A smoothing criterion for families of curves

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Abstract

We study some of the general principles underlying the geometry of families of nodal curves. In particular: i) we prove a smoothing criterion for nodal curves in a given family, ii) we derive from it the existence of nodal curves on a general K3 surface according to Mumford.

1 Introduction

This aim of this paper is to give a clear statement and proof of a smoothing result for families of nodal curves which has been used, and sometimes proved, in several special cases (see e.g. [3], [9] and [1], §23) and which *vox populi* considers to be true more or less by obvious general reasons. It states in precise terms that *in a family of projective curves with at most nodes as singularities the locus of δ-nodal curves, if non-empty, has codimension $\leq \delta$ and if equality holds then the family contains smooth curves among its fibres* (Theorem 6.3).

The reason for writing down a version of its proof is that the principle underlying this result is very likely to hold in more general situations like, just to mention a few: a) for families of projective curves with more complicated singularities than just nodes; b) for families of surfaces with ordinary double points.

This expectation calls for a transparent explanation of why the theorem holds, which could serve as a starting point for its generalizations. This is what Theorem 6.3 hopefully does.

As an illustration of the applications of Theorem 6.3 we give a proof of the well known (see [1], §23) existence of irreducible nodal curves with any number of nodes between 0 and the maximum on a general primitively polarized K3 surface.

2 Multiple-point schemes

All schemes will be defined over $\mathbf{k} = \bar{\mathbf{k}}$ of characteristic 0, and noetherian.

Definition 2.1 *A morphism $f : X \to Y$ is called* curvilinear *if the differential*

$$f^*\Omega^1_Y(x) \to \Omega^1_X(x)$$

has corank ≤ 1 at every point $x \in X$.

A more manageable condition is given by the following:

Proposition 2.2 *(i) A morphism $f : X \to Y$ is curvilinear if and only if every point $x \in X$ has an open neighborhood U such that the restriction $f_{|U}$ factors through an embedding of U into the affine Y-line $\mathbf{A}^1_Y$.*

(ii) The property of being curvilinear for a morphism is invariant under base change.

Proof. (i) See [6], Prop. 2.7, p. 12. (ii) follows immediately from (i) or from the definition. □

Let $f : X \to Y$ be a finite morphism. We can define *the first iteration* $f_1 : X_2 \to X$ *of* f as follows:

$$X_2 = I\!P(\mathcal{I}_\Delta), \qquad f_1 : X_2 \to X \times_Y X \xrightarrow{p_2} X$$

where $\Delta \subset X \times_Y X$ is the diagonal. X_2 is the so-called *residual scheme of* Δ; its properties are described in [5], §2. Moreover we define:

$$N_r(f) = \mathrm{Fitt}^Y_{r-1}(f_*\mathcal{O}_X), \qquad M_r = f^{-1}(N_r(f))$$

(see [10], p. 200). $N_r(f)$ is called *the r-th multiple point scheme* of f. We will need the following:

Proposition 2.3 *If f is finite and curvilinear then f_1 is also finite and curvilinear and*

$$M_r(f) = N_{r-1}(f_1)$$

Proof. See [6], Lemma 3.9 and Lemma 3.10. □

Proposition 2.4 *Assume that $f : X \to Y$ is finite and curvilinear, local complete intersection of codimension 1. Assume moreover that f is birational onto its image and that X and Y are local complete intersections and pure-dimensional. Then:*

(i) *$f_1 : X_2 \to X$ is a local complete intersection of codimension 1, finite and curvilinear, birational onto its image and X_2 is a pure-dimensional local complete intersection.*

(ii) *Each component of $M_r(f)$ (resp. $N_r(f)$) has codimension at most $r-1$ in X (resp. at most r in Y).*

(iii) *Assume moreover that Y is nonsingular. If $N_r(f) \neq \emptyset$ and has pure codimension r in Y for some $r \geq 2$, then $N_s(f) \neq \emptyset$ and has pure codimension s in Y for all $1 \leq s \leq r-1$. Moreover:*

$$N_r(f) \subset \overline{N_{r-1}(f) \backslash N_r(f)} \subset \cdots \subset \overline{N_1(f) \backslash N_2(f)} \tag{1}$$

Proof. (i) See [6], Lemma 3.10. (ii) See [6], Theorem 3.11.

(iii) by induction on r. The assumption that f is birational onto its image implies that $N_1(f)$ has pure codimension 1 and that $N_2(f)$ has pure codimension 2 in Y ([6] , Prop. 3.2(ii)). Therefore the dimensionality assertion is true for $r = 2$. Assume $r \geq 3$ and that $N_r(f)$ has pure codimension r in Y. Then $N_{r-1}(f_1) = M_r(f)$ has pure codimension $r-1$ in X. By part (i) we can apply the inductive hypothesis to f_1: it follows that $M_s(f) = N_{s-1}(f_1) \neq \emptyset$ and has pure codimension $s-1$ in X for all $1 \leq s-1 \leq r-2$. This implies that $N_s(f) \neq \emptyset$ and has pure codimension s in Y for all such s. The chain of inclusions (1) is a consequence of the fact that the non-emptyness of all the $N_s(f)$'s is local around every point of $N_r(f)$ because the argument holds for the restriction of f above an arbitrary open subset of Y. □

