Deformations and Extensions of Gorenstein Weighted Projective Spaces

Thomas Dedieu and Edoardo Sernesi

Abstract We study the existence of deformations of all 14 Gorenstein weighted projective spaces $\mathbf{P}$ of dimension 3 by computing the number of times their general anticanonical divisors are extendable. In favorable cases (8 out of 14), we find that $\mathbf{P}$ deforms to a 3-dimensional extension of a general non-primitively polarized $K3$ surface. On our way, we show that each such $\mathbf{P}$ in its anticanonical model satisfies property N_2, i.e., its homogeneous ideal is generated by quadrics, and the first syzygies are generated by linear syzygies, and we compute the deformation space of the cone over $\mathbf{P}$. This gives as a byproduct the exact number of times $\mathbf{P}$ is extendable.

Keywords Weighted projective space · Canonical curve · K3 surface · Fano variety

1 Introduction

Some topics related to this paper have been discussed and worked out with our friend and colleague Ciro Ciliberto. It is a great pleasure for us to dedicate this work to him.

It is well known that there are precisely 14 Gorenstein weighted projective spaces of dimension 3 (see [29]; we give the list in Table 1). In this paper, we introduce a method in the study of their deformations, consisting in studying simultaneously

Dedicated to Ciro Ciliberto on the occasion of his 70th birthday.

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the deformations and the extendability of their general anticanonical divisors. The underlying philosophy goes back to Pinkham [28], and then Wahl [35] who showed the close connection between the existence of extensions of a projective variety $X \subset \mathbf{P}^r$ and the deformation theory of its affine cone $CX \subset \mathbf{A}^{r+1}$. We discuss and recall this connection in Sect. 4. Wahl's interest was focused on canonical curves, aiming at a characterization of those curves that are hyperplane sections of a $K3$ surface ("$K3$ curves") by means of the behavior of their Gaussian map, thereafter called "Wahl map". His program was carried out in [2] and applied in [8] to the extendability of canonical curves, $K3$ surfaces and Fano varieties. The present work follows the same direction, as surface (resp. curve) linear sections of Gorenstein weighted projective 3-spaces in their anticanonical embeddings are $K3$ surfaces with at worst canonical singularities (resp. canonical curves).

Our starting point is the observation that most polarized general anticanonical divisors $(S, -K_{\mathbf{P}}|_S)$ of the weighted projective spaces $\mathbf{P}$ of Table 1 are non-primitively polarized (and singular). This suggests considering a 1-parameter smoothing $(\mathcal{S}, \mathcal{L})$ of $(S, -K_{\mathbf{P}}|_S)$ and exploiting the fact that the extendability of non-primitively polarized $K3$ surfaces is well understood, thanks to work of Ciliberto–Lopez–Miranda [9], Knutsen [18], and Ciliberto–Dedieu [5, 6] (see Sect. 3 for details). This plan works fine when the extendability of $(S, -K_{\mathbf{P}}|_S)$ coincides with that of the general non-primitively polarized $K3$ surface, i.e., the invariant

$$\alpha(S_t, L_t) = h^0(S_t, N_{S_t/\mathbf{P}^g} \otimes L_t^{-1}) - g - 1$$

introduced in Theorem 3.3 takes the same value for all fibers of $(\mathcal{S}, \mathcal{L})$. What we get in this case is that $\mathbf{P}$ deforms to a threefold extension of the general member of $(\mathcal{S}, \mathcal{L})$. The final output (see Sect. 7) is an understanding of the deformation properties of Cases $\#i$, $i \in \{1, \ldots, 7, 9\}$, in the notation of Tables 1 and 3. A finer analysis is required for the other cases, which we don't try to carry out here, although we make a couple of observations at the end of Sect. 7.

Our strategy involves various substantial technical verifications. The main point is controlling the deformation theory of the affine and projective cones over possibly singular $K3$ surfaces. In the nonsingular case, this is a well-known chapter of deformation theory, due to Schlessinger [34]: we extend it to the singular case in Sect. 4. It is a non-trivial task to compute the relevant deformation spaces in our examples, and for this purpose we took advantage of the computational power of Macaulay2 [23]. Still this leaves some obstacles to the human user (see the proof of Proposition 6.2 and the comments thereafter), which we have found are best coped with by considering more generally deformations of cones over arithmetically Cohen–Macaulay surfaces or arithmetically Gorenstein curves.

