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Some Remarks on Deformations of Invertible Sheaves.

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Sunto. – *Su una varietà proiettiva sia dato un fascio invertibile con la proprietà di essere generato dalle sue sezioni. Si studia il comportamento per deformazione di questa proprietà, sotto certe ipotesi.*

1. – Let $\pi: \mathfrak{X} \to S$ be a proper and flat morphism of irreducible complex spaces and $\mathfrak{L}$ an invertible sheaf on $\mathfrak{X}$. If the restriction $\mathfrak{L}_0$ to a fibre $\mathfrak{X}_0 = \pi^{-1}(0)$, $0 \in S$, is ample then it is well known that $\mathfrak{L}$ is π-ample over a neighborhood U of $0 \in S$ [3]. Suppose instead that $\mathfrak{L}_0$ satisfies a weaker condition: there is a tensor power $\mathfrak{L}_0^{\otimes k}$, with $k > 0$, whose global sections define a birational morphism f of $\mathfrak{X}_0$ onto a projective irreducible variety Y. Under the assumptions that Y is normal and that S is a nonsingular curve we prove that there is an open neighborhood U of $0 \in S$, a commutative diagram of proper maps

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\psi} & \mathfrak{Y} \\ & {\scriptstyle\pi}\searrow \quad \swarrow{\scriptstyle\tau} & \\ & U & \end{array}$$

such that τ is projective and an invertible sheaf $\mathcal{H}$ on $\mathfrak{Y}$ very ample relative to τ with $\psi^* \mathcal{H} = \mathfrak{L}^{\otimes h}$ for some $h > 0$, if and only if a certain graded $\boldsymbol{C}$-algebra, naturally defined by the data, is of finite type. Moreover we show that τ is a flat family of deformations of a projective variety whose normalization is Y (for a precise statement of our result see § 2). As a corollary we prove that the above finiteness condition is satisfied if the first direct image sheaf $R^1 f_* \mathcal{O}_X$ is zero. These results have analogies with theorems of MARKOE-ROSSI [6] and of RIEMENSCHNEIDER [8] which show the possibility of « blowing down » certain families of strongly pseudoconvex manifolds.

In § 2 we introduce some terminology, describe our basic situation and state the theorem, and in § 3 we give the proof. The corollary is proved in § 4; in § 5 we give a well known example, due to Zariski, of a family where the conditions of the theorem are not satisfied.

This work is included in the author's 1976 Brandeis University thesis.

2. – If $R=\sum_{i\geqslant 0} R_i$ is a graded $\boldsymbol{C}$-algebra ($\boldsymbol{C}$ is the field of complex numbers; we only consider positively graded $\boldsymbol{C}$-algebras) and n is a positive integer, we denote by $R^{(n)}$ the graded subalgebra $\sum_{i\geqslant 0} R_{ni}$ of R. If R is finitely generated over $\boldsymbol{C}$ by its elements of degree one we call R a homogeneous $\boldsymbol{C}$-algebra (or a homogeneous $\boldsymbol{C}$-domain if moreover it is an integral domain).

All varieties and spaces we consider are defined over the complex numbers. To each finitely generated $\boldsymbol{C}$-algebra R one associates a projective variety Proj(R) in a well known way.

We will be concerned with the following situation. Y is an irreducible normal projective variety, H is a very ample invertible sheaf on Y (i.e. the global sections of H define a projective embedding of Y). Let $f\colon X\to Y$ be a birational morphism from an irreducible projective variety X and let $L=f^*H$, the pullback of H on X. Since Y is normal $f_*\mathcal{O}_X=\mathcal{O}_Y$ and $H^0(X, L^{\otimes i})=H^0(Y, H^{\otimes i})$ for all $i>0$. Therefore the global sections of $L^{\otimes i}$ define a morphism of X into a projective space which is just the composition of f with the projective embedding of Y defined by the global sections of $H^{\otimes i}$. Put $R_i=H^0(X, L^{\otimes i})$; then $R=\sum_{i\geqslant 0} R_i$ is a finitely generated graded $\boldsymbol{C}$-domain under tensor product [7] and $Y=\mathrm{Proj}(R)$.

