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Small Deformations of Global Complete Intersections.

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Sunto. – *Si studiano le «piccole deformazioni» di una varietà proiettiva non singolare V che sia globalmente un'intersezione completa in P^n. Si dimostra che se* $\dim V \geqslant 2$ *allora ogni deformazione abbastanza piccola di V è globalmente un'intersezione completa in P^n se e solo se V non è una superficie K-3.*

0. – Introduction.

The purpose of this note is to study the deformations of global complete intersections in complex projective space P^n, extending the calculations of KODAIRA and SPENCER ([5] theorem 18.5) and of GORI [1]. By a global complete intersection we mean a connected nonsingular variety V of codimension p, $0 < p < n$, in P^n, $(n \geqslant 2)$ defined by the vanishing of p homogeneous polynomials $f_1, \ldots, f_p$ say, of degrees $d_1, \ldots, d_p$ respectively. We will always order the d_j's so that $d_j \leqslant d_{j+1}$, $j = 1, \ldots, p-1$, and assume that $d_1 \geqslant 2$ (V does not lie in a hyperplane); call $\mathbf{d} = (d_1, \ldots, d_p)$ the *multidegree* of V.

We investigate the problem of finding all sufficiently small deformations of V as an abstract compact complex manifold; our main result is the following:

THEOREM. – *Let V be a global complete intersection of multidegree $\mathbf{d} = (d_1, \ldots, d_p)$ in P^n and suppose that* $\dim V \geqslant 2$. *Then all sufficiently small deformations of V are again global complete intersections of multidegree $\mathbf{d}$ in P^n if and only if one of the following conditions is satisfied:*

I) $p < n-2$

II) $p = n-2$ *and* $\mathbf{d} \neq (4), (2, 3), (2, 2, 2)$.

It turns out that only those V which are K-3 surfaces have been excluded (see § 2).

In § 1 we consider a global complete intersection V in P^n and we construct a family $\mathfrak{V}$ of submanifolds of P_n parametrized by a polydisc M_ε such that all fibres V_t, $t \in M_\varepsilon$, are global complete intersections and $\mathfrak{V}_0 = V$. We introduce a complex $K^\bullet$ of locally free sheaves on P^n which we call the *Koszul complex of* V. Using this complex we can prove by spectral sequences that the infinitesimal displacement map of $(\mathfrak{V}, M_\varepsilon)$ at $0 \in M_\varepsilon$ is always surjective (see § 1 for a precise definition).

The proof of the theorem is given in § 2 and it is based on the notion of complete complex analytic family, due to KODAIRA and SPENCER [5]; we study the family $(\mathfrak{V}, M_\varepsilon)$ using the completeness theorem of [6] and certain spectral sequences deduced from the complex $K^\bullet$.

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1. – Preliminaries.

Let $\boldsymbol{o}$ be the structure sheaf on P^n, $\boldsymbol{o}(1)$ the canonical positive invertible sheaf and $\boldsymbol{o}(d) = \boldsymbol{o}(1)^{\otimes d}$, $d \geqslant 0$. Recall that sections of $\boldsymbol{o}(d)$ are homogeneous polynomials of degree d. If $\boldsymbol{d} = (d_1, \dots, d_p)$, $(2 \leqslant d_1 \leqslant \dots \leqslant d_p$ and $0 < p < n$ as always) a global complete intersection V in P^n of multidegree $\boldsymbol{d}$ can be viewed as the zero set of a section $f = (f_1, \dots, f_p)$ of the locally free sheaf $E = \boldsymbol{o}(d_1) \oplus \oplus \dots \oplus \boldsymbol{o}(d_p)$. Let $g^1, \dots, g^m$, with $g^h = (g_1^h, \dots, g_p^h)$, $h = 1, \dots, m$, be a basis of $H^0(P^n, E)$. Consider the space C^m of m complex variables $t^1, \dots, t^m$; if $t = (t^1, \dots, t^m) \in C^m$ we define $|t| = \max\limits_h |t^h|$ and denote $M_\varepsilon = \{t \in C^m \colon |t| < \varepsilon\}$, the polycylinder of center $0 \in C^m$ and radius $\varepsilon > 0$. When ε is sufficiently small the couple $(\mathfrak{V}, M_\varepsilon)$ with