Assume now that $f : X \to Y$ is finite and projective with X, Y algebraic. Then we have two stratifications of Y. The first one is defined by the multiple-point schemes of f:

$$M : \coprod_{r \geq 0} (N_r(f) \backslash N_{r+1}(f)) \to Y$$

The second one is the flattening stratification of the sheaf $\mathcal{O}_X$:

$$\Phi : \coprod_{i \geq 0} W_i \to Y$$

Lemma 2.5 *If $f : X \to Y$ is a finite and projective morphism of algebraic schemes, then the stratifications M and Φ coincide.*

Proof. Recalling that a finite morphism $g : V \to U$ is flat if and only if $g_*\mathcal{O}_V$ is locally free, the strata W_i of Φ are indexed by the nonnegative integers, corresponding to the ranks of the locally free sheaves $f_*(\mathcal{O}_{X|f^{-1}(W_i)}) = (f_*\mathcal{O}_X)_{|W_i}$ (see [10], Note 6 p. 205 for this equality). On the other hand the stratification M is the one defined by the sheaf $f_*\mathcal{O}_X$ in the sense considered in [10], Theorem 4.2.7, p. 199, and its strata are also characterized by the fact that $f_*\mathcal{O}_X$ has a locally free restriction to each of them. Therefore M and Φ are equal. $\square$

Remark 2.6 Assume that $f : X \to Y$ is projective, finite and birational onto its image. Then Zariski's Main Theorem implies that its image $f(X) = N_1(f)$ cannot be normal unless f is an embedding, i.e. unless $N_r(f) = \emptyset$ for all $r \geq 2$. This means that, in presence of higher multiple point schemes, we must expect that $f(X)$ has non-normal singularities.

Definition 2.7 *Let $q : \mathcal{C} \to B$ be a flat family of projective curves. Assume that all the fibres of q are reduced curves having locally planar singularities, and let T^1_q be the first relative cotangent sheaf of q. Then $T^1_q = \mathcal{O}_Z$ for a closed subscheme $Z \subset \mathcal{C}$. We will call Z the* critical scheme *of q.*

It is well known and easy to show that, under the conditions of the definition, Z is finite over B and commutes with base change ([10], Lemma 4.7.5 p. 258). It is supported on the locus where q is not smooth. We will be interested in the multiple-point schemes of $f : Z \to B$, the restriction of q to Z.

Example 2.8 *Planar double point singularities* are those l.c.i. curve singularities whose local ring $\mathcal{O}_p$ has as completion:

$$\widehat{\mathcal{O}}_p \cong \mathbf{k}[[x,y]]/(y^2 + x^m), \qquad \text{for some } m \geq 2$$

The semiuniversal deformation of $\widehat{\mathcal{O}}_p$ is

$$\mathbf{k}[[t_1, \ldots, t_{m-1}]] \longrightarrow \frac{\mathbf{k}[[t_1,\ldots,t_{m-1},x,y]]}{(y^2+x^m+t_1x^{m-2}+\cdots+t_{m-2}x+t_{m-1})} \tag{2}$$

We will denote by $\pi : \mathcal{C} \to B$ the corresponding family of schemes, by $Z \subset \mathcal{C}$ the critical scheme of π, and by $f : Z \to B$ the restriction of π.

The special cases of nodes, cusps and tacnodes are respectively:

$$\widehat{\mathcal{O}}_p \cong \begin{cases} \mathbf{k}[[x,y]]/(x^2+y^2) & \text{(node)} \\ \mathbf{k}[[x,y]]/(y^2+x^3) & \text{(cusp)} \\ \mathbf{k}[[x,y]]/(y^2+x^4) & \text{(tacnode)} \end{cases} \tag{3}$$

The corresponding semiuniversal deformations are:

$$\mathbf{k}[[t]] \to \mathbf{k}[[t,x,y]]/(x^2+y^2+t) \qquad \text{(node)}$$

$$\mathbf{k}[[u_1,u_2]] \to \mathbf{k}[[u_1,u_2,x,y]]/(y^2+x^3+u_1x+u_2) \qquad \text{(cusp)}$$

$$\mathbf{k}[[v_1,v_2,v_3]] \to \mathbf{k}[[v_1,v_2,v_3,x,y]]/(y^2+x^4+v_1x^2+v_2x+v_3) \quad \text{(tacnode)}$$

For the deformation (2) of the general planar double point, Z is defined by:

$$\frac{\mathbf{k}[[t_1,\dots,t_{m-1},x]]}{(x^m+t_1x^{m-2}+\cdots+t_{m-2}x+t_{m-1}, mx^{m-1}+(m-2)t_1x^{m-3}+\cdots+t_{m-2})}$$

In the special cases Z is defined by the following family:

$$\mathbf{k}[[t,x,y]]/(x^2+y^2+t,x,y) \cong \mathbf{k} \qquad \text{(node)}$$

$$\mathbf{k}[[u_1,u_2,x]]/(x^3+u_1x+u_2, 3x^2+u_1) \qquad \text{(cusp)}$$

$$\mathbf{k}[[v_1,v_2,v_3,x]]/(x^4+v_1x^2+v_2x+v_3, 4x^3+2v_1x+v_2) \quad \text{(tacnode)}$$

All these are clearly finite and curvilinear and moreover Z is nonsingular of relative dimension -1 over B.

