A smoothing criterion for families of curves

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Abstract

We study some of the general principles underlying the geometry of families of nodal curves. In particular: i) we prove a smoothing criterion for nodal curves in a given family, ii) we derive from it the existence of nodal curves on a general K3 surface according to Mumford.

1 Introduction

This aim of this paper is to give a clear statement and proof of a smoothing result for families of nodal curves which has been used, and sometimes proved, in several special cases (see e.g. [3], [9] and [1], §23) and which vox populi considers to be true more or less by obvious general reasons. It states in precise terms that in a family of projective curves with at most nodes as singularities the locus of $\delta$-nodal curves, if non-empty, has codimension $\leq \delta$ and if equality holds then the family contains smooth curves among its fibres (Theorem 6.3).

The reason for writing down a version of its proof is that the principle underlying this result is very likely to hold in more general situations like, just to mention a few: a) for families of projective curves with more complicated singularities than just nodes; b) for families of surfaces with ordinary double points.

This expectation calls for a transparent explanation of why the theorem holds, which could serve as a starting point for its generalizations. This is what Theorem 6.3 hopefully does.

As an illustration of the applications of Theorem 6.3 we give a proof of the well known (see [1], §23) existence of irreducible nodal curves with any number of nodes between 0 and the maximum on a general primitively polarized K3 surface.
2 Multiple-point schemes

All schemes will be defined over \(k = \bar{k}\) of characteristic 0, and noetherian.

**Definition 2.1** A morphism \(f : X \to Y\) is called curvilinear if the differential
\[
f^*\Omega^1_Y(x) \to \Omega^1_X(x)
\]
has corank \(\leq 1\) at every point \(x \in X\).

A more manageable condition is given by the following:

**Proposition 2.2** (i) A morphism \(f : X \to Y\) is curvilinear if and only if every point \(x \in X\) has an open neighborhood \(U\) such that the restriction \(f|_U\) factors through an embedding of \(U\) into the affine \(Y\)-line \(A^1_Y\).

(ii) The property of being curvilinear for a morphism is invariant under base change.

**Proof.** (i) See [6], Prop. 2.7, p. 12. (ii) follows immediately from (i) or from the definition.

Let \(f : X \to Y\) be a finite morphism. We can define the first iteration \(f_1 : X_2 \to X\) of \(f\) as follows:
\[
X_2 = \mathbb{P}(I_\Delta), \quad f_1 : X_2 \to X \times_Y X \to X
\]
where \(\Delta \subset X \times_Y X\) is the diagonal. \(X_2\) is the so-called residual scheme of \(\Delta\); its properties are described in [5], §2. Moreover we define:
\[
N_r(f) = \text{Fitt}_{r-1}(f_*\mathcal{O}_X), \quad M_r = f^{-1}(N_r(f))
\]
(see [10], p. 200). \(N_r(f)\) is called the \(r\)-th multiple point scheme of \(f\). We will need the following:

**Proposition 2.3** If \(f\) is finite and curvilinear then \(f_1\) is also finite and curvilinear and
\[
M_r(f) = N_{r-1}(f_1)
\]

**Proof.** See [6], Lemma 3.9 and Lemma 3.10.
Proposition 2.4 Assume that \( f : X \to Y \) is finite and curvilinear, local complete intersection of codimension 1. Assume moreover that \( f \) is birational onto its image and that \( X \) and \( Y \) are local complete intersections and pure-dimensional. Then:

(i) \( f_1 : X_2 \to X \) is a local complete intersection of codimension 1, finite and curvilinear, birational onto its image and \( X_2 \) is a pure-dimensional local complete intersection.

(ii) Each component of \( M_r(f) \) (resp. \( N_r(f) \)) has codimension at most \( r-1 \) in \( X \) (resp. at most \( r \) in \( Y \)).

(iii) Assume moreover that \( Y \) is nonsingular. If \( N_r(f) \neq \emptyset \) and has pure codimension \( r \) in \( Y \) for some \( r \geq 2 \), then \( N_s(f) \neq \emptyset \) and has pure codimension \( s \) in \( Y \) for all \( 1 \leq s \leq r-1 \). Moreover:

\[
N_r(f) \subset \overline{N_{r-1}(f) \setminus N_r(f)} \subset \cdots \subset \overline{N_1(f) \setminus N_2(f)} \quad (1)
\]

Proof. (i) See [6], Lemma 3.10. (ii) See [6], Theorem 3.11.

(iii) by induction on \( r \). The assumption that \( f \) is birational onto its image implies that \( N_1(f) \) has pure codimension 1 and that \( N_2(f) \) has pure codimension 2 in \( Y \) ([6], Prop. 3.2(iii)). Therefore the dimensionality assertion is true for \( r = 2 \). Assume \( r \geq 3 \) and that \( N_r(f) \) has pure codimension \( r \) in \( Y \). Then \( N_{r-1}(f_1) = M_r(f) \) has pure codimension \( r-1 \) in \( X \). By part (i) we can apply the inductive hypothesis to \( f_1 \): it follows that \( M_s(f) = N_{s-1}(f_1) \neq \emptyset \) and has pure codimension \( s-1 \) in \( X \) for all \( 1 \leq s-1 \leq r-2 \). This implies that \( N_s(f) \neq \emptyset \) and has pure codimension \( s \) in \( Y \) for all such \( s \). The chain of inclusions \((1)\) is a consequence of the fact that the non-emptiness of all the \( N_s(f) \)'s is local around every point of \( N_r(f) \) because the argument holds for the restriction of \( f \) above an arbitrary open subset of \( Y \).