As a further reward of this computation, we obtain the exact number of times each Gorenstein weighted projective space of dimension 3 is extendable (Corollary 6.4 and Table 3). We have not been able however to identify the maximal extension in all cases.

Another condition we had to verify in order to apply Wahl's criterion (Theorem 4.8) is that the projective schemes X involved satisfy condition N_2 (Definition 4.7), so that “each first order ribbon over X is integrable to at most one extension of X”. We carry this out again with the computer and Macaulay2 (see Proposition 6.1), by explicitly computing the homogeneous ideals of all Gorenstein weighted projective spaces in their anticanonical embeddings, as well as their first syzygy modules.

Some of our end results about Gorenstein Fano threefolds in Sect. 7 can also be obtained by direct calculations, using computational tricks on weighted projective spaces similar to those employed by Hacking in [15, Sect. 11], and showed to us by the referee. We find it nice that the observations we made indirectly using deformation theory may be confirmed by direct computations of a different nature. Let us also mention the article [24] (which has been continued in [15]), in which degenerations of the projective plane to various weighted projective planes are exhibited: this is similar in spirit to what we do in Sect. 7.

The organization of the article is as follows. In Sect. 2, we gather some elementary facts about weighted projective geometry and give the list of all 14 Gorenstein weighted projective spaces of dimension 3. In Sect. 3, we give a synthetic account of the extension theory of non-primitive polarized $K3$ surfaces along the lines of [8] and taking advantage of [18]. Section 4 is the technical heart of the paper and is devoted to the deformation theory of cones and its application to extensions. This leads to our main technical result Theorem 5.2 in the following Sect. 5. In Sect. 6, we carry out the explicit computations required for our application of Theorem 5.2 to Gorenstein weighted projective spaces, and in the final Sect. 7, we give the explicit output of this application.

2 Gorenstein Weighted Projective Spaces

We will consider some weighted projective spaces (WPS for short) of dimension 3. In this section, we collect some preliminary definitions and basic facts. The authoritative reference is [11]; we will also rely on [3, 13].

2.1 Consider a weighted projective 3-space of the form $\mathbf{P} := \mathbf{P}(a_0, a_1, a_2, a_3)$, where the a_i's are relatively prime positive integers.

It is not restrictive to further assume, and we will do it, that any three of the a_i's are relatively prime, in which case one says that $\mathbf{P}$ is *well formed*. Let:

$$m := \mathrm{lcm}(a_0, a_1, a_2, a_3), \quad s := a_0 + a_1 + a_2 + a_3.$$

The following holds:

(1) For all $d \in \mathbf{Z}$ the sheaf $\mathcal{O}(d)$ is reflexive of rank 1, and it is invertible if and only if $d = km$ for some $k \in \mathbf{Z}$ [3, Sect. 4].
(2) $\mathrm{Pic}(\mathbf{P}) = \mathbf{Z} \cdot [\mathcal{O}(m)]$ [3, Theorem 7.1, p. 152].
(3) $\mathbf{P}$ is Cohen–Macaulay and its dualizing sheaf is $\omega_{\mathbf{P}} = \mathcal{O}(-s)$. Therefore $\mathbf{P}$ is Gorenstein if and only if $m|s$ [3, Corollary 6B.10, p. 151]. In this case, $\mathbf{P}$ has

canonical singularities because it is a Gorenstein orbifold. This follows for example from [32, Proposition 1.7].

(4) The intersection product in $\mathbf{P}$ is determined by (see [19, p. 240]):

$$\mathcal{O}(1)^3 = \frac{1}{a_0a_1a_2a_3}.$$

Lemma 2.2 *Let $S \subset \mathbf{P}(a_0, a_1, a_2, a_3)$ be a general hypersurface of degree d, such that $\mathcal{O}(d)$ is locally free on $\mathbf{P}$. For all $k \in \mathbf{Z}$, the restriction to S of $\mathcal{O}(k)$ is locally free if and only if*

$$\forall i \neq j : \quad \gcd(a_i, a_j) | k.$$

Proof The local freeness needs only to be checked at the singular points. As S is general it may be singular only along the singular locus of $\mathbf{P}$, hence only along the lines joining two coordinate points $P_i = (0 : \ldots : 1 : \ldots : 0)$. Moreover, S avoids all coordinate points themselves thanks to our assumption on the degree.

