in T_{S,0}$.

It follows that if $H^1(X, \Theta_X) = (0)$ then every first order deformation of X is trivial. It turns out that every infinitesimal deformation of X is trivial as well, i.e. X is *rigid*. For example $\mathbb{P}^r$ is rigid because $H^1(\mathbb{P}^r, \Theta_{\mathbb{P}^r}) = (0)$.

c) Line bundles on a fixed nonsingular projective variety. Let L be a line bundle on a nonsingular projective variety X. A first order deformation of L is a line bundle L_ϵ on $X \times \mathrm{Spec}(k[\epsilon])$ which restricts to L on the closed fibre $X = (X \times \mathrm{Spec}(k[\epsilon])) \times_{\mathrm{Spec}(k[\epsilon])} \mathrm{Spec}(k)$. Assume that L is given by a system of transition functions $\{f_{\alpha\beta}\}$ with respect to an open covering $\{U_\alpha\}$ of X, $f_{\alpha\beta} \in \Gamma(U_\alpha \cap U_\beta, \mathcal{O}^*_X)$. Then L_ϵ can be represented, in the same covering $\{U_\alpha\}$ of $X \times \mathrm{Spec}(k[\epsilon])$ by transition functions:

$$\tilde{f}_{\alpha\beta} \in \Gamma(U_\alpha \cap U_\beta, O^*_{X\times\mathrm{Spec}(k[\epsilon])})$$

such that

$$\tilde{f}_{\alpha\beta}\tilde{f}_{\beta\gamma} = \tilde{f}_{\alpha\gamma} \tag{3}$$

and wich restrict to the $f_{\alpha\beta}$'s modulo ϵ.

Since $\mathcal{O}^*_{X\times \mathrm{Spec}(k[\epsilon])} = \mathcal{O}^*_X + \epsilon\mathcal{O}_X$ we can write

$$\tilde{f}_{\alpha\beta} = f_{\alpha\beta}(1+\epsilon\Phi_{\alpha\beta})$$

for suitable $\Phi_{\alpha\beta} \in \Gamma(U_\alpha \cap U_\beta, \mathcal{O}_X)$. Identity (3) gives

$$\Phi_{\alpha\beta} + \Phi_{\beta\gamma} = \Phi_{\alpha\gamma}$$

and therefore the system $\{\Phi_{\alpha\beta}\}$ defines an element of $H^1(X, \mathcal{O}_X)$. It is easy to check that this element does not depend on the choices made and that conversely each element of $H^1(X, \mathcal{O}_X)$ defines a first order deformation of L.

The class of all line bundles on X has the structure of a locally finite type scheme, denoted $\mathrm{Pic}(X)$, and we have computed its Zariski tangent space at L:

$$T_{\mathrm{Pic}(X),L} \cong H^1(X, \mathcal{O}_X)$$

3. HIGHER ORDER DEFORMATIONS - OBSTRUCTIONS - So far we have discovered that we can compute various tangent spaces to deformation problems as cohomology vector spaces. This is of course only a first step towards the description of the local structure of our moduli problems. For the next step we need to push the local point of view a little further.

Suppose that we need to study infinitesimal deformations of a geometrical object X inside a class $\mathcal{M}$. Let's assume that a moduli space M for $\mathcal{M}$ exists and let $\xi \in F(M)$ be the universal family. Then letting $[X] = m \in M$ be the point corresponding to X, to every infinitesimal deformation of X there corresponds a morphism

$$\begin{array}{rccc} \varphi: & \mathrm{Spec}(A) & \to & M \\ & \text{closed pt} & \mapsto & m \end{array}$$

which induces the given deformation by pullback. In turn φ corresponds to a homomorphism of local k-algebras

$$\tilde{\varphi} : \mathcal{O} = \mathcal{O}_{M,m} \to A$$

Since A is artinian, $\tilde{\varphi}$ factors through the completion $\mathcal{O} \to \hat{\mathcal{O}}$ with respect to the maximal ideal and therefore the properties of $\mathcal{O}$ detected by the study of infinitesimal deformations will be *analytic properties*, i.e. properties preserved under completion.