3 The generic deformation of a curve

Let C be a connected, reduced projective local complete intersection (l.c.i.) curve of arithmetic genus $p_a(C)$. Then we have:

$$\mathrm{Ext}^2(\Omega^1_C, \mathcal{O}_C) = 0$$

and $\mathrm{Def}_C(\mathbf{k}[\epsilon]) = \mathrm{Ext}^1(\Omega^1_C, \mathcal{O}_C)$ so that Def_C is unobstructed and C has a versal formal deformation

$$\mathcal{X} \longrightarrow \mathrm{Specf}\,(\mathbf{k}[[z_1,\dots,z_n]])$$

where

$$n = \dim_{\mathbf{k}} \left[\mathrm{Ext}^1(\Omega^1_C, \mathcal{O}_C) \right] = 3p_a(C) - 3 + h^0(T_C)$$

and where $T_C = \mathcal{H}om(\Omega^1_C, \mathcal{O}_C)$ (see [10]). By Grothendieck's effectivity theorem ([10], Theorem 2.5.13, p. 82) $\mathcal{X}$ is the formal completion of a unique scheme projective and flat over $\mathcal{M} := \mathrm{Spec}(\mathbf{k}[[z_1, \ldots, z_n]])$, which we will also denote by $\mathcal{X}$. We thus obtain a deformation

$$\begin{tikzcd} C \arrow[r] \arrow[d] & \mathcal{X} \arrow[d, "\pi"] \\ \mathrm{Spec}(\mathbf{k}) \arrow[r] & \mathcal{M} \end{tikzcd}$$

which we call *the generic deformation of* C, conforming to the terminology introduced in [8], p. 64.

Let $p \in C$ be a closed singular point. Since $\mathrm{Ext}^2(\Omega_{\mathcal{O}_p/\mathbf{k}}, \mathcal{O}_p) = 0$, the local ring $\mathcal{O}_p = \mathcal{O}_{C,p}$ has a semiuniversal formal deformation $\widetilde{\mathcal{O}}_p$ which is an algebra over the smooth parameter algebra

$$A_p = \mathbf{k}[[t_1, \ldots, t_{r(p)}]]$$

The number of parameters is equal to

$$r(p) := \dim_{\mathbf{k}} \left[\mathrm{Ext}^1(\Omega_{\mathcal{O}_p/\mathbf{k}}, \mathcal{O}_p) \right]$$

because $\mathrm{Ext}^1(\Omega_{\mathcal{O}_p/\mathbf{k}}, \mathcal{O}_p) = T^1_{\mathcal{O}_p}$ is the first cotangent space of $\mathcal{O}_p$, which is naturally identified with the space of first order deformations of $\mathcal{O}_p$. We have an obvious restriction morphism of functors

$$\mathrm{Def}_C \to \mathrm{Def}_{\mathcal{O}_p}$$

which corresponds, by semiuniversality, to a homomorphism

$$\psi_p : A_p \to \mathbf{k}[[z_1, \ldots, z_n]]$$

inducing an isomorphism

$$\mathcal{O}_{\mathcal{X},p} \cong \widetilde{\mathcal{O}}_p \otimes_{A_p} \mathbf{k}[[z_1, \ldots, z_n]]$$

Putting all these local informations together we obtain a morphism of functors:

$$\Psi : \mathrm{Def}_C \to \prod_{p \in C} \mathrm{Def}_{\mathcal{O}_p}$$

which corresponds to a morphism we will denote with the same letter:

$$\Psi := \prod_{p\in C} \mathrm{Spec}(\psi_p) : \mathcal{M} \to \mathcal{M}_{\mathrm{lo}} = \mathrm{Spec}(\mathbf{k}[[t_1, \ldots, t_r]]) = \mathrm{Spec}(\widehat{\otimes}_p A_p) \tag{4}$$

where

$$r = h^0(C, Ext^1(\Omega^1_C, \mathcal{O}_C)) = h^0(T^1_C) = \sum_{p\in C} r(p)$$

Lemma 3.1 *The morphism Ψ is smooth. Therefore, up to a change of variables, it is dual to an inclusion*

$$\mathbf{k}[[t_1, \ldots, t_r]] \hookrightarrow \mathbf{k}[[t_1, \ldots, t_r, z_{r+1}, \ldots, z_{r+m}]]$$

Proof. Because of the smoothness of its domain, the smoothness of Ψ is equivalent to the surjectivity of its differential. But

$$d\Psi : \mathrm{Ext}^1(\Omega^1_C, \mathcal{O}_C) \to H^0(C, Ext^1(\Omega^1_C, \mathcal{O}_C))$$

is a hedge-homomorphism in the spectral sequence for Ext's, and it is surjective because $H^2(C, Ext^0(\Omega^1_C, \mathcal{O}_C)) = 0$. □

Remark 3.2 Lemma 3.1 is Proposition (1.5) of [4]. It holds for any reduced curve, without assuming that C is a l.c.i.. For obvious reasons the number m of extra variables z_j appearing in the statement of the lemma is

$$\begin{aligned} m = \dim(\ker(d\Psi)) &= h^1(C, Ext^0(\Omega^1_C, \mathcal{O}_C)) = h^1(C, T_C) \\ &= 3p_a(C) - 3 + h^0(T_C) - h^0(T^1_C) \end{aligned}$$

They correspond to the locus $t_1 = \cdots = t_r = 0$ in $\mathcal{M}$, parametrizing locally trivial deformations of C.