Let $\pi\colon \mathfrak{X}\to S$ be a proper and flat family of deformations of X over an irreducible space S: there is given a point $0\in S$ and an isomorphism $X\xrightarrow{\sim}\pi^{-1}(0)=\mathfrak{X}_0$ of X with the fibre of π over 0. Moreover let an invertible sheaf $\mathfrak{L}$ on $\mathfrak{X}$ be given such that $L\xrightarrow{\sim}\mathfrak{L}|\mathfrak{X}_0=\mathfrak{L}_0$ under the isomorphism $X\xrightarrow{\sim}\mathfrak{X}_0$. We identify X and L with $\mathfrak{X}_0$ and $\mathfrak{L}_0$ using these isomorphisms, and briefly we write that $(\mathfrak{X},\mathfrak{L})\xrightarrow{\pi} S$ is a (proper and flat) family of deformations of (X, L) with $(X, L)=(\mathfrak{X}_0,\mathfrak{L}_0)$. To such family there is associated the graded $\boldsymbol{C}$-algebra $\hat{R}=\sum_{i\geqslant 0}\hat{R}_i$, where $\hat{R}_i=(\pi_*\mathfrak{L}^{\otimes i})_0/\underline{m}(0)(\pi_*\mathfrak{L}^{\otimes i})_0$, with $(\pi_*\mathfrak{L}^{\otimes i})_0$ being the stalk at $0\in S$ of the sheaf $\pi_*\mathfrak{L}^{\otimes i}$ and $m(0)$ the maximal ideal of the local ring $\mathcal{O}_{S,0}$; the product in $\hat{R}$ is induced by the tensor product of germs of sections.

THEOREM. – *Let $f: X \to Y$ be a birational morphism of irreducible projective varieties with Y normal, H a very ample invertible sheaf on Y and $L = f^*H$. Suppose that $(\mathfrak{X}, \mathfrak{L}) \xrightarrow{\pi} S$ is a family of deformations of (X, L) with $(X, L) = (\mathfrak{X}_0, \mathfrak{L}_0)$, $0 \in S$, such that the parameter space S is an irreducible nonsingular curve. Then the following conditions are equivalent:*

i) *the graded* $\boldsymbol{C}$*-algebra*

$$\hat{R}^{(k)} = \sum_{i \geq 0} (\pi_* \mathfrak{L}^{\otimes ki})_0 / \underline{m}(0)(\pi_* \mathfrak{L}^{\otimes ki})_0$$

is finitely generated for some $k > 0$.

ii) *there is an open neighborhood U of $0 \in S$ and a commutative diagram of proper morphisms of complex spaces*

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\psi} & \mathcal{Y} \\ & {}_{\pi}\searrow \quad \swarrow_{\tau} & \\ & U & \end{array}$$

such that the map τ is projective and there is a very ample relative to U invertible sheaf $\mathcal{H}$ on $\mathcal{Y}$ such that $\psi^* \mathcal{H} = \mathfrak{L}^{\otimes h}$ for some $h > 0$.

Moreover if conditions i) *and* ii) *are satisfied the map τ is flat, Y is the normalization of $\tau^{-1}(0)$ and the restriction $\psi|_{\mathfrak{X}_0}: \mathfrak{X}_0 \to \tau^{-1}(0)$ is the composition $p \circ f$, where $p: Y \to \tau^{-1}(0)$ is the normalization map.*

3. – PROOF OF THE THEOREM. – For each $i > 0$ we have a natural $\boldsymbol{C}$-linear map

$$t_{0,i}: (\pi_* \mathfrak{L}^{\otimes i})_0 / \underline{m}(0)(\pi_* \mathfrak{L}^{\otimes i})_0 \to H^0(X, L^{\otimes i})$$

(the « base change map ») which defines a homomorphism of graded $\boldsymbol{C}$-algebras $t: \hat{R} = \sum_{i \geq 0} \hat{R}_i \to \sum_{0 \geq i} R_i = R$.