Assume now that \( f : X \to Y \) is finite and projective with \( X, Y \) algebraic. Then we have two stratifications of \( Y \). The first one is defined by the multiple-point schemes of \( f \):

\[
M : \coprod_{r \geq 0} (N_r(f) \setminus N_{r+1}(f)) \to Y
\]

The second one is the flattening stratification of the sheaf \( \mathcal{O}_X \):

\[
\Phi : \coprod_{i \geq 0} W_i \to Y
\]
Lemma 2.5 If $f : X \to Y$ is a finite and projective morphism of algebraic schemes, then the stratifications $M$ and $\Phi$ coincide.

Proof. Recalling that a finite morphism $g : V \to U$ is flat if and only if $g_*\mathcal{O}_V$ is locally free, the strata $W_i$ of $\Phi$ are indexed by the nonnegative integers, corresponding to the ranks of the locally free sheaves $f_*(\mathcal{O}_X|_{f^{-1}(W_i)}) = (f_*\mathcal{O}_X)|_{W_i}$ (see [10], Note 6 p. 205 for this equality). On the other hand the stratification $M$ is the one defined by the sheaf $f_*\mathcal{O}_X$ in the sense considered in [10], Theorem 4.2.7, p. 199, and its strata are also characterized by the fact that $f_*\mathcal{O}_X$ has a locally free restriction to each of them. Therefore $M$ and $\Phi$ are equal.

Remark 2.6 Assume that $f : X \to Y$ is projective, finite and birational onto its image. Then Zariski’s Main Theorem implies that its image $f(X) = N_1(f)$ cannot be normal unless $f$ is an embedding, i.e. unless $N_r(f) = \emptyset$ for all $r \geq 2$. This means that, in presence of higher multiple point schemes, we must expect that $f(X)$ has non-normal singularities.

Definition 2.7 Let $q : C \to B$ be a flat family of projective curves. Assume that all the fibres of $q$ are reduced curves having locally planar singularities, and let $T^1_q$ be the first relative cotangent sheaf of $q$. Then $T^1_q = \mathcal{O}_Z$ for a closed subscheme $Z \subset C$. We will call $Z$ the critical scheme of $q$.

It is well known and easy to show that, under the conditions of the definition, $Z$ is finite over $B$ and commutes with base change ([10], Lemma 4.7.5 p. 258). It is supported on the locus where $q$ is not smooth. We will be interested in the multiple-point schemes of $f : Z \to B$, the restriction of $q$ to $Z$.

Example 2.8 Planar double point singularities are those l.c.i. curve singularities whose local ring $\mathcal{O}_p$ has as completion:

$$\hat{\mathcal{O}}_p \cong k[[x, y]]/(y^2 + x^m), \quad \text{for some } m \geq 2$$

The semiuniversal deformation of $\hat{\mathcal{O}}_p$ is

$$k[[t_1, \ldots, t_{m-1}]] \to \frac{k[[t_1, \ldots, t_{m-1}, x, y]]}{(y^2 + x^m + t_1x^{m-2} + \cdots + t_{m-2}x + t_{m-1})} \quad (2)$$
We will denote by \( \pi : C \rightarrow B \) the corresponding family of schemes, by \( Z \subset C \) the critical scheme of \( \pi \), and by \( f : Z \rightarrow B \) the restriction of \( \pi \).

The special cases of nodes, cusps and tacnodes are respectively:

\[
\hat{O}_p \cong \begin{cases} 
\kappa[[x,y]]/(x^2 + y^2) & \text{(node)} \\
\kappa[[x,y]]/(y^2 + x^3) & \text{(cusp)} \\
\kappa[[x,y]]/(y^2 + x^4) & \text{(tacnode)} 
\end{cases}
\] 

The corresponding semiuniversal deformations are:

\[
\kappa[[t]] \rightarrow \kappa[[t,x,y]]/(x^2 + y^2 + t) 
\]

\[
\kappa[[u_1, u_2]] \rightarrow \kappa[[u_1, u_2, x, y]]/(y^2 + x^3 + u_1 x + u_2) 
\]

\[
\kappa[[v_1, v_2, v_3]] \rightarrow \kappa[[v_1, v_2, v_3, x, y]]/(y^2 + x^4 + v_1 x^2 + v_2 x + v_3) 
\]

For the deformation (2) of the general planar double point, \( Z \) is defined by:

\[
\kappa[[t_1, \ldots, t_{m-1}, x]]/
\]

\[
(x^m + t_1 x^{m-2} + \cdots + t_{m-1} x^2 + t_{m-1}, m x^{m-1} + (m-2) t_1 x^{m-3} + \cdots + t_{m-2}) 
\]

In the special cases \( Z \) is defined by the following family:

\[
\kappa[[t, x, y]]/(x^2 + y^2 + t, x, y) \cong \kappa 
\]

\[
\kappa[[u_1, u_2, x]]/(x^3 + u_1 x + u_2, 3x^2 + u_1) 
\]

\[
\kappa[[v_1, v_2, v_3, x]]/(x^4 + v_1 x^2 + v_2 x + v_3, 4x^3 + 2v_1 x + v_2) 
\]

All these are clearly finite and curvilinear and moreover \( Z \) is nonsingular of relative dimension \(-1\) over \( B \).