4 Local properties of families of curves

Consider a flat projective family of deformations of C:

$$\begin{tikzcd} C \arrow[r] \arrow[d] & \mathcal{C} \arrow[d, "\varphi"] \\ \mathrm{Spec}(\mathbf{k}) \arrow[r, "b"] & B \end{tikzcd} \tag{5}$$

Denote by $Z \subset \mathcal{C}$ the critical scheme, and by $f : Z \to B$ the restriction of φ:

$$\begin{tikzcd} Z \arrow[r, hook] \arrow[dr, "f"'] & \mathcal{C} \arrow[d, "\varphi"] \\ & B \end{tikzcd} \tag{6}$$

Let $\widehat{\mathcal{O}}_b$ the completion of the local ring $\mathcal{O}_{B,b}$, let $T := \mathrm{Spec}(\widehat{\mathcal{O}}_b)$ and denote by b the closed point of T too. Pulling back the family (5) by the morphism $h : T \to B$ induced by $\mathcal{O}_{B,b} \to \widehat{\mathcal{O}}_b$ we obtain a family

$$\begin{tikzcd} C \arrow[r] \arrow[d] & \widetilde{\mathcal{C}} \arrow[r, equal] \arrow[d, "\tilde{\varphi}"] & T \times_B \mathcal{C} \\ \mathrm{Spec}(\mathbf{k}) \arrow[r, "b"] & T & \end{tikzcd} \tag{7}$$

whose fibre over b is again C. Let

$$\begin{tikzcd} \widetilde{Z} \arrow[r, hook] \arrow[dr, "\tilde{f}"'] & \widetilde{\mathcal{C}} \arrow[d, "\tilde{\varphi}"] \\ & T \end{tikzcd}$$

be the critical scheme of $\tilde{\varphi}$.

By versality there is a morphism $\mu : T \to \mathcal{M}$ with uniquely determined differential such that $\widetilde{\mathcal{C}} = T \times_{\mathcal{M}} \mathcal{X}$, so that we also have the composition:

$$\Psi \circ \mu : T \xrightarrow{\mu} \mathcal{M} \xrightarrow{\Psi} \mathcal{M}_{\mathrm{lo}}$$

Infinitesimally μ and $\Psi \circ \mu$ can be described as follows. We have an exact sequence of locally free sheaves on $\mathcal{C}$:

$$0 \to \varphi^* \Omega^1_B \to \Omega^1_{\mathcal{C}} \to \Omega^1_{\mathcal{C}/B} \to 0$$

(see [10], Theorem D.2.8) which dualizes as:

$$\begin{tikzcd} 0 \arrow[r] & T_{\mathcal{C}/B} \arrow[r] & T_{\mathcal{C}} \arrow[r] & \varphi^* T_B \arrow[r, "u"] & Ext^1_{\mathcal{C}}(\Omega^1_{\mathcal{C}/B}, \mathcal{O}_{\mathcal{C}}) \arrow[r] \arrow[d, equal] & 0 \\ & & & & \mathcal{O}_Z & \end{tikzcd} \tag{8}$$

and this gives a global description of $T^1_\varphi = \mathcal{O}_Z$ as the structure sheaf of the critical scheme. If we push u down to B we obtain a factorization:

$$\begin{tikzcd}
T_B \arrow[rr, "\varphi_*(u)"] \arrow[dr] & & f_*\mathcal{O}_Z = f_* Ext^1_{\mathcal{C}}(\Omega^1_{\mathcal{C}/B}, \mathcal{O}_{\mathcal{C}}) \\
& Ext^1_\varphi(\Omega^1_{\mathcal{C}/B}, \mathcal{O}_{\mathcal{C}}) \arrow[ur] &
\end{tikzcd}$$

and at the point $b \in B$ this gives:

$$\begin{tikzcd}
T_bB \arrow[rr, "d(\Phi\circ\mu)"] \arrow[dr, "d\mu"'] & & {[f_*\mathcal{O}_Z](b) = H^0(T^1_C)} \\
& \mathrm{Ext}^1(\Omega^1_C, \mathcal{O}_C) \arrow[ur] &
\end{tikzcd}$$

Here we used the fact that the critical scheme commutes with base change for a l.c.i. morphism (generalizing [10], Lemma 4.7.5 p. 258). The right diagonal arrows are edge homomorphisms of the respective spectral sequences. Here we are especially interested in the differential of $\Phi\circ\mu$, so we will not insist in investigating $d\mu$. The map $d(\Phi\circ\mu)$ can be analyzed by means of the restriction of (8) to the fibre $C = \mathcal{C}(b)$:

$$\begin{tikzcd}
0 \arrow[r] & T_C \arrow[r] & T_{\mathcal{C}|C} \arrow[r] & T_bB \otimes_{\mathbf{k}} \mathcal{O}_C \arrow[r, "u(b)"] \arrow[d, equal] & T^1_C \arrow[r] & 0 \\
& & & N_{C/\mathcal{C}} & &
\end{tikzcd} \tag{9}$$

which gives:

$$d(\Phi\circ\mu) = H^0(u(b)) : T_bB \longrightarrow [f_*\mathcal{O}_Z](b) = H^0(T^1_C) \tag{10}$$