$$\mathfrak{V} = \{(x, t) \in P^n \times M \colon f(x) + \sum_{h=1}^{m} t^h g^h(x) = 0\} =$$

$$= \{(x, t) \in P^n \times M \colon f_1(x) + \sum_{h=1}^{m} t^h g_1^h(x) = \dots = f_p(x) + \sum_{h=1}^{m} t^h g_p^h(x) = 0\}$$

defines an analytic family of compact submanifolds of P^n in the sense of [5]; the fibres $\mathfrak{V}_t$ of this family are global complete intersections of multidegree $\boldsymbol{d}$ and $\mathfrak{V}_0 = V$.

By « evaluation » at the section f of E we get a surjective map of sheaves $'f'\colon E^* \to I_V$ from $E^* = \operatorname{Hom}_{\boldsymbol{o}}(E, \boldsymbol{o})$ onto the ideal sheaf of V in P^n; $'f'$ extends to a map from the second exterior power

$\overset{2}{\bigwedge} E$ to E: and the sequence of sheaves on P^n

$$\overset{2}{\bigwedge} E^* \to E^* \to I_V \to 0$$

is exact (because V is a global complete intersection, cf. [9]). Therefore the induced sequence of sheaves on V

$$0 \to \mathrm{Hom}_{\boldsymbol{o}}(I_V, \boldsymbol{o}_V) \to \mathrm{Hom}_{\boldsymbol{o}}(E^*, \boldsymbol{o}_V) \xrightarrow{e} \mathrm{Hom}_{\boldsymbol{o}}\left(\overset{2}{\bigwedge} E^*, \boldsymbol{o}_V\right)$$

is exact, $\boldsymbol{o}_V$ denoting the structure sheaf on V; the map e is the zero map so that $\mathrm{Hom}_{\boldsymbol{o}}(I_V, \boldsymbol{o}_V) = N_V$, the sheaf of holomorphic sections of the normal bundle of V in P^n, is isomorphic to $\mathrm{Hom}_{\boldsymbol{o}}(E^*, \boldsymbol{o}_V) = E \underset{\boldsymbol{o}}{\otimes} \boldsymbol{o}_V$, the sheaf theoretic restriction of E to V. If we identify these two sheaves using this isomorphism and we also identify the tangent space $T_{M_{\varepsilon,0}}$ of M_ε at 0 with $H^0(P^n, E)$ sending $\partial/\partial t^h \mapsto g^h$ we get a linear map

$$\sigma_0 \colon T_{M_{\varepsilon,0}} \to H^0(V, N_V)$$

induced by the restriction $E \to E \otimes \boldsymbol{o}_V$. It is immediate to check that σ_0 coincides with the infinitesimal displacement map of $\mho$ at 0 defined in [5], § 12; the subspace $\sigma_0(T_{M_{\varepsilon,0}})$ of $H^0(V, N_V)$ is the characteristic system on V of the family $\mho$ [4].

Extending the map $'f'\colon E^* \to \boldsymbol{o}$ to the full exterior algebra $\overset{\bullet}{\bigwedge} E^*$ we get a complex of locally free sheaves of $\boldsymbol{o}$-modules

$$K^\bullet \colon 0 \to \overset{p}{\bigwedge} E^* \to \overset{p-1}{\bigwedge} E^* \to \dots \to \overset{2}{\bigwedge} E^* \to E^* \xrightarrow{'f'} \boldsymbol{o} \to 0\,.$$