For example, if $A = k[\epsilon]$ then $F(\mathrm{Spec}(k[\epsilon]))$ is the Zariski tangent space of $\mathcal{O}$, which coincides with that of $\hat{\mathcal{O}}$.

We can rephrase all the above by considering the category

$$\mathcal{A} = (\text{local artinian } k\text{-algebras with residue field } k)$$

and saying that our deformation problem defines a covariant functor

$$F_{\mathcal{A}} : \mathcal{A} \to (\mathrm{Sets})$$

i.e. a *functor of Artin rings*, defined by

$$F_{\mathcal{A}} = \{\text{infinitesimal deformations of } X \text{ over } \mathrm{Spec}(A)\} = \mathrm{Hom}(\mathcal{O}, A)$$

The most important analytic property is nonsingularity. We can investigate the nonsingularity of M at m by means of the functor $F_{\mathcal{A}}$ and applying the following

LEMMA *Let $\mathcal{O}$ be a local noetherian k-algebra with residue field k. The following conditions are equivalent:*

(i) $\mathcal{O}$ is a regular local ring.

(ii) $\hat{\mathcal{O}}$ is a regular local ring.

(iii) There is an isomorphism

$$\hat{\mathcal{O}} \cong k[[X_1, \ldots, X_d]]$$

where d is the Krull dimension of $\mathcal{O}$, and $X_1, \ldots, X_d$ are indeterminates.

(iv) For every commutative diagram:

$$\begin{array}{ccc} k & \rightarrow & A' \\ \downarrow & & \downarrow \\ \mathcal{O} & \rightarrow & A \end{array}$$

where the right vertical arrow is a surjection of local artinian k-algebras, there is a k-algebra homomorphism $\mathcal{O} \rightarrow A'$ keeping the diagram

$$\begin{array}{ccc} k & \rightarrow & A' \\ \downarrow & \nearrow & \downarrow \\ \mathcal{O} & \rightarrow & A \end{array}$$

commutative.

Condition (iv) of the Lemma states that $\mathrm{Hom}(\mathcal{O}, A') \rightarrow \mathrm{Hom}(\mathcal{O}, A)$ is surjective for all surjections $A' \rightarrow A$ in $\mathcal{A}$. This condition has an immediate translation into a property of the functor $F_{\mathcal{A}}$:

PROPOSITION *M is nonsingular at m if and only if for every surjection $A' \rightarrow A$ in $\mathcal{A}$ the corresponding map*

$$F_{\mathcal{A}}(A') \rightarrow F_{\mathcal{A}}(A)$$

is surjective.

If this condition is satisfied the functor $F_{\mathcal{A}}$ is said to be smooth.

The condition of the Proposition has the following deformation-theoretic interpretation. Given a surjection $A' \to A$ in $\mathcal{A}$ and any deformation (1) there is a deformation

$$\begin{array}{ccc} X & \to & \mathcal{X}' \\ \downarrow & & \downarrow \pi \\ \mathrm{Spec}(k) & \subset & \mathrm{Spec}(A') \end{array} \tag{3}$$

extending (1), i.e. such that (1) is induced by (3) by pulling it back via $\mathrm{Spec}(A) \to \mathrm{Spec}(A')$. If the extension (3) exists for each surjection $A' \to A$ the deformation (1) is called *unobstructed*; otherwise it is *obstructed*.

If all infinitesimal deformations of X are unobstructed then X is called *unobstructed*; otherwise X is *obstructed*.

It turns out that in order to check (un)obstructedness it suffices to consider surjections $q : A' \to A$ in $\mathcal{A}$ such that $\ker(q) \cong k$ (called *small extensions*).

Let's denote by t_R the Zariski tangent space $(m_R/m_R^2)^\vee$ of a local ring (R, m_R). We have the following

DEFINITION *Let (R, m_R) be a complete local k-algebra with residue field k. Write $R = k[[X_1, \dots, X_n]]/J$ where $J \subset (\underline{X})^2$. Then the k-vector space*

$$o(R) := (J/(\underline{X})J)^\vee$$

is called the obstruction space of R.