A typical example of a family (5) is when C is contained in a projective scheme X, B is the Hilbert scheme of X and φ is the universal family. In this case $T_bB = H^0(C, N_{C/X})$ and the map (10) is induced by the natural map of sheaves

$$N_{C/X} \to T^1_C$$

Proposition 4.1 *Assume that the fibres of φ have at most planar double point singularities. Then f is finite, projective and curvilinear. If $r = h^0(T^1_C)$ then $b \in N_r(f)\backslash N_{r+1}(f)$ and*

$$T_b[N_r(f)] = \ker[H^0(u(b))] \tag{11}$$

Proof. Z is supported at the singular points of the fibres of φ, therefore it is a closed subscheme of $\mathcal{X}$, projective over B, and having a zero-dimensional intersection with every fibre. This implies that f is finite and projective.

It suffices to prove curvilinearity locally above b and, since h is etale, it suffices to prove it for $\tilde{f}$. Locally around a point $z \in \widetilde{Z}$, $\tilde{\varphi}$ is the pullback of the semiuniversal deformation of a locally planar singularity. Since for such singularities taking the critical scheme commutes with base change ([10], Lemma 4.7.5 p. 258), it follows that Z is locally the pullback of one of the families described in Example 2.8, which are curvilinear.

Since $[f_*\mathcal{O}_Z](b) = H^0(T^1_C)$, then $b \in N_r(f)\backslash N_{r+1}(f)$ by definition of the support of $N_r(f)$. We are left to prove (11). Consider a tangent vector $\theta \in T_bB$ and the pullback of (6) over $\mathrm{Spec}(\mathbf{k}[\epsilon])$ via θ:

$$\begin{tikzcd} Z \otimes_{\mathbf{k}} \mathbf{k}[\epsilon] \arrow[r, hook] \arrow[dr, "f_\theta"] & \mathcal{C} \otimes_{\mathbf{k}} \mathbf{k}[\epsilon] \arrow[r] \arrow[d, "\varphi_\theta"] & \mathcal{C} \arrow[d, "\varphi"] \\ & \mathrm{Spec}(\mathbf{k}[\epsilon]) \arrow[r, "\theta"] & B \end{tikzcd}$$

The usual deformation-theoretic interpretation of the exact sequence (9) shows that a tangent vector $\theta \in T_bB$ is in $\ker[H^0(u(b))]$ if and only if φ_θ is a first order deformation of C which is trivial locally at every singular point. In turn this is equivalent to the flatness of f_θ (see [13], Lemma 3.3.7). But this means precisely that θ is a tangent vector to the stratum containing b of the flattening stratification of $\mathcal{O}_Z$ over B. By applying Lemma 2.5 we conclude. $\square$

5 The stratification of the generic deformation - nodal case

Consider the simplest case, in which C has nodes $p_1, \dots, p_\delta$ and no other singularities. Then

$$\mathcal{M}_{\mathrm{lo}} = \mathrm{Spec}(\mathbf{k}[[t_1, \dots, t_\delta]]), \qquad \mathcal{M} = \mathrm{Spec}(\mathbf{k}[[t_1, \dots, t_\delta, z_{\delta+1}, \dots, z_{\delta+m}]]$$

t_j is the parameter appearing in the versal deformation $x^2+y^2+t_j = 0$ of the j-th node. The union of the coordinate hyperplanes $t_1 \cdots t_\delta = 0$ is a normal crossing divisor defining a stratification

$$\mathcal{M}_{\mathrm{lo}} = \coprod_{r=0,\dots,\delta} \mathcal{V}^r \tag{12}$$

where $\mathcal{V}^r$ is the locally closed nonsingular subscheme supported on the set of points which belong to exactly r coordinate hyperplanes. The stratum $\mathcal{V}^r$ has pure codimension r and is in turn a disjoint union:

$$\mathcal{V}^r = \coprod_{1 \leq j_1 < \cdots < j_r \leq \delta} \mathcal{V}(j_1, \ldots, j_r)$$

where $\mathcal{V}(j_1, \ldots, j_r)$ is the locus of points where precisely the coordinates $j_1, \ldots, j_r$ vanish. Obviously $\overline{\mathcal{V}(j_1, \ldots, j_r)}$ is the linear subspace $V(t_{j_1}, \ldots, t_{j_r})$.

The stratification (12) pulls back to $\mathcal{M}$ to an analogous one, defined by the union of the coordinate hyperplanes $t_1 \cdots t_\delta = 0$, and having nonsingular strata:

$$\mathcal{M} = \coprod_{r=0,\ldots,\delta} \mathcal{V}^r(\pi) \tag{13}$$

with

$$\mathcal{V}^r(\pi) = \coprod_{1 \leq j_1 < \cdots < j_r \leq \delta} \mathcal{V}(j_1, \ldots, j_r)(\pi) \tag{14}$$

The $\mathcal{V}^r(\pi)$'s and the $\mathcal{V}(j_1, \ldots, j_r)(\pi)$'s are defined by the same conditions as the $\mathcal{V}^r$'s and the $\mathcal{V}(j_1, \ldots, j_r)$'s, and share with them the same properties of codimension and nonsingularity, being obtained from them by taking the cartesian product with $\mathrm{Spec}(\mathbf{k}[[z_{\delta+1}, \ldots, z_{\delta+m}]])$. The following statement is an obvious consequence of the previous analysis (compare also with [4], Corollary (1.9)).