We define $K^{-r} = \overset{r}{\bigwedge} E$ so that $K^\bullet$ is a cohomological complex; we call it « the Koszul complex of V ». Notice that

$$E^* = \boldsymbol{o}(-d_1) \oplus \dots \oplus \boldsymbol{o}(-d_p)$$

where $\boldsymbol{o}(-k) = \mathrm{Hom}_{\boldsymbol{o}}(\boldsymbol{o}(1), \boldsymbol{o})^{\otimes k}$ for a nonnegative integer k, and

$$\overset{r}{\bigwedge} E^* = \overset{r}{\bigwedge}\Big[\bigoplus_{j=1}^{p} \boldsymbol{o}(-d_j)\Big] = \bigoplus_{1 \leqslant j_1 < \dots < j_r \leqslant p} \boldsymbol{o}(-d_{j_1} - \dots - d_{j_r})\,.$$

by linear algebra.

(1.1) Proposition. – *The map* $\sigma_0 \colon T_{M_{\varepsilon,0}} \to H^0(V, N_V)$ *is surjective, i.e. the characteristic system on* V *of the family* $\mho$ *is complete.*

PROOF. – Consider the two spectral sequences of hypercohomology [3] of the complex $K^\bullet \underset{o}{\otimes} E$

$$A_1^{r,s} = H^s(P^n, K^r \otimes E) \Rightarrow \mathbf{H}^\bullet(K^\bullet \otimes E)$$
$$'A_2^{r,s} = H^r(P^n, \mathcal{H}^s(K^\bullet \otimes E)) \Rightarrow \mathbf{H}^\bullet(K^\bullet \otimes E)$$

where $\mathcal{H}^s(K^\bullet \otimes E)$ denotes the s-th cohomology sheaf of the complex $K^\bullet \otimes E$. Since V is a global complete intersection the complex $K^\bullet$ is exact at all nonzero dimensions (cf. [9]) and the same is true for the complex $K^\bullet \otimes E$ because E is locally free, i.e. $\mathcal{H}^s(K^\bullet \otimes E) = 0$ if $s \neq 0$; therefore $'A_2^{r,s}$ degenerates and

$$\mathbf{H}^\bullet(K^\bullet \otimes E) = H^\bullet(P^n, \mathcal{H}^0(K^\bullet \otimes E)) = H^\bullet(V, E \otimes \boldsymbol{o}_V)\,.$$

Now consider the other spectral sequence:

$$A_1^{r,s} = H^s(P^n, K^r \otimes E) \Rightarrow H^{r+s}(V, E \otimes \boldsymbol{o}_V)\,.$$

Since

$$K^r \otimes E \begin{cases} = \text{a direct sum of sheaves } \boldsymbol{o}(k) & \text{for } -p \leqslant r \leqslant 0 \\ = 0 \text{ otherwise} & \end{cases}$$

and since for a sheaf $\boldsymbol{o}(k)$ on P^n $H^s(P^n, \boldsymbol{o}(k)) = 0$ if $s \neq 0, n$ ([8], § 65) we see that

$$A_1^{r,s} = 0 \begin{cases} \text{if } s \neq 0, n \text{ and } r \text{ arbitrary} \\ \text{if } s = 0 \text{ or } n \text{ and } r < -p \text{ or } r > 0 \end{cases}$$

Hence from $p < n$ it follows that the only possibly nonzero term abutting to $H^0(V, E \otimes \boldsymbol{o}_V)$ is $A_1^{0,0}$ and therefore $A_\infty^{0,0} = H^0(V, E \otimes \boldsymbol{o}_V)$; it is clear that the edge homomorphism

$$H^0(P^n, E) = A_1^{0\,0} \to A_\infty^{0,0} = H^0(V, E \otimes \boldsymbol{o}_V)$$

is surjective and it coincides with the infinitesimal displacement map σ_0.

REMARK. – In the terminology of [2] (1.1) implies that the Hilbert scheme of P^n is smooth at the point corresponding to V.