Clearly $o(R) = 0$ if and only if $R \cong k[[X_1, \dots, X_n]]$. We have the following inequalities:

$$\dim(t_R) \geq \dim(R) \geq \dim(t_R) - \dim(o(R))$$

Moreover for each A in $\mathcal{A}$ and for each $\varphi : R \to A$ there is a map which associates to each small extension $q : A' \to A$ an element $v(q) \in o(R)$ which is 0 if and only if φ can be lifted to $\varphi' : R \to A'$.

If we have a "sufficiently well behaved" deformation functor $F_{\mathcal{A}}$ then it is possible to define the *obstruction* to find an extension in $F_{\mathcal{A}}(A')$ of a given $\eta \in F_{\mathcal{A}}(A)$; this obstruction is usually an element of a cohomology vector space H. The deformation η will then be unobstructed precisely if the obstruction vanishes for each small extension q. If the deformation functor is $F_{\mathcal{A}} = \mathrm{Hom}(\mathcal{O}, -)$ where $\mathcal{O} = \mathcal{O}_{M,m}$ as above, then it follows by general nonsense that $o(\hat{\mathcal{O}}) \subset H$. This implies that M is nonsingular at m if the vector space H vanishes and, more generally, that

$$\dim(\mathcal{O}) \geq \dim(t_{\mathcal{O}}) - \dim(H) = \dim(F_{\mathcal{A}}(k[\epsilon]) - \dim(H)$$

Let's illustrate this principle with an example.

Nonsingular varieties Assume that we have a class $\mathcal{M}$ of nonsingular varieties for which the moduli space M exists. Let X be in $\mathcal{M}$. Assume that we have a small extension $A' \to A$ and an infinitesimal deformation (1) of X. We want to find conditions for the extendability to a deformation of X over $\mathrm{Spec}(A')$.

Let $\{U_\alpha\}$ be an affine open cover of X, $\theta_\alpha : \mathcal{X}_{|U_\alpha} \cong U_\alpha \times \mathrm{Spec}(A)$ be $\mathrm{Spec}(A)$-isomorphisms inducing the identity on U_α, and let

$$\theta_{\alpha\beta} = \theta_\alpha \theta_\beta^{-1} : U_{\alpha\beta} \times \mathrm{Spec}(A) \to U_{\alpha\beta} \times \mathrm{Spec}(A)$$

be the induced $\mathrm{Spec}(A)$-automorphisms. Then the existence of a deformation $\pi' : \mathcal{X}' \to \mathrm{Spec}(A')$ extending (1) is equivalent to the existence of a system of automorphisms

$$\theta'_{\alpha\beta} : U_{\alpha\beta} \times \mathrm{Spec}(A') \to U_{\alpha\beta} \times \mathrm{Spec}(A')$$

which restrict to the automorphisms $\theta_{\alpha\beta}$ on $U_{\alpha\beta} \times \mathrm{Spec}(A)$, and such that

$$\theta'_{\alpha\beta}\theta'_{\beta\gamma} = \theta'_{\alpha\gamma} \tag{4}$$

on $U_{\alpha\beta\gamma}$. Let's choose arbitrarily automorphisms $\theta'_{\alpha\beta}$ which extend the $\theta_{\alpha\beta}$'s (they exist by the nonsingularity of the affine varieties $U_{\alpha\beta}$), and let's consider the $\mathrm{Spec}(A')$-automorphisms of $U_{\alpha\beta\gamma} \times \mathrm{Spec}(A')$:

$$\theta'_{\alpha\beta\gamma} := \theta'_{\alpha\beta}\theta'_{\beta\gamma}(\theta'_{\alpha\gamma})^{-1}$$

Each of these restricts to the identity on $U_{\alpha\beta\gamma} \times \mathrm{Spec}(A)$ and therefore, by the Lemma, is an element of $\Gamma(U_{\alpha\beta\gamma}, \Theta_X)$. The system $\{\theta'_{\alpha\beta\gamma}\}$ is therefore a 2-cocycle with coefficients in Θ_X and defines an element $\theta \in H^2(X, \Theta_X)$.