### 3 The generic deformation of a curve

Let \( C \) be a connected, reduced projective local complete intersection (l.c.i.) curve of arithmetic genus \( p_a(C) \). Then we have:

\[
\text{Ext}^2(\Omega_C^1, \mathcal{O}_C) = 0
\]

and \( \text{Def}_C(\kappa[\epsilon]) = \text{Ext}^1(\Omega_C^1, \mathcal{O}_C) \) so that \( \text{Def}_C \) is unobstructed and \( C \) has a versal formal deformation

\[
\mathcal{X} \longrightarrow \text{Spec} \, (\kappa[[z_1, \ldots, z_n]])
\]
where
\[ n = \dim_k \left[ \text{Ext}^1(\Omega^1_C, \mathcal{O}_C) \right] = 3p_a(C) - 3 + \mu_0(T_C) \]
and where \( T_C = \text{Hom}(\Omega^1_C, \mathcal{O}_C) \) (see [10]). By Grothendieck’s effectiveness theorem ([10], Theorem 2.5.13, p. 82) \( \mathcal{X} \) is the formal completion of a unique scheme projective and flat over \( \mathcal{M} := \text{Spec}(k[[z_1, \ldots, z_n]]) \), which we will also denote by \( \mathcal{X} \). We thus obtain a deformation

\[
\begin{array}{ccc}
C & \longrightarrow & \mathcal{X} \\
\downarrow & & \downarrow \pi \\
\text{Spec}(k) & \longrightarrow & \mathcal{M}
\end{array}
\]

which we call the generic deformation of \( C \), conforming to the terminology introduced in [8], p. 64.

Let \( p \in C \) be a closed singular point. Since \( \text{Ext}^2(\Omega_{\mathcal{O}_p/k}, \mathcal{O}_p) = 0 \), the local ring \( \mathcal{O}_p = \mathcal{O}_{C,p} \) has a semiuniversal formal deformation \( \tilde{\mathcal{O}}_p \) which is an algebra over the smooth parameter algebra

\[ A_p = k[[t_1, \ldots, t_{r(p)}]] \]

The number of parameters is equal to

\[ r(p) := \dim_k \left[ \text{Ext}^1(\Omega_{\mathcal{O}_p/k}, \mathcal{O}_p) \right] \]

because \( \text{Ext}^1(\Omega_{\mathcal{O}_p/k}, \mathcal{O}_p) = T^1_{\mathcal{O}_p} \) is the first cotangent space of \( \mathcal{O}_p \), which is naturally identified with the space of first order deformations of \( \mathcal{O}_p \). We have an obvious restriction morphism of functors

\[ \text{Def}_C \to \text{Def}_{\mathcal{O}_p} \]

which corresponds, by semiuniversality, to a homomorphism

\[ \psi_p : A_p \to k[[z_1, \ldots, z_n]] \]

inducing an isomorphism

\[ \mathcal{O}_{\mathcal{X}, p} \cong \tilde{\mathcal{O}}_p \otimes_{A_p} k[[z_1, \ldots, z_n]] \]

Putting all these local informations together we obtain a morphism of functors:

\[ \Psi : \text{Def}_C \to \prod_{p \in C} \text{Def}_{\mathcal{O}_p} \]
which corresponds to a morphism we will denote with the same letter:

$$\Psi := \prod_{p \in C} \text{Spec}(\psi_p) : \mathcal{M} \to \mathcal{M}_b = \text{Spec}(k[[t_1, \ldots, t_r]]) = \text{Spec}(\widehat{\otimes}_p A_p)$$

(4)

where

$$r = h^0(C, \text{Ext}^1(\Omega^1_C, \mathcal{O}_C)) = h^0(T^1_C) = \sum_{p \in C} r(p)$$

**Lemma 3.1** The morphism $$\Psi$$ is smooth. Therefore, up to a change of variables, it is dual to an inclusion

$$k[[t_1, \ldots, t_r]] \hookrightarrow k[[t_1, \ldots, t_r, z_{r+1}, \ldots, z_{r+m}]]$$

**Proof.** Because of the smoothness of its domain, the smoothness of $$\Psi$$ is equivalent to the surjectivity of its differential. But

$$d\Psi : \text{Ext}^1(\Omega^1_C, \mathcal{O}_C) \to H^0(C, \text{Ext}^1(\Omega^1_C, \mathcal{O}_C))$$

is a hedge-homomorphism in the spectral sequence for Ext’s, and it is surjective because $$H^2(C, \text{Ext}^0(\Omega^1_C, \mathcal{O}_C)) = 0$$. 