LEMMA 1. – *Let $\varphi: \mathcal{V} \to B$ be a proper morphism of complex spaces, $\mathcal{F}$ a coherent sheaf on $\mathcal{V}$ which is $\mathcal{O}_B$-flat. Assume that B is a nonsingular curve. Then $\varphi_* \mathcal{F}$ is a locally free sheaf of finite rank on B and at each point $b \in B$ the following are true:*

(α) *the base change map*

$$t_b: (\varphi_* \mathcal{F})_b / \underline{m}(b)(\varphi_* \mathcal{F})_b \to H^0(\mathcal{V}_b, \mathcal{F}_b)$$

is injective;

(β) $\operatorname{rank}_{\mathcal{O}_B}(\varphi_* \mathcal{F}) \geq \dim_{\boldsymbol{C}} H^0(\mathcal{V}_b, \mathcal{F}_b) - \dim_{\boldsymbol{C}} H^1(\mathcal{V}_b, \mathcal{F}_b)$.

PROOF. – See [5].

By part (α) of Lemma 1, $\hat{R}$ can be identified with a graded subalgebra of R.

Suppose that condition ii) is satisfied. Then the canonical homomorphism $\pi^*\pi_*\mathcal{L}^{\otimes h} \to \mathcal{L}^{\otimes h}$ is surjective on $\pi^{-1}(U)$ and it defines the U-map $\pi^{-1}(U) \to \mathcal{Y} \hookrightarrow P(\pi_*\mathcal{L}^{\otimes h}) = P(\tau_*\mathcal{H})$, because $\mathcal{L}^{\otimes h} = \psi^*\mathcal{H}$. In particular the rational map defined on X by

$$\hat{R}_h = (\pi_*\mathcal{L}^{\otimes h})_0/\underline{m}(0)(\pi_*\mathcal{L}^{\otimes h})_0$$

has no base points; therefore $\hat{R}^{(h)} = \sum_{i\geqslant 0} \hat{R}_{hi}$ is finitely generated [7] and i) is satisfied with $k = h$.

Viceversa suppose that condition (i) is satisfied and put $Z = \operatorname{Proj}(\hat{R}^{(k)})$.

LEMMA 2. – *The rational map $p\colon Y \dashrightarrow Z$ induced by the inclusion $\hat{R}^{(k)} \subseteq R^{(k)}$ is everywhere defined, finite and birational. Therefore p is the normalization map of Z.*

PROOF OF LEMMA 2. – By the projection formula $R^j f_* L^{\otimes i} = (R^j f_* \mathcal{O}_X) \otimes_{\mathcal{O}_Y} H^{\otimes i}$; since Y is normal $f_*\mathcal{O}_X = \mathcal{O}_Y$, therefore

$$H^1(Y, f_*L^{\otimes i}) = H^1(Y, H^{\otimes i}) = 0 \qquad \text{if } i \gg 0 .$$

So by the Leray spectral sequence we have

$$H^1(X, L^{\otimes i}) = H^0(Y, (R^1 f_* \mathcal{O}_X) \otimes_{\mathcal{O}_Y} H^{\otimes i}) \qquad \text{when } i \gg 0 ;$$

hence by [9] there is a polynomial $Q(T)$ with rational coefficients that $\dim_{\boldsymbol{C}} H^1(X, L^{\otimes i}) = Q(i)$ when $i \gg 0$ and degree $(Q(T)) = \dim(\operatorname{Supp}(R^1 f_* \mathcal{O}_X)) < \dim(Y)$, because f is birational. Let n be a positive integer such that both $\hat{R}^{(nk)}$ and $R^{(nk)}$ are homogeneous $\boldsymbol{C}$-domains (n exists because $\hat{R}^{(k)}$ and $R^{(k)}$ are finitely generated $\boldsymbol{C}$-domains [1]). Now let $\hat{P}(T)$ and $P(T)$ be the Hilbert polynomials of $\hat{R}^{(nk)}$ and $R^{(nk)}$ respectively. By Lemma 1, when $i \gg 0$ we have

$$\hat{P}(i) = \dim_{\boldsymbol{C}} \hat{R}_{nki} \geqslant \dim_{\boldsymbol{C}} R_{nki} - \dim_{\boldsymbol{C}} H^1(X, L^{\otimes nki}) = P(i) - Q(nki) .$$