The embedding of V in P^n gives an exact sequence of locally free sheaves on V

(1) $$0 \to \Theta_V \to \Theta_{P^n|V} \to N_V \to 0$$

where Θ_V is the sheaf of germs of vector fields on V and $\Theta_{P^n|V}$ is the sheaf theoretic restriction to V of the analogous sheaf on P^n; one has an induced « connecting homomorphism »

$$\delta\colon H^0(V, N_V) \to H^1(V, \Theta_V)\,.$$

(1.2) PROPOSITION. – *The triangle of linear maps*

$$\begin{array}{ccc} H^0(V, N_V) & \xrightarrow{\delta} & H^1(V, \Theta_V) \\ \sigma_0 \nwarrow & & \nearrow \\ & T_{M_\varepsilon, 0} & \end{array}$$

where ϱ is the Kodaira-Spencer map [5] *of the family* $(\mathcal{V}, M_\varepsilon)$, *is commutative.*

PROOF. – [5] (12.4) page 388.

(1.3) COROLLARY. – *The family* $(\mathcal{V}, M_\varepsilon)$ *is complete at* $0 \in M$ *if and only if the linear map* δ *is surjective.*

PROOF. – By (1.1) and (1.2) δ is surjective if and only if ϱ is; therefore the corollary is a consequence of the completeness theorem of KODAIRA and SPENCER [6].

(1.4) COROLLARY. – *If* $H^1(V, \Theta_{P^n|V}) = 0$ *the family* $(\mathcal{V}, M_\varepsilon)$ *is complete at* $0 \in M_\varepsilon$:

PROOF. – From the long cohomology sequence of (1) we deduce an exact sequence of groups

$$H^0(V, N_V) \xrightarrow{\delta} H^1(V, \Theta_V) \to H^1(V, \Theta_{P^n|V})\,.$$

The corollary is therefore a consequence of (1.3).

2. – Proof of the theorem.

We keep the notations of § 1. The completeness at $0 \in M_\varepsilon$ of the family $(\mathcal{V}, M_\varepsilon)$ is obviously a sufficient condition for every sufficiently small deformation of V to be again in P^n. Proposition (1.1) shows that $(\mathcal{V}, M_\varepsilon)$ contains all small deformation of V in P^n, hence the condition is also necessary. Since all fibres of $\mathcal{V}$ are global complete intersections of multidegree $\boldsymbol{d}$ we will prove the theorem if we will show that when $\dim V \geq 2$ the family $\mathcal{V}$ is complete at $0 \in M$ if and only if one of the conditions I), II) are satisfied.

Consider the two spectral sequences of hypercohomology of the complex $K^\bullet \underset{o}{\otimes} \Theta_{P^n}$:

$$B_1^{r,s} = H^s(P^n, K^r \otimes \Theta_{P^n}) \Rightarrow \mathbf{H}^\bullet(K^\bullet \otimes \Theta_{P^n})$$
$$'B_2^{r,s} = H^r(P^n, \mathcal{H}^s(K^\bullet \otimes \Theta_{P^n})) \Rightarrow \mathbf{H}^\bullet(K^\bullet \otimes \Theta_{P^n}) .$$

Since $K^\bullet$ is exact at nonzero dimensions and Θ_{P^n} is locally free we see that $\mathcal{H}^s(K^\bullet \otimes \Theta_{P^n}) = 0$ is $s \neq 0$, so that $'B_2^{r,s} = 0$ if $s \neq 0$; moreover $\mathcal{H}^0(K^\bullet \otimes \Theta_{P^n}) = \Theta_{P^n|V}$. Therefore the spectral sequence $'B_2^{r,s}$ degenerates and

$$\mathbf{H}^r(K^\bullet \otimes \Theta_{P^n}) = {}'B_1^{r,0} = H^r(V, \Theta_{P^n|V}) .$$