**Remark 3.2** Lemma 3.1 is Proposition (1.5) of [4]. It holds for any reduced curve, without assuming that $$C$$ is a l.c.i. For obvious reasons the number $$m$$ of extra variables $$z_j$$ appearing in the statement of the lemma is

$$m = \dim(\ker(d\Psi)) = h^1(C, \text{Ext}^0(\Omega^1_C, \mathcal{O}_C)) = h^1(C, T_C)$$

$$= 3p_a(C) - 3 + h^0(T_C) - h^0(T^1_C)$$

They correspond to the locus $$t_1 = \cdots = t_r = 0$$ in $$\mathcal{M}$$, parametrizing locally trivial deformations of $$C$$.

### 4 Local properties of families of curves

Consider a flat projective family of deformations of $$C$$:

$$\begin{array}{c}
C \xrightarrow{\varphi} \mathcal{C} \\
\text{Spec}(k) \xrightarrow{\psi} B
\end{array}$$

(5)
Denote by $Z \subset C$ the critical scheme, and by $f : Z \to B$ the restriction of $\varphi$:

\[
\begin{array}{ccc}
Z & \hookrightarrow & C \\
\downarrow f & & \downarrow \varphi \\
\downarrow & & \downarrow B
\end{array}
\quad (6)
\]

Let $\hat{O}_b$ the completion of the local ring $O_{B,b}$, let $T := \text{Spec}(\hat{O}_b)$ and denote by $b$ the closed point of $T$ too. Pulling back the family (5) by the morphism $h : T \to B$ induced by $O_{B,b} \to \hat{O}_b$ we obtain a family

\[
\begin{array}{ccc}
C & \longrightarrow & \tilde{C} \\
\downarrow & & \downarrow \tilde{\varphi} \\
\text{Spec}(k) & \longrightarrow & T
\end{array}
\quad (7)
\]

whose fibre over $b$ is again $C$. Let

\[
\begin{array}{ccc}
\tilde{Z} & \hookrightarrow & \tilde{C} \\
\downarrow f & & \downarrow \tilde{\varphi} \\
\downarrow & & \downarrow T
\end{array}
\]

be the critical scheme of $\tilde{\varphi}$.

By versality there is a morphism $\mu : T \to \mathcal{M}$ with uniquely determined differential such that $\tilde{C} = T \times_\mathcal{M} \mathcal{X}$, so that we also have the composition:

\[
\Psi \circ \mu : T \longrightarrow \mathcal{M} \longrightarrow \mathcal{M}_{\text{lo}}
\]

Infinitesimally $\mu$ and $\Psi \circ \mu$ can be described as follows. We have an exact sequence of locally free sheaves on $C$:

\[
0 \to \varphi^* \Omega^1_B \to \Omega^1_C \to \Omega^1_{C/B} \to 0
\]

(see [10], Theorem D.2.8) which dualizes as:

\[
\begin{array}{c}
0 \longrightarrow T_{C/B} \longrightarrow T_C \longrightarrow \varphi^* T_B \longrightarrow \text{Ext}^1_{\mathcal{C}}(\Omega^1_{C/B}, \mathcal{O}_C) \longrightarrow 0
\end{array}
\quad (8)
\]

\[
0 \rightarrow \mathcal{O}_Z
\]
and this gives a global description of $T^1_\varphi = \mathcal{O}_Z$ as the structure sheaf of the critical scheme. If we push $u$ down to $B$ we obtain a factorization:

$$
\begin{array}{ccccccccc}
T_B & \xrightarrow{\varphi^*(u)} & f_*\mathcal{O}_Z = f_*\text{Ext}^1_\mathcal{C}(\Omega^1_{\mathcal{C}/B}, \mathcal{O}_\mathcal{C}) & \xrightarrow{\text{Ext}^1_\varphi(\Omega^1_{\mathcal{C}/B}, \mathcal{O}_\mathcal{C})} & \\
\end{array}
$$

and at the point $b \in B$ this gives:

$$
\begin{array}{ccccccccc}
T_bB & \xrightarrow{d(\Phi\circ\mu)} & [f_*\mathcal{O}_Z](b) = H^0(T^1_C) & \xrightarrow{\text{Ext}^1(\Omega^1_{\mathcal{C}}, \mathcal{O}_\mathcal{C})} & \\
\end{array}
$$

Here we used the fact that the critical scheme commutes with base change for a l.c.i. morphism (generalizing [10], Lemma 4.7.5 p. 258). The right diagonal arrows are edge homomorphisms of the respective spectral sequences. Here we are especially interested in the differential of $\Phi \circ \mu$, so we will not insist in investigating $d\mu$. The map $d(\Phi \circ \mu)$ can be analyzed by means of the restriction of (8) to the fibre $C = \mathcal{C}(b)$:

$$
\begin{array}{cccccccccc}
0 & \longrightarrow & T_C & \longrightarrow & T_C|C & \longrightarrow & T_bB \otimes_k \mathcal{O}_C & \xrightarrow{u(b)} & T^1_C & \longrightarrow & 0 \\
& & & \bigg| & & & \bigg| & & & \bigg| \\
& & & & & & & & & \mathcal{N}_{C/C} \\
\end{array}
$$

which gives:

$$
d(\Phi \circ \mu) = H^0(u(b)) : T_bB \longrightarrow [f_*\mathcal{O}_Z](b) = H^0(T^1_C) \quad (10)
$$