Theorem 5.1 *Let $\mathcal{Z} \subset \mathcal{X}$ be the critical scheme of the generic deformation $\pi : \mathcal{X} \to \mathcal{M}$. Denote by $\phi : \mathcal{Z} \to \mathcal{M}$ the restriction of π. Then:*

(i) *(13) is the flattening stratification of ϕ.*

(ii) *Each stratum $\mathcal{V}^r(\pi)$ is nonsingular of pure codimension r.*

(iii) *Let R be a complete local $\mathbf{k}$-algebra R with $R/m_R \cong \mathbf{k}$ and let*

$$\mathbf{k}[[t_1, \ldots, t_\delta, z_{\delta+1}, \ldots, z_{\delta+m}]] \to R$$

be a local homomorphism. The induced morphism $\mathrm{Spec}(R) \to \mathcal{M}$ factors through $\overline{\mathcal{V}(j_1, \ldots, j_r)(\pi)}$ if and only if the flat family of deformations of C:

$$\begin{array}{ccc} C & \longrightarrow & \mathrm{Spec}(R) \times_{\mathcal{M}} \mathcal{X} \\ \downarrow & & \downarrow \\ \mathrm{Spec}(\mathbf{k}) & \longrightarrow & \mathrm{Spec}(R) \end{array}$$

is locally trivial at the points $p_{j_1}, \dots, p_{j_r}$.

6 Families of nodal curves

In this section we will consider a flat projective family of curves $\varphi : \mathcal{C} \to B$ parametrized by a scheme B. *We will assume that all fibres of φ have at most nodes as singularities.* Let $f : Z \to B$ be the restriction of φ to its critical scheme $Z \subset \mathcal{C}$. By Proposition 4.1 f is projective, finite and curvilinear and we can consider the multiple-point schemes of f.

Definition 6.1 *For any $r \geq 0$ the r-th stratum $N_r(f) \backslash N_{r+1}(f)$ of the multiple-point stratification of f is called* the Severi variety of curves with r nodes *of the family φ, and denoted by $\mathcal{V}^r_\varphi$. The stratification*

$$\coprod_r \mathcal{V}^r_\varphi$$

is called the Severi stratification *of B.*

Lemma 6.2 *Let $\lambda : Y \to B$ be any morphism, and let:*

$$\begin{tikzcd} \mathcal{C}_Y \arrow[r] \arrow[d, "\psi"] & \mathcal{C} \arrow[d, "\varphi"] \\ Y \arrow[r, "\lambda"] & B \end{tikzcd}$$

be the induced cartesian diagram. Then

$$\lambda^{-1}(\mathcal{V}^r_\varphi) = \mathcal{V}^r_\psi$$

foa all $r \geq 0$. In other words forming the Severi stratification commutes with base change.

Proof. The lemma is a rephrasing of the fact that the critical scheme and the multiple-point stratification both commute with base change. □

The following result is a generalized version of Theorem 2.2 of [3].

Theorem 6.3 *Let $\varphi : \mathcal{C} \to B$ be a flat projective family of curves having at most nodes as singularities, with B algebraic and integral. Assume that $b \in B$ is a* **k***-rational point such that the fibre $C = \mathcal{C}(b)$*

has precisely $\delta \geq 0$ nodes and no other singularities, so that $b \in \mathcal{V}_{\varphi}^{\delta}$. Then

$$\operatorname{codim}_B(\mathcal{V}_{\varphi}^{\delta}) \leq \delta$$

Assume moreover $\delta \geq 1$ and that one of the following conditions is satisfied:

(i) *$\mathcal{V}_{\varphi}^{\delta}$ has pure codimension δ in B at b.*

(ii) *b is a nonsingular point of B and the map (10) is surjective.*

Then there is a neighborhood U of b where $\mathcal{V}_{\varphi}^{r}$ is non-empty and of pure codimension r for all $0 \leq r \leq \delta$. Moreover in case (ii) all the Severi varieties $\mathcal{V}_{\varphi}^{r}$ are nonsingular in a neighborhood of $b \in U$. In particular the general fibre of φ is nonsingular.

Proof. Consider the scheme $T = \operatorname{Spec}(\widehat{\mathcal{O}}_{B,b})$ and the deformation (7) induced by φ. Since multiple-point stratifications commute with base change, the Severi stratification of φ pulls back to the Severi stratification of $\tilde{\varphi}$. Since $T \to B$ is an etale neighborhood of b, the hypothesis and the conclusion are valid on T if and only if they are valid on B. Hence it suffices to prove the theorem for $\tilde{\varphi}$. We have an induced morphism

$$\Phi \circ \mu : T \to \mathcal{M}_{\mathrm{lo}}$$

whose differential is (10). The Severi stratification $\coprod_r \mathcal{V}_{\tilde{\varphi}}^{r}$ is obtained by pulling back the stratification $\coprod_r \mathcal{V}^{r}$ of $\mathcal{M}_{\mathrm{lo}}$ by $\Phi \circ \mu$. Since this stratification is defined by the regular system of parameters $t_1, \ldots, t_{\delta}$, the Severi stratification is defined by

$$(\Phi \circ \mu)^*(t_1), \ldots, (\Phi \circ \mu)^*(t_{\delta}) \tag{15}$$

This implies in particular that $\mathcal{V}_{\tilde{\varphi}}^{\delta}$ cannot have codimension larger than δ. If $\mathcal{V}_{\tilde{\varphi}}^{\delta}$ has codimension δ then (15) is a regular sequence, and therefore each stratum $\mathcal{V}_{\tilde{\varphi}}^{r}$ is non-empty and of pure codimension r. This proves the theorem in case (i).