We now investigate the other spectral sequence

$$B_1^{-r,s} = H^s(P^n, \bigwedge^r E \otimes \Theta_{P^n}) =$$
$$= \bigoplus_{1 \leq j_1 < \ldots < j_r \leq p} H^s(P^n, \Theta_{P^n}(-d_{j_1} - \ldots - d_{j_r})) \Rightarrow H^{s-r}(V, \Theta_{P^n|V})$$

where we have denoted $\Theta_{P^n}(k) = \Theta_{P^n} \otimes \boldsymbol{o}(k)$ for any integer k. To compute these groups we use the well known exact sequence of locally free sheaves on P^n

(2) $$0 \to \boldsymbol{o} \to \boldsymbol{o}(1)^{(n+1)} \xrightarrow{\theta} \Theta_{P^n} \to 0$$

Identifying sections of $\boldsymbol{o}(1)$ with rational functions on C^{n+1}, homogeneous of degree 1, the map θ is given by

$$\theta(R_0, \ldots, R_n) = \sum_{i=0}^{m} R_i \, \partial/\partial x_i$$

if $R_0, R_1, \ldots, R_n$ are sections of $\boldsymbol{o}(1)$ ($x_0, \ldots, x_n$ being the coordinate functions on C^{n+1}), which defines a degree preserving derivation on the homogeneous coordinate ring, hence a derivation on P^n. By tensorization with $\boldsymbol{o}(k)$ we obtain, for each integer k, the exact sequence of sheaves

$$0 \to \boldsymbol{o}(k) \to \boldsymbol{o}(k+1)^{(n+1)} \to \Theta_{P^n}(k) \to 0$$

and the long cohomology sequence provides us with the following exact sequence of groups, for each $s \geqslant 0$:

$$(3)_s \qquad H^s(P, \mathbf{o}(k)) \to H^s(P^n, \mathbf{o}(k+1))^{(n+1)} \to H^s(P^n, \Theta_{P^n}(k)) \to \\ \to H^{s+1}(P^n, \mathbf{o}(k)) \to H^{s+1}(P^n, \mathbf{o}(k+1))^{(n+1)} .$$

From $(3)_s$ we deduce

$$(4) \qquad H^s(P^n, \Theta_{P^n}(k)) = 0 \quad \text{for each integer } k \text{ if } 1 \leqslant s \leqslant n-2$$

because $H^s(P^n, \mathbf{o}(k)) = H^{s+1}(P^n, \mathbf{o}(k)) = 0$ for such values of s [8]. We can distinguish two cases:

(i) $p \leqslant n-3$. In this case $H^1(V, \Theta_{P^n|V}) = 0$ and the family $(\mathcal{V}, M_\varepsilon)$ is complete at 0 by (1.4). Infact the terms abutting to $H^1(V, \Theta_{P^n|V})$ are of the form $B^{-r,r+1}$, $0 \leqslant r \leqslant p$, and they are all zero by (4) because $p \leqslant n-3$.

(ii) $p = n-2$. In this case the only possibly nonzero term abutting to $H^1(V, \Theta_{P^n|V})$ is

$$B_1^{-n+2,n-1} = H^{n-1}(P^n, \bigwedge^{n-2} E^* \otimes \Theta_{P^n}) = H^{n-1}(P^n, \Theta_{P^n}(-d_1 - \dots - d_{n-2})) .$$

To compute it we use the sequence $(3)_{n-1}$ with $k = -d_1 - \dots - d_{n-2}$.

$$0 \to H^{n-1}(P^n, \Theta_{P^n}(-d_1 - \dots - d_{n-2})) \to \\ \to H^n(P^n, \mathbf{o}(-d_1 - \dots - d_{n-2})) \to H^n(P^n, \mathbf{o}(-d_1 - \dots - d_{n-2} + 1))^{(n+1)} .$$