A typical example of a family (5) is when $C$ is contained in a projective scheme $X$, $B$ is the Hilbert scheme of $X$ and $\varphi$ is the universal family. In this case $T_bB = H^0(C, \mathcal{N}_{C/X})$ and the map (10) is induced by the natural map of sheaves $\mathcal{N}_{C/X} \rightarrow T^1_C$

**Proposition 4.1** Assume that the fibres of $\varphi$ have at most planar double point singularities. Then $f$ is finite, projective and curvilinear. If $r = h^0(T^1_C)$ then $b \in N_r(f) \setminus N_{r+1}(f)$ and

$$
T_b[N_r(f)] = \ker[H^0(u(b))] \quad (11)
$$
Proof. \(Z\) is supported at the singular points of the fibres of \(\varphi\), therefore it is a closed subscheme of \(X\), projective over \(B\), and having a zero-dimensional intersection with every fibre. This implies that \(f\) is finite and projective.

It suffices to prove curvilinearity locally above \(b\) and, since \(h\) is etale, it suffices to prove it for \(\tilde{f}\). Locally around a point \(z \in \tilde{Z}\), \(\tilde{\varphi}\) is the pullback of the semiuniversal deformation of a locally planar singularity. Since for such singularities taking the critical scheme commutes with base change ([10], Lemma 4.7.5 p. 258), it follows that \(Z\) is locally the pullback of one of the families described in Example 2.8, which are curvilinear.

Since \([f_\theta \mathcal{O}_Z](b) = H^0(T^1_b)\), then \(b \in N_r(f) \setminus N_{r+1}(f)\) by definition of the support of \(N_r(f)\). We are left to prove (11). Consider a tangent vector \(\theta \in T_b B\) and the pullback of (6) over \(\text{Spec}(k[\epsilon])\) via \(\theta:\)

\[
\begin{array}{ccc}
Z \otimes_k k[\epsilon] & \xrightarrow{f_\theta} & C \otimes_k k[\epsilon] \\
& & \downarrow \varphi_\theta \\
\text{Spec}(k[\epsilon]) & \xrightarrow{\theta} & B
\end{array}
\]

The usual deformation-theoretic interpretation of the exact sequence (9) shows that a tangent vector \(\theta \in T_b B\) is in \(\ker[H^0(u(b))]\) if and only if \(\varphi_\theta\) is a first order deformation of \(C\) which is trivial locally at every singular point. In turn this is equivalent to the flatness of \(f_\theta\) (see [13], Lemma 3.3.7). But this means precisely that \(\theta\) is a tangent vector to the stratum containing \(b\) of the flattening stratification of \(\mathcal{O}_Z\) over \(B\). By applying Lemma 2.5 we conclude. \(\square\)

5 The stratification of the generic deformation - nodal case

Consider the simplest case, in which \(C\) has nodes \(p_1, \ldots, p_\delta\) and no other singularities. Then

\[
\mathcal{M}_{\text{lo}} = \text{Spec}(k[[t_1, \ldots, t_\delta]]), \quad \mathcal{M} = \text{Spec}(k[[t_1, \ldots, t_\delta, z_{\delta+1}, \ldots, z_{\delta+m}]])
\]

\(t_j\) is the parameter appearing in the versal deformation \(x^2+y^2+t_j = 0\) of the \(j\)-th node. The union of the coordinate hyperplanes \(t_1 \cdots t_\delta = 0\) is a normal crossing divisor defining a stratification

\[
\mathcal{M}_{\text{lo}} = \prod_{r=0, \ldots, \delta} \psi^r
\]

(12)
where $\mathcal{V}^r$ is the locally closed nonsingular subscheme supported on the set of points which belong to exactly $r$ coordinate hyperplanes. The stratum $\mathcal{V}^r$ has pure codimension $r$ and is in turn a disjoint union:

$$\mathcal{V}^r = \bigsqcup_{1 \leq j_1 < \cdots < j_r \leq \delta} \mathcal{V}(j_1, \ldots, j_r)$$

where $\mathcal{V}(j_1, \ldots, j_r)$ is the locus of points where precisely the coordinates $j_1, \ldots, j_r$ vanish. Obviously $\mathcal{V}(j_1, \ldots, j_r)$ is the linear subspace $V(t_{j_1}, \ldots, t_{j_r})$.

The stratification (12) pulls back to $\mathcal{M}$ to an analogous one, defined by the union of the coordinate hyperplanes $t_1 \cdots t_{\delta} = 0$, and having nonsingular strata:

$$\mathcal{M} = \bigsqcup_{r=0, \ldots, \delta} \mathcal{V}^r(\pi)$$

with

$$\mathcal{V}^r(\pi) = \bigsqcup_{1 \leq j_1 < \cdots < j_r \leq \delta} \mathcal{V}(j_1, \ldots, j_r)(\pi)$$

The $\mathcal{V}^r(\pi)$'s and the $\mathcal{V}(j_1, \ldots, j_r)(\pi)$'s are defined by the same conditions as the $\mathcal{V}^r$'s and the $\mathcal{V}(j_1, \ldots, j_r)$'s, and share with them the same properties of codimension and nonsingularity, being obtained from them by taking the cartesian product with Spec($k[[z_{\delta+1}, \ldots, z_{\delta+m}]]$).