The hypothesis that the map (10) is surjective and that B is nonsingular at b implies that $\Phi \circ \mu$ is smooth. It follows that (15) is a regular sequence, and moreover that all the strata of the Severi stratification are nonsingular. $\square$

Definition 6.4 *If $\mathcal{V}_{\varphi}^{\delta}$ is nonsingular and of codimension δ at a* **k***-rational point b we say that $\mathcal{V}_{\varphi}^{\delta}$ is* regular *at b; otherwise we say that*

$\mathcal{V}_{\varphi}^{\delta}$ *is* superabundant at b. *If an irreducible component* $\mathcal{V}$ *of* $\mathcal{V}_{\varphi}^{\delta}$ *is regular at all its* $\mathbf{k}$*-rational points then* $\mathcal{V}$ *is called* regular. *Otherwise* $\mathcal{V}$ *is called* superabundant.

With this terminology, we can say that Theorem 6.3(ii) gives a criterion of local regularity for the Severi varieties $\mathcal{V}_{\varphi}^{r}$.

When one needs to apply Theorem 6.3, in practise it often happens that one can construct a subvariety $Y \subset \mathcal{V}_{\varphi}^{\delta}$ of codimension δ in B such that the restriction $\varphi_Y : Y \times_B \mathcal{C} \to Y$ is a family of *reducible* curves having δ nodes and not contained in a larger such family. In order to apply Theorem 6.3 one would need to know that $\dim(Y) = \dim(\mathcal{V}_{\varphi}^{\delta})$, i.e. that the family φ_Y is not contained in a larger family generically parametrizing *irreducible* curves having the same number δ of nodes. This is guaranteed by the following useful result, classically called *splitting principle* ("principio di spezzamento").

Proposition 6.5 *Let* $\varphi : \mathcal{C} \to B$ *be a flat projective family of curves, with* B *a normal connected algebraic scheme. Suppose that all the geometric fibres of* φ *have precisely* δ *nodes and no other singularities for some* $\delta \geq 0$. *Then the number of irreducible components of the geometric fibres of* φ *is constant.*

Proof. Since B is normal and all fibres of φ have the same geometric genus we can apply [12], Theorem 1.3.2, to normalize simultaneously the fibres of φ. We obtain a commutative diagram:

$$\begin{tikzcd} \overline{\mathcal{C}} \arrow[rr] \arrow[dr, "\bar{\varphi}"'] & & \mathcal{C} \arrow[dl, "\varphi"] \\ & B & \end{tikzcd}$$

where $\bar{\varphi}$ is a smooth projective family of curves and for each $\mathbf{k}$-rational point $b \in B$ the induced morphism $\overline{\mathcal{C}}(b) \to \mathcal{C}(b)$ is the normalization. The number of irreducible components of $\mathcal{C}(b)$ is the same as the number of connected components of $\overline{\mathcal{C}}(b)$: thus it suffices to prove that this is constant.

Since any two $\mathbf{k}$-rational points of B can be joined by a chain of algebraic integral curves, it suffices to prove what we need in the case when B is an integral curve. We can even assume that B is a nonsingular algebraic curve by pulling back to its normalization if necessary. Considering the Stein factorization of $\bar{\varphi}$:

$$
\begin{tikzcd}
\overline{\mathcal{C}} \arrow[d, "h"'] \arrow[dr, "\bar{\varphi}"] & \\
Z \arrow[r, "\psi"'] & B
\end{tikzcd}
$$

we see that the number of connected components of $\overline{\mathcal{C}}(b)$ equals the degree of the 0-dimensional scheme $Z(b)$, and this is constant. In fact $Z = \mathrm{Spec}(\bar{\varphi}_*\mathcal{O}_{\overline{\mathcal{C}}})$ and $\bar{\varphi}_*\mathcal{O}_{\overline{\mathcal{C}}}$ is locally free because it is torsion free and B is a nonsingular curve. $\Box$

In §7 we will give an application of Proposition 6.5 .

Example 6.6 (The Severi varieties of a linear system) This example is the most basic one. Consider a globally generated line bundle L on a projective connected surface S and the diagram

$$
\begin{tikzcd}
\mathcal{C} \arrow[r, hook] \arrow[d, "\pi"] & S \times |L| \arrow[dl] \\
|L| &
\end{tikzcd}
\tag{16}
$$

where $|L| = I\!P(H^0(L))$ is the complete linear system defined by L,

$$\mathcal{C} = \{(x, [C]) : x \in C\}$$

is the tautological family, π is the second projection. We have

$$T_{[C]}(|L|) = H^0(C, N_{C/S}) = H^0(C, \mathcal{O}_C(C))$$

for any curve C in the linear system $|L|$. The map (10) is the natural:

$$H^0(C, N_{C/S}) \to H^0(T^1_C)$$

so that its kernel, the space of first order locally trivial deformations of C, is identified with $T_{[C]}N_r(f)$, where $r = h^0(T^1_C)$.