Another choice of the automorphisms $\theta'_{\alpha\beta}$ is of the form

$$\bar\theta'_{\alpha\beta} = \theta'_{\alpha\beta}\delta_{\alpha\beta}$$

for some $\delta_{\alpha\beta} \in \Gamma(U_{\alpha\beta}, \Theta_X)$. Therefore:

$$\bar\theta'_{\alpha\beta\gamma} = \theta'_{\alpha\beta\gamma}\delta_{\alpha\beta}\delta_{\beta\gamma}(\delta_{\alpha\gamma})^{-1}$$

and therefore $\{\theta'_{\alpha\beta\gamma}\}$ and $\{\bar\theta'_{\alpha\beta\gamma}\}$ define the same cohomology class in $H^2(X, \Theta_X)$.

The class $\theta \in H^2(X, \Theta_X)$ is the *obstruction* to extend the deformation (3) to $\mathrm{Spec}(A')$.

In particular we see that *if* $H^2(X, \Theta_X) = 0$ *then* M *is nonsingular at* $[X]$. For example, nonsingular projective curves are unobstructed.

4. VERSAL AND UNIVERSAL FORMAL FAMILIES - We have seen how one can study the infinitesimal properties of a moduli space M at a point m using functorial methods and cohomological techniques. We now want to consider a local moduli problem and see whether it is possible to study its infinitesimal properties and to give it a local structure of some kind. From an infinitesimal point of view a local moduli problem corresponds to a (covariant) functor of Artin rings

$$F : \mathcal{A} \to (\mathrm{Sets})$$

such that $F(k)$ consists of one element. In the best possible case there will be a local k-algebra $\mathcal{O}$ with residue field k and an isomorphism of functors

$$\mathrm{Hom}(\mathcal{O}, -) = \mathrm{Hom}(\hat{\mathcal{O}}, -) \to F$$

(the equality on the left is because, as we observed already, $A = \hat{A}$ for every A in $\mathcal{A}$. Since $\hat{\mathcal{O}}$ is not in $\mathcal{A}$, such a functor is not quite representable: it is called *prorepresentable.* Representable functors of Artin rings are not so interesting in this context, but prorepresentable ones are, and prorepresentability is the reachest structure such a functor can have.

Weaker structures can be introduced by requiring that there exists a morphism of functors (a "natural transformation")

$$f : \mathrm{Hom}(R, -) \to F$$

for some complete local k-algebra R with residue field k, which is not quite an isomorphism, but has some weaker property. Before discussing these properties let's see for a moment how a morphism f as above can be interpreted.

Let's denote by $\hat{\mathcal{A}}$ the category of complete local k-algebras with residue field k. Every functor of Artin rings $F : \mathcal{A} \to (\mathrm{Sets})$ can be extended to a functor

$$\hat{F} : \hat{\mathcal{A}} \to (\mathrm{Sets})$$

by letting, for every (R, m) in $\hat{\mathcal{A}}$:

$$\hat{F}(R) = \varprojlim F(R/m^{n+1})$$

and for every $\varphi : (R, m) \to (S, p)$:

$$\hat{F}(\varphi) : \hat{F}(R) \to \hat{F}(S)$$

to be the map induced by the maps $F(R/m^n) \to F(S/p^n)$, $n \geq 1$.

An element $\hat{u} \in \hat{F}(R)$ is called a *formal element* of F. By definition $\hat{u}$ can be represented as a system of elements $\{u_n \in F(R/m^{n+1})\}_{n \geq 0}$ such that for every $n \geq 0$ the map

$$F(R/m^{n+1}) \to F(R/m^n)$$

induced by the projection $R/m^{n+1} \to R/m^n$ sends

$$u_n \longmapsto u_{n-1} \tag{5}$$

If for example F is the functor of infinitesimal deformations of a nonsingular variety X, each u_n is an infinitesimal deformation of X parametrized by $\mathrm{Spec}(R/m^{n+1})$. The compatibility condition (5) is that u_n pulls back to u_{n-1} under the closed embedding

$$\mathrm{Spec}(R/m^n) \subset \mathrm{Spec}(R/m^{n+1})$$

In this case the formal element $\hat{u}$ is also called a *formal family of deformations* of X.