The following statement is an obvious consequence of the previous analysis (compare also with [4], Corollary (1.9)).

**Theorem 5.1** Let $\mathcal{Z} \subset \mathcal{X}$ be the critical scheme of the generic deformation $\pi : \mathcal{X} \to \mathcal{M}$. Denote by $\phi : \mathcal{Z} \to \mathcal{M}$ the restriction of $\pi$.

Then:

(i) (13) is the flattening stratification of $\phi$.

(ii) Each stratum $\mathcal{V}^r(\pi)$ is nonsingular of pure codimension $r$.

(iii) Let $R$ be a complete local $k$-algebra $R$ with $R/m_R \cong k$ and let

$$k[[t_1, \ldots, t_{\delta}, z_{\delta+1}, \ldots, z_{\delta+m}]] \to R$$

be a local homomorphism. The induced morphism Spec($R$) $\to$ $\mathcal{M}$ factors through $\mathcal{V}(j_1, \ldots, j_r)(\pi)$ if and only if the flat family of deformations of $C$:

$$\begin{array}{ccc}
C & \longrightarrow & \text{Spec}(R) \times_{\mathcal{M}} \mathcal{X} \\
\downarrow & & \downarrow \\
\text{Spec}(k) & \longrightarrow & \text{Spec}(R)
\end{array}$$

11
is locally trivial at the points $p_{j_1}, \ldots, p_{j_r}$.

6 Families of nodal curves

In this section we will consider a flat projective family of curves $\varphi : C \to B$ parametrized by a scheme $B$. We will assume that all fibres of $\varphi$ have at most nodes as singularities. Let $f : Z \to B$ be the restriction of $\varphi$ to its critical scheme $Z \subset C$. By Proposition 4.1 $f$ is projective, finite and curvilinear and we can consider the multiple-point schemes of $f$.

**Definition 6.1** For any $r \geq 0$ the $r$-th stratum $N_r(f) \setminus N_{r+1}(f)$ of the multiple-point stratification of $f$ is called the Severi variety of curves with $r$ nodes of the family $\varphi$, and denoted by $V^r_{\varphi}$. The stratification

$$\coprod_r V^r_{\varphi}$$

is called the Severi stratification of $B$.

**Lemma 6.2** Let $\lambda : Y \to B$ be any morphism, and let:

$$\begin{array}{ccc}
C_Y & \longrightarrow & C \\
\downarrow \psi & & \downarrow \varphi \\
Y & \longrightarrow & B \\
& \lambda & \\
\end{array}$$

be the induced cartesian diagram. Then

$$\lambda^{-1}(V^r_{\varphi}) = V^r_{\psi}$$

for all $r \geq 0$. In other words forming the Severi stratification commutes with base change.

**Proof.** The lemma is a rephrasing of the fact that the critical scheme and the multiple-point stratification both commute with base change. $\square$

The following result is a generalized version of Theorem 2.2 of [3].

**Theorem 6.3** Let $\varphi : C \to B$ be a flat projective family of curves having at most nodes as singularities, with $B$ algebraic and integral. Assume that $b \in B$ is a $k$-rational point such that the fibre $C = \mathcal{C}(b)$
has precisely $\delta \geq 0$ nodes and no other singularities, so that $b \in \mathcal{V}_\varphi^\delta$. Then
\[
\text{codim}_B(\mathcal{V}_\varphi^\delta) \leq \delta
\]
Assume moreover $\delta \geq 1$ and that one of the following conditions is satisfied:

(i) $\mathcal{V}_\varphi^\delta$ has pure codimension $\delta$ in $B$ at $b$.
(ii) $b$ is a nonsingular point of $B$ and the map (10) is surjective.

Then there is a neighborhood $U$ of $b$ where $\mathcal{V}_\varphi^r$ is non-empty and of pure codimension $r$ for all $0 \leq r \leq \delta$. Moreover in case (ii) all the Severi varieties $\mathcal{V}_\varphi^r$ are nonsingular in a neighborhood of $b \in U$. In particular the general fibre of $\varphi$ is nonsingular.

**Proof.** Consider the scheme $T = \text{Spec}(\hat{O}_{B,b})$ and the deformation (7) induced by $\varphi$. Since multiple-point stratifications commute with base change, the Severi stratification of $\varphi$ pulls back to the Severi stratification of $\tilde{\varphi}$. Since $T \to B$ is an etale neighborhood of $b$, the hypothesis and the conclusion are valid on $T$ if and only if they are valid on $B$. Hence it suffices to prove the theorem for $\tilde{\varphi}$. We have an induced morphism
\[
\Phi \circ \mu : T \to \mathcal{M}_{k_0}
\]
whose differential is (10). The Severi stratification $\bigsqcup_r \mathcal{V}_\varphi^r$ is obtained by pulling back the stratification $\bigsqcup_r \mathcal{V}_\varphi^r$ of $\mathcal{M}_{k_0}$ by $\Phi \circ \mu$. Since this stratification is defined by the regular system of parameters $t_1, \ldots, t_\delta$, the Severi stratification is defined by
\[
(\Phi \circ \mu)^*(t_1), \ldots, (\Phi \circ \mu)^*(t_\delta)
\]
This implies in particular that $\mathcal{V}_\varphi^\delta$ cannot have codimension larger than $\delta$. If $\mathcal{V}_\varphi^\delta$ has codimension $\delta$ then (15) is a regular sequence, and therefore each stratum $\mathcal{V}_\varphi^r$ is non-empty and of pure codimension $r$. This proves the theorem in case (i).