In general π has fibres with arbitrary planar singularities, so that, in order to apply the previous theory, we will need to restrict π above a conveniently chosen open subset $B \subset |L|$ parametrizing curves with at most nodes as singularities. We will denote by $\mathcal{V}^\delta_L$ the corresponding Severi varieties; they are locally closed subschemes of $|L|$ and will be called *Severi varieties of the linear system* $|L|$. In the case $S = I\!P^2$

and $L = \mathcal{O}(d)$ we obtain the classical Severi varieties of plane nodal curves of degree d; they are denoted by $\mathcal{V}_d^\delta$. It is well known that all irreducible components of the $\mathcal{V}_d^\delta$'s are regular (see [10]). The same is true for the Severi varieties of globally generated L's on a K3 surface [11] and on an abelian surface [7].

7 Existence of nodal curves on $K3$ surfaces

As a simple application of Theorem 6.3 one can prove the existence of nodal irreducible curves with any number of nodes between 0 and the maximum, belonging to the primitive polarization on a general $K3$ surface.

Theorem 7.1 *A general primitively polarized algebraic $K3$ surface (X, L) of genus $g \geq 2$ contains irreducible nodal curves of geometric genus $g-\delta$ for every $0 \leq \delta \leq g$. Equivalently, $\mathcal{V}_L^\delta \neq \emptyset$ for all $0 \leq \delta \leq g$.*

Proof. We start from a Kummer surface X_0 and a reducible nodal curve $C_0 \subset X_0$ such that

$$C_0 = G + S$$

where G, S are two nonsingular rational curves meeting transversally at $g+1$ points. The existence of X_0 and C_0 is explained for example in [1], p. 366. Since $C_0^2 = 2g-2$, we have $\dim(|C_0|) = g$. There is a pair $(\mathcal{X}, \mathcal{L})$ and a projective morphism $\psi : \mathcal{X} \to U$ defining a family of polarized $K3$ surfaces, with U nonsingular of dimension 19, with isomorphic Kodaira-Spencer map at every point, and such that $(X_0, \mathcal{O}(C_0)) \cong (\mathcal{X}(u_0), \mathcal{L}(u_0))$ for some point $u_0 \in U$ (see [1], Thm. VIII, 7.3 and p. 366). Letting $V = \mathbb{P}(\psi_* \mathcal{L})$ and letting $\tau : V \to U$ be the projection, we obtain a diagram

$$\begin{tikzcd} \mathcal{C} \arrow[r] \arrow[d, "\varphi"] & \mathcal{X} \arrow[d, "\psi"] \\ V \arrow[r, "\tau"] & U \end{tikzcd}$$

where τ is smooth, $\dim(V) = g + 19$, and φ is a family of curves of aritmetic genus g, such that, for every $u \in U$, $\tau^{-1}(u) = |\mathcal{L}(u)|$, and the restriction

$$\begin{array}{ccc} \mathcal{C} \times_V |\mathcal{L}(u)| & \hookrightarrow & \mathcal{X}(u) \times |\mathcal{L}(u)| = |\mathcal{L}(u)| \times_U \mathcal{X} \\ \downarrow & & \\ |\mathcal{L}(u)| & & \end{array}$$

is the g-dimensional tautological family parametrized by $|\mathcal{L}(u)|$. Let $v_0 \in \tau^{-1}(u_0)$ be such that $\mathcal{C}(v_0) = C_0$. Then, after restricting to a neighborhood W of v_0, we may assume that all fibres of φ have at most nodes as singularities. Consider the Severi variety $\mathcal{V}_\varphi^{g+1} \subset W$. By Lemma 6.2 we have:

$$\tau^{-1}(u_0) \cap \mathcal{V}_\varphi^{g+1} = \mathcal{V}_{\mathcal{L}(u_0)}^{g+1}$$

which is zero-dimensional, and the reducible curve C_0 does not extend to a surface $X(u)$ for a general $u \in U$ because $\mathcal{L}(u)$ is primitive. Therefore, by applying Proposition 6.5 we deduce that $\mathcal{V}_\varphi^{g+1}$ has codimension (at least and therefore equal to) $g+1$ in W. By applying Theorem 6.3, we deduce that $\mathcal{V}_\varphi^\delta \neq \emptyset$ for all $0 \leq \delta \leq g$. But since $\overline{\mathcal{V}_\varphi^\delta} \supset V_\varphi^{g+1}$, a general curve in $\mathcal{V}_\varphi^\delta$ must be irreducible, having less nodes that C_0 and degenerating to C_0, which has two nonsingular irreducible components. Moreover $\mathcal{V}_\varphi^\delta$ has pure dimension $19 + g - \delta$ (Theorem 6.3(ii)) and $\mathcal{V}_\varphi^\delta \cap |\mathcal{L}(u)| = \mathcal{V}_{\mathcal{L}(u)}^\delta$ for $u \in U$ (Lemma 6.2). □

Note that the Severi varieties $\mathcal{V}_L^\delta$ whose existence is proved in Theorem 7.1 are regular [11]. Therefore in particular Theorem 7.1 asserts the existence of nodal rational curves and of a 1-dimensional family of elliptic curves in $|L|$, thus recovering Theorem 23.1, p. 365, of [1].

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