If $f : F \to G$ is a morphism of functors of Artin rings then it can be extended in an obvious way to a morphism of functors $\hat{f} : \hat{F} \to \hat{G}$.

LEMMA *Let R be in $\hat{\mathcal{A}}$. There is a $1-1$ correspondence between $\hat{F}(R)$ and the set of morphisms of functors*

$$\mathrm{Hom}(R, -) \longrightarrow F \tag{6}$$

Proof
To a formal element $\hat{u} \in \hat{F}(R)$ there is associated a morphism of functors (6) in the following way. Each $u_n \in F(R/m^{n+1})$ defines a morphism of functors $\mathrm{Hom}(R/m^{n+1}, -) \to F$. The compatibility conditions (5) imply that the following diagram commutes:

$$\begin{array}{ccc} \mathrm{Hom}(R/m^n, -) & \to & \mathrm{Hom}(R/m^{n+1}, -) \\ & \searrow & \downarrow \\ & & F \end{array}$$

for every n. Since for each A in $\mathcal{A}$

$$\mathrm{Hom}(R/m^n, A) \to \mathrm{Hom}(R/m^{n+1}, A)$$

is a bijection for all $n \gg 0$ we may define

$$\mathrm{Hom}(R, A) \to F(A)$$

as

$$\lim_{n\to\infty} [\mathrm{Hom}(R/m^{n+1}, A) \to F(A)]$$

Conversely each morphism (6) defines a formal element $\hat{u} \in \hat{F}(R)$, where $u_n \in F(R/m^{n+1})$ is the image of the canonical projection $R \to R/m^{n+1}$ via the map

$$\mathrm{Hom}(R, R/m^{n+1}) \to F(R/m^{n+1})$$

q.e.d.

If $\hat{u} \in \hat{F}(R)$ is such that the induced morphism (6) is an isomorphism, then F is prorepresentable, and we say that F is *prorepresented by the pair* $(R, \hat{u})$. In this case $\hat{u}$ is called a *universal formal element* for F, and $(R, \hat{u})$ is a *universal pair.*

If for example F is the functor of infinitesimal deformations of a nonsingular variety X belonging to a class $\mathcal{M}$ which has a moduli space M, then the universal family $\mathcal{Y} \to M$ induces by restriction to the schemes $\mathrm{Spec}(\hat{O}/m^{n+1})$ a universal formal element for F (or a *universal formal family*).

Note that all prorepresentable functors have the following property:

N_0) $F(k)$ contains exactly one element.

All functors we will consider will have property N_0 and from now on this will be implicitly assumed unless otherwise specified.

DEFINITION *Let $f : F \to G$ be a morphism of functors of Artin rings. f is called* smooth *if for every surjection $\mu : B \to A$ in $\mathcal{A}$ the natural map:*

$$F(B) \to F(A) \times_{G(A)} G(B)$$

induced by the diagram:

$$\begin{array}{ccc} F(B) & \to & G(B) \\ \downarrow & & \downarrow \\ F(A) & \to & G(A) \end{array}$$

is surjective.

Note that the smoothness condition applied to the surjection $k[\epsilon] \to k$ states that the map

$$F(k[\epsilon]) \to G(k[\epsilon])$$

is surjective. This map is denoted *df* and called the *differential* of f.

Let F be a functor of Artin rings. A formal element $\hat{u} \in \hat{F}(R)$, for some R in $\hat{\mathcal{A}}$, is called *versal* if the morphism $\mathrm{Hom}(R, -) \to F$ defined by $\hat{u}$ is smooth; $\hat{u}$ is called *semiuniversal* if it is versal and moreover the differential $\mathrm{Hom}(R, \mathbf{k}[\epsilon]) \to F(\mathbf{k}[\epsilon])$ is an isomorphism.

We will call the pair $(R, \hat{u})$ a *versal pair* (respectively a *semiuniversal pair*, a *universal pair*) if $\hat{u}$ is versal (respectively semiuniversal, universal).