The hypothesis that the map (10) is surjective and that $B$ is nonsingular at $b$ implies that $\Phi \circ \mu$ is smooth. It follows that (15) is a regular sequence, and moreover that all the strata of the Severi stratification are nonsingular.

**Definition 6.4** If $\mathcal{V}_\varphi^\delta$ is nonsingular and of codimension $\delta$ at a $k$-rational point $b$ we say that $\mathcal{V}_\varphi^\delta$ is regular at $b$; otherwise we say that
\( \mathcal{V}_\varphi^\delta \) is superabundant at \( b \). If an irreducible component \( \mathcal{V} \) of \( \mathcal{V}_\varphi^\delta \) is regular at all its \( k \)-rational points then \( \mathcal{V} \) is called regular. Otherwise \( \mathcal{V} \) is called superabundant.

With this terminology, we can say that Theorem 6.3(ii) gives a criterion of local regularity for the Severi varieties \( \mathcal{V}_\varphi^r \).

When one needs to apply Theorem 6.3, in practice it often happens that one can construct a subvariety \( Y \subset \mathcal{V}_\varphi^\delta \) of codimension \( \delta \) in \( B \) such that the restriction \( \varphi_Y : Y \times_B \mathcal{C} \to Y \) is a family of reducible curves having \( \delta \) nodes and not contained in a larger such family. In order to apply Theorem 6.3 one would need to know that \( \dim(Y) = \dim(\mathcal{V}_\varphi^\delta) \), i.e. that the family \( \varphi_Y \) is not contained in a larger family generically parametrizing irreducible curves having the same number \( \delta \) of nodes. This is guaranteed by the following useful result, classically called splitting principle (“principio di spezzamento”).

**Proposition 6.5** Let \( \varphi : \mathcal{C} \to B \) be a flat projective family of curves, with \( B \) a normal connected algebraic scheme. Suppose that all the geometric fibres of \( \varphi \) have precisely \( \delta \) nodes and no other singularities for some \( \delta \geq 0 \). Then the number of irreducible components of the geometric fibres of \( \varphi \) is constant.

**Proof.** Since \( B \) is normal and all fibres of \( \varphi \) have the same geometric genus we can apply [12], Theorem 1.3.2, to normalize simultaneously the fibres of \( \varphi \). We obtain a commutative diagram:

\[
\begin{array}{ccc}
\mathcal{C} & \xrightarrow{\bar{\varphi}} & \mathcal{C} \\
\downarrow \varphi & & \downarrow \varphi \\
B & \xrightarrow{\varphi} & B
\end{array}
\]

where \( \bar{\varphi} \) is a smooth projective family of curves and for each \( k \)-rational point \( b \in B \) the induced morphism \( \mathcal{C}(b) \to \mathcal{C}(b) \) is the normalization. The number of irreducible components of \( \mathcal{C}(b) \) is the same as the number of connected components of \( \mathcal{C}(b) \): thus it suffices to prove that this is constant.

Since any two \( k \)-rational points of \( B \) can be joined by a chain of algebraic integral curves, it suffices to prove what we need in the case when \( B \) is an integral curve. We can even assume that \( B \) is a nonsingular algebraic curve by pulling back to its normalization if
necessary. Considering the Stein factorization of \( \tilde{\varphi} \):

\[
\begin{array}{c}
\overset{\psi}{Z} \\
\downarrow \downarrow \\
\overset{\varphi}{C} \\
\downarrow h
\end{array}
\]

we see that the number of connected components of \( \tilde{C}(b) \) equals the degree of the 0-dimensional scheme \( Z(b) \), and this is constant. In fact \( Z = \text{Spec}(\tilde{\varphi}_* \mathcal{O}_C) \) and \( \tilde{\varphi}_* \mathcal{O}_C \) is locally free because it is torsion free and \( B \) is a nonsingular curve. \( \square \)

In §7 we will give an application of Proposition 6.5.

**Example 6.6 (The Severi varieties of a linear system)** This example is the most basic one. Consider a globally generated line bundle \( L \) on a projective connected surface \( S \) and the diagram

\[
\begin{array}{ccc}
C & \hookrightarrow & S \times |L| \\
\downarrow \pi & & \downarrow |L| \\
|L|
\end{array}
\]

where \( |L| = \mathbb{P}(H^0(L)) \) is the complete linear system defined by \( L \),

\[
C = \{(x, [C]) : x \in C\}
\]

is the tautological family, \( \pi \) is the second projection. We have

\[
T_{[C]}(|L|) = H^0(C, N_{C/S}) = H^0(C, \mathcal{O}_C(C))
\]

for any curve \( C \) in the linear system \( |L| \). The map (10) is the natural:

\[
H^0(C, N_{C/S}) \to H^0(T^1_C)
\]

so that its kernel, the space of first order locally trivial deformations of \( C \), is identified with \( T_{[C]}N_r(f) \), where \( r = h^0(T^1_C) \).