Since degree $(Q(T)) < \dim(Y) = \operatorname{degree}(P(T))$, we see that $\hat{P}(T)$ and $P(T)$ have the same degree and the same leading coefficient. Letting

$$N + 1 = \dim_{\boldsymbol{C}} R_{nk} , \qquad M + 1 = \dim_{\boldsymbol{C}} \hat{R}_{nk} ,$$

we see that $Y = \mathrm{Proj}(R^{(nk)})$ and $Z = \mathrm{Proj}(\hat{R}^{(nk)})$ can be embedden in P^N and P^M respectively and the map $p\colon Y \dashrightarrow Z$ is the restriction to Y of a rational projection of P^N onto P^M with center ad $(N-M-1)$-dimensional linear subspace of P^N; the Hilbert polynomials $P(T)$ and $\hat{P}(T)$ of $R^{(nk)}$ and $\hat{R}^{(nk)}$ respectively have the same degree and the same leading coefficient, hence Y and Z have the same dimension and the same degree in their respective ambient spaces. It is well known that these facts imply that p must be everywhere defined on Y, finite and birational, and Lemma 2 is proved.

Let $n > 0$ be as in the proof of Lemma 2 and put $h = nk$. Then the rational map defined on X by $(\pi_* \mathcal{L}^{\otimes h})_0/\underline{m}(0)\cdot(\pi_*\mathcal{L}^{\otimes h})_0$ has $Z = \mathrm{Proj}(\hat{R}^{(h)})$ as image and is everywhere defined on X by Lemma 2. Therefore the canonical homomorphism $\pi^*\pi_*\mathcal{L}^{\otimes h} \to \mathcal{L}^{\otimes h}$ is surjective on $\mathfrak{X}_0$ and by the properness of π it is surjective on $\pi^{-1}(U)$ for some open neighborhood U of $0 \in S$. This gives a U-morphism $\psi\colon \pi^{-1}(U) \to P(\pi_*\mathcal{L}^{\otimes h})$ and denoting by $\mathfrak{Y}$ the image of ψ we obtain a commutative diagram

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\psi} & \mathfrak{Y} \\ & {\scriptstyle\pi}\searrow \quad \swarrow{\scriptstyle\tau} & \\ & U & \end{array}$$

(with τ being the map induced by the projection $P(\pi_*\mathcal{L}^{\otimes h}) \to U$.) which clearly satisfies condition (ii). Since $\psi|\mathfrak{X}_0$ is defined by $(\pi^*\mathcal{L}^{\otimes h})_0/m(0)(\pi^*\mathcal{L}^{\otimes h})_0$ we see from Lemma 2 that Y is the normalization of $\tau^{-1}(0)$ and $\psi|\mathfrak{X}_0 = p\circ f$ ($p\colon Y \to \tau^{-1}(0)$ is the normalization map). To show that τ is flat we use the criterion of KERNER [4]. Since the map τ is open (because π is open) we must show that all its fibres are reduced. Since π is flat and $\mathfrak{X}_0 = \pi^{-1}(0)$ is reduced, $\mathfrak{X}_s = \pi^{-1}(s)$ is also reduced, $s \in U$, provided U is sufficiently small [4]. Therefore the $\boldsymbol{C}$-algebra $\sum_{i\geqslant 0} H^0(\mathfrak{X}_s, \mathcal{L}_s^{\otimes hi})$ and its subalgebra $\sum_{i\geqslant 0}(\pi_*\mathcal{L}^{\otimes hi})_s/\underline{m}(s)(\pi_*\mathcal{L}^{\otimes hi})_s$ are reduced. Since

$$\tau^{-1}(s) = \mathrm{Proj}\Big(\sum_{i\geqslant 0}(\pi_*\mathcal{L}^{\otimes hi})_s/\underline{m}(s)(\pi_*\mathcal{L}^{\otimes hi})_s\Big)$$

we see that $\tau^{-1}(s)$ is reduced, so τ is flat and the theorem is proved.