By Serre duality this is dual to the following exact sequence of C-vector spaces:

$$\xleftarrow{t} H^0(P^n, \mathbf{o}(d_1 + \dots + d_{n-2} - n - 2))^{(n+1} \\ \leftarrow H^0(P^n, \mathbf{o}(d_1 + \dots + d_{n-2} - n - 1)) \xleftarrow{t} \\ 0 \leftarrow H^{n-1}(P^n, \Theta_{P^n}(-d_1 - \dots - d_{n-2}))^V \leftarrow$$

where the map t is defined by

$$t(g_0, \dots, g_n) = \sum_{i=0}^n x_i g_i \qquad \bigl(g_i \in H^0(P_n, \mathbf{o}(d_1 + \dots + d_{n-2} - n - 2))\bigr) .$$

It is clear that t is surjective if and only if $d_1 + \dots + d_{n-2} \neq n+1$.

We deduce that $H^1(V, \Theta_{P^n|V}) = B_1^{-n+2,n-1} = 0$ if condition (II) of the theorem is satisfied, and by (1.4) the family $(\mathcal{V}, M_\varepsilon)$ is complete at 0 in that case.

Therefore we see that $(\mathcal{V}, M_\varepsilon)$ is complete at 0 if one of the conditions I), II) is satisfied. To conclude the proof we have to show that those are the only cases when we have the completeness. This is a consequence of (1.3) and of the following:

(2.1) PROPOSITION. – *Let V be a global complete intersection in P^n of multidegree $\mathbf{d} = (d_1, \ldots, d_{n-2})$ and assume that one of the following is true:*

$$n = 3 \quad \textit{and} \quad \mathbf{d} = (4)$$

$$n = 4 \quad \textit{and} \quad \mathbf{d} = (2, 3)$$

$$n = 5 \quad \textit{and} \quad \mathbf{d} = (2, 2, 2)$$

Then

$$\dim H^1(V, \Theta_V) = 20 \qquad \textit{and} \qquad \dim\left(H^0(V, N_V)\right) = 19 .$$

PROOF. – V is a regular surface with trivial canonical bundle, i.e. V is a K-3 surface [10]. Infact $H^1(V, \boldsymbol{o}_V) = 0$ is proposition 5, § 78 of [8] and $K_V = \boldsymbol{o}_V$ follows at once from the adjunction formula [7]. The proposition is now a consequence of [10], chapter IX § 4.

BIBLIOGRAPHY

[1] F. GORI, *Sulle deformazioni delle varietà algebriche intersezioni complete nonsingolari di codimensione* 2, Ann. Scuola Norm. Sup. Pisa, **24** (1970), pp. 103-110.

[2] A. GROTHENDIECK, *Techniques de construction et théorèmes d'existence en géométrie algébrique*, Sem. Bourbaki, 1960/61, exp. 221, Benjamin, New York.

[3] A. GROTHENDIECK, *Elements de géométrie algébrique*, Inst. Hautes Études Sci. Publ. Math., **14** (1961).

[4] K. KODAIRA, *A theorem of completeness of characteristic systems for analytic families of compact submanifolds of complex manifolds*, Ann. of Math., **75** (1962), pp. 146-162.

[5] K. KODAIRA - D. C. SPENCER, *On deformations of complex analytic structures, I, II*, Ann. of Math., **67** (1958), pp. 328-466.

[6] K. KODAIRA - D. C. SPENCER, *A theorem of completeness for complex analytic fibre spaces*, Acta Math., **400** (1958), pp. 281-294.

[7] F. LAZZERI, *Some concepts of algebraic geometry*, Schr. Math. Inst. Univ. Münster, (2), **2** (1970).

[8] J. P. SERRE, *Fasceaux algébriques cohérents*, Ann. of Math., **64** (1955), pp. 197-268.

[9] J. P. SERRE, *Algébre locale et multiplicitès*, Lecture Notes in Math. n. 11, Springer, Berlin, 1965.

[10] I. R. SAFAREVIC, *Algebraic surfaces*, Proc. Steklov Inst. Math., **75** (1965).

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