In general \( \pi \) has fibres with arbitrary planar singularities, so that, in order to apply the previous theory, we will need to restrict \( \pi \) above a conveniently chosen open subset \( B \subset |L| \) parametrizing curves with at most nodes as singularities. We will denote by \( \mathcal{Y}^n_L \) the corresponding Severi varieties; they are locally closed subschemes of \( |L| \) and will be called *Severi varieties of the linear system* \( |L| \). In the case \( S = \mathbb{P}^2 \)

...
and \( L = O(d) \) we obtain the classical Severi varieties of plane nodal curves of degree \( d \); they are denoted by \( Y_d^S \). It is well known that all irreducible components of the \( Y_d^S \)'s are regular (see [10]). The same is true for the Severi varieties of globally generated \( L \)'s on a K3 surface [11] and on an abelian surface [7].

7 Existence of nodal curves on \( K3 \) surfaces

As a simple application of Theorem 6.3 one can prove the existence of nodal irreducible curves with any number of nodes between 0 and the maximum, belonging to the primitive polarization on a general \( K3 \) surface.

Theorem 7.1 A general primitively polarized algebraic \( K3 \) surface \((X, L)\) of genus \( g \geq 2 \) contains irreducible nodal curves of geometric genus \( g - \delta \) for every \( 0 \leq \delta \leq g \). Equivalently, \( V_\delta^L \neq \emptyset \) for all \( 0 \leq \delta \leq g \).

Proof. We start from a Kummer surface \( X_0 \) and a reducible nodal curve \( C_0 \subset X_0 \) such that

\[ C_0 = G + S \]

where \( G, S \) are two nonsingular rational curves meeting transversally at \( g + 1 \) points. The existence of \( X_0 \) and \( C_0 \) is explained for example in [1], p. 366. Since \( C_0^2 = 2g - 2 \), we have \( \dim(|C_0|) = g \). There is a pair \((\mathcal{X}, \mathcal{L})\) and a projective morphism \( \psi : \mathcal{X} \to U \) defining a family of polarized \( K3 \) surfaces, with \( U \) nonsingular of dimension 19, with isomorphic Kodaira-Spencer map at every point, and such that \((X_0, O(C_0)) \cong (\mathcal{X}(u_0), \mathcal{L}(u_0))\) for some point \( u_0 \in U \) (see [1], Thm. VIII, 7.3 and p. 366). Letting \( V = \mathcal{P}(\psi_* \mathcal{L}) \) and letting \( \tau : V \to U \) be the projection, we obtain a diagram

\[
\begin{array}{ccc}
C & \longrightarrow & \mathcal{X} \\
\varphi \downarrow & & \psi \downarrow \\
V & \tau \longrightarrow & U
\end{array}
\]

where \( \tau \) is smooth, \( \dim(V) = g + 19 \), and \( \varphi \) is a family of curves of arithmetic genus \( g \), such that, for every \( u \in U \), \( \tau^{-1}(u) = |\mathcal{L}(u)| \), and
the restriction
\[
\mathcal{C} \times_V |\mathcal{L}(u)| \to X(u) \times |\mathcal{L}(u)| = |\mathcal{L}(u)| \times_U X
\]
is the \(g\)-dimensional tautological family parametrized by \(|\mathcal{L}(u)|\). Let \(v_0 \in \tau^{-1}(u_0)\) be such that \(\mathcal{C}(v_0) = C_0\). Then, after restricting to a neighborhood \(W\) of \(v_0\), we may assume that all fibres of \(\phi\) have at most nodes as singularities. Consider the Severi variety \(V_{g+1}^\delta \subset W\). By Lemma 6.2 we have:
\[
\tau^{-1}(u_0) \cap V_{g+1}^\delta = V_{g+1}^\delta_{\mathcal{L}(u_0)}
\]
which is zero-dimensional, and the reducible curve \(C_0\) does not extend to a surface \(X(u)\) for a general \(u \in U\) because \(\mathcal{L}(u)\) is primitive. Therefore, by applying Proposition 6.5 we deduce that \(V_{g+1}^\delta\) has codimension \((\text{at least and therefore equal to})\) \(g + 1\) in \(W\). By applying Theorem 6.3, we deduce that \(V_{g+1}^\delta \neq \emptyset\) for all \(0 \leq \delta \leq g\). But since \(\overline{V_{\phi}^\delta} \supset V_{g+1}^\delta\), a general curve in \(V_{\phi}^\delta\) must be irreducible, having less nodes that \(C_0\) and degenerating to \(C_0\), which has two nonsingular irreducible components. Moreover \(V_{\phi}^\delta\) has pure dimension \(19 + g - \delta\) (Theorem 6.3(ii)) and \(V_{\phi}^\delta \cap |\mathcal{L}(u)| = V_{\mathcal{L}(u)}^\delta\) for \(u \in U\) (Lemma 6.2).

Note that the Severi varieties \(V_{\mathcal{L}}^\delta\) whose existence is proved in Theorem 7.1 are regular [11]. Therefore in particular Theorem 7.1 asserts the existence of nodal rational curves and of a 1-dimensional family of elliptic curves in \(|\mathcal{L}|\), thus recovering Theorem 23.1, p. 365, of [1].

References


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