4. – COROLLARY. – *Under the same hypothesis of theorem, assume moreover that* $R^1 f_* \mathcal{O}_X = 0$. *Then the equivalent conditions* (i) *and* (ii) *of the theorem are satisfied.*

PROOF. – Since $R^1 f_* \mathcal{O}_X = 0$ the Leray spectral sequence for f and $L^{\otimes i}$ gives $H^1(X, L^{\otimes i}) = H^1(Y, H^{\otimes i}) = 0$ if $i \gg 0$ and Lemma 1 shows that $t_{0,i}: (\pi_*\mathcal{L}^{\otimes i})_0/\underline{m}(0)(\pi_*\mathcal{L}^{\otimes i})_0 \to H^0(X, L^{\otimes i})$ is an isomorphism for all $i \gg 0$. Therefore if k is sufficiently large $\hat{R}^{(k)} = R^{(k)}$ is finitely generated; so condition (i) is satisfied.

It is clear from the proof of the theorem that $\tau^{-1}(0) = Y$ if $\hat{R}^{(k)} = R^{(k)}$ for some $k > 0$. Therefore if $R^1 f_* \mathcal{O}_X = 0$ then every flat deformation of (X, L) over a nonsingular curve S can be blown down to a deformation of (Y, H), locally with respect to $0 \in S$. This statement has analogies with results of MARKOE-ROSSI [6] and RIEMENSCHNEIDER [8]; in the surface case it has already been proved by BURNS-WAHL [2].

5. – Let F be a nonsingular projective surface; on F consider a nonsingular irreducible curve S of positive genus g and denote by C a hyperplane section of F. Fix an integer $h \geqslant 1$ such that the following conditions are satisfied:

1) $H^1(F, \mathcal{O}(nC)) = H^2(F, \mathcal{O}(nC)) = 0$ for each $n \geqslant h$;

2) the intersection number $m = (hC + S \cdot S)$ is greater than $2g$;

3) $(hC \cdot S) > 2g - 2$.

Let $P_1, \ldots, P_m$ be distinct points on S such that the invertible sheaves $\mathcal{O}_S(hC + S)$ and $\mathcal{O}_S(P_1 + \ldots + P_m)$ on S are isomorphic. Moreover let Γ be an irreducible nonsingular curve of F linearly equivalent to hC and not passing through any of the points $P_1, \ldots, P_m$.

We obtain a proper and smooth family of projective surfaces $\pi: \mathfrak{X} \to S$ by taking $\mathfrak{X}$ to be the nonsingular variety obtained after successively blowing up $S \times F$ along the m nonsingular curves $\gamma_1 = S \times P_1, \ldots, \gamma_{m-1} = S \times P_{m-1}$, $\gamma_m = \{(s, s) \in S \times F\}$, and π to be the composition of the m blowing downs with the projection $S \times \times F \to S$. Moreover we let T_1 and T_2 be the proper transforms in $\mathfrak{X}$ of the surfaces $S \times \Gamma$ and $S \times S$ respectively in $S \times F$; consider the invertible sheaf $\mathcal{L} = \mathcal{O}_{\mathfrak{X}}(T_1 + T_2)$ on $\mathfrak{X}$. It is easy to check that the restriction $\mathcal{L}_{P_m}$ of $\mathcal{L}$ to the fibre $\mathfrak{X}_{P_m} = \pi^{-1}(P_m)$ is generated by its global sections, which define a birational morphism of $\mathfrak{X}_{P_m}$ onto a projective variety. In [10] ZARISKI has shown that in an arbitrarily small neighborhood of $P_m \in S$ there are points P such that $\mathcal{L}_P^{\otimes n}$ is not generated by its global sections on $\mathfrak{X}_P$ for all $n > 0$; *a fortiori* the equivalent conditions of our theorem are not satisfied in this example.

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