It is clear from the definitions that:

$$\hat{u} \quad \text{universal} \quad \Rightarrow \quad \hat{u} \quad \text{semiuniversal} \quad \Rightarrow \quad \hat{u} \quad \text{versal}$$

but none of the inverse implications is true.

What does it mean that a functor F has a versal pair $(R, \hat{u})$? Frome the definition of smoothness it follows easily that the map

$$\text{Hom}(R, S) \to \hat{F}(S) \tag{7}$$

induced by $\hat{u}$ is surjective for every S in $\hat{A}$. This means that every formal element $\hat{v} \in F(S)$ is induced by $\hat{u} \in F(R)$ by pullback. So we see that this is a property, weaker than universality, which is a sort of "completeness" of the formal element $\hat{u}$, in the sense that it induces every other by pullback.

Semiuniversality is stronger than versality: the bijectivity of the differential implies a sort of minimality among all possible versal pairs.

A theorem of Schlessinger gives conditions, easy to verify in practise, for the existence of a formal semiuniversal element for a functor F. It turns out that most functors of Artin rings arising in deformation theory satisfy Schlessinger's conditions, even though they seldom have a universal formal element; therefore all such functors have a structure weaker than prorepresentability, but very close to it.

Examples of functors satysfying Schlessinger conditions are:

$F = Pic(X)_L =$ deformations of a line bundle L on a fixed scheme X (the *local Picard functor* of X at L)

$F =$ deformations of a projective scheme X

$F =$ deformations of an affine variety with isolated singularities

$F = Hilb^Y_X =$ the local Hilbert functor of a closed embedding $X \subset Y$

$F = Quot^F_G =$ the local Quot scheme of a quotient $F \to G$ of sheaves on a scheme X

5. ALGEBRIZATION - Suppose we know that a functor of Artin rings F has a (semi)universal pair $(R, \hat{u})$, and that F extends to the category $\mathcal{A}^*$ of local noetherian k-algebras. Then we should ask if there is a pair (S, u), where $u \in F(S)$, having the following properties:

i) S is in $\mathcal{A}^*$, and has some finiteness properties (e.g. it is essentially of finite type, it is henselian, etc.).

ii) $\hat{S} = R$.

iii) u induces $\hat{u}$.

This question is an abstract version of a natural problem in local deformation theory. Consider for example a projective nonsingular variety X. We can consider local deformations of X, i.e. families of the form

$$\xi : \begin{array}{c} \mathcal{X} \\ \downarrow \pi \\ \text{Spec}(S) \end{array}$$

where (S, m_S) is in $\mathcal{A}^*$, and with an isomorphism $X \cong \mathcal{X}(m_S)$. Then we want to know if there is a (semi)universal such family ξ, i.e. a family which induces every other by pullback, and has a (semi)universal property. Applying the theory outlined before to the functor of Artin rings defined by X we obtain a formal (semi)universal pair $(R, \hat{u})$, and we now want to see if we can lift this pair to a pair (S, u) as above.

This is an algebraic version of the original problem studied and solved by Kodaira, Niremberg, Spencer and Kuranishi in the analytic case. Their final result is the following.

THEOREM *Let X be a compact complex manifold. Then there is a germ of complex space $(B, 0)$, with $\dim(B) \geq h^1(X, \Theta_X) - h^2(X, \Theta_X)$ and a smooth and proper family*

$$\begin{array}{c} \mathcal{X} \\ \xi : \downarrow \pi \\ B \end{array}$$

such that $X \cong \mathcal{X}(0)$, which is a semiuniversal family of deformations of X. If $H^2(X, \Theta_X) = 0$ then B is nonsingular of dimension $h^1(X, \Theta_X)$. If $H^0(X, \Theta_X) = 0$ then ξ is universal.

In the algebraic case there is no such general result. The most general algebrizability result is due to M. Artin. It gives sufficient conditions for the existence of a pair (S, u) as above with S an henselian ring, i.e. the local ring of an algebraic space (for an exposition see Artin(1971)).

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