Let P be a point on the line P_iP_j, off P_i and P_j. In the local ring $\mathcal{O}_P$ we have the invertible monomials $x_i^{n_i}x_j^{n_j}$ whose degrees sweep out

$$a_i\mathbf{Z} + a_j\mathbf{Z} = \gcd(a_i, a_j)\mathbf{Z}.$$

This shows that if $\gcd(a_i, a_j)$ divides k then $\mathcal{O}(k)$ is invertible at all points of P_iP_j but P_i and P_j themselves, hence the "if" part of the statement. The "only if" part follows in the same way. □

2.3 It follows from 2.1 that there are exactly 14 distinct 3-dimensional weighted projective spaces which are Gorenstein, see [29]. We list them in Table 1, together with the following information. For each $\mathbf{P}$ in the list, we denote by S a general anticanonical divisor: it is a $K3$ surface with ADE singularities [33]. We also use the following notation:

- m is the lcm of the weights, so that $\mathcal{O}(m)$ generates $\mathrm{Pic}(\mathbf{P})$;
- s is the sum of the weights, so that $\omega_{\mathbf{P}} = \mathcal{O}(-s)$;
- i_S denotes the divisibility of $K_{\mathbf{P}}|_S$ in $\mathrm{Pic}(S)$, which is readily computed with Lemma 2.2 above;
- g_1 is the genus of the primitively polarized $K3$ surface (S, L_1), where $L_1 = -\frac{1}{i_S} K_{\mathbf{P}}|_S$; we reserve the symbol g to the common genus of the Fano variety $\mathbf{P}$ and the polarized $K3$ surface $(S, -K_{\mathbf{P}}|_S)$, i.e., $2g - 2 = -K_{\mathbf{P}}^3$.

The rows are ordered according to g_1, then i_S (decreasing), then the weights. We also indicate the singularities of S, which may be found following [13].

Table 1 Gorenstein 3-dimensional weighted projective spaces

#	Weights	$-K_{\mathbf{P}}^3$	m	s	i_S	g_1	Sings(S)
1	(1, 1, 1, 3)	72	3	6	6	2	Smooth
2	(1, 1, 4, 6)	72	12	12	6	2	A_1
3	(1, 2, 2, 5)	50	10	10	5	2	$5A_1$
4	(1, 1, 1, 1)	64	1	4	4	3	Smooth
5	(1, 1, 2, 4)	64	4	8	4	3	$2A_1$
6	(1, 3, 4, 4)	36	12	12	3	3	$3A_3$
7	(1, 1, 2, 2)	54	2	6	3	4	$3A_1$
8	(1, 2, 6, 9)	54	18	18	3	4	$3A_1, A_2$
9	(2, 3, 3, 4)	24	12	12	2	4	$3A_1, 4A_2$
10	(1, 4, 5, 10)	40	20	20	2	6	$A_1, 2A_4$
11	(1, 2, 3, 6)	48	6	12	2	7	$2A_1, 2A_2$
12	(1, 3, 8, 12)	48	24	24	2	7	$2A_2, A_3$
13	(2, 3, 10, 15)	30	30	30	1	16	$3A_1, 2A_2, A_4$
14	(1, 6, 14, 21)	42	42	42	1	22	A_1, A_2, A_6

3 Extendability of Non-primitive Polarized $K3$ Surfaces

Let us first recall the following.

Definition 3.1 A projective variety $X \subset \mathbf{P}^r$ of dimension d is called *n-extendable* for some $n \geq 1$ if there exists a projective variety $\widetilde{X} \subset \mathbf{P}^{r+n}$ of dimension $d+n$, not a cone, such that $X = \widetilde{X} \cap \mathbf{P}^r$ for some linear embedding $\mathbf{P}^r \subset \mathbf{P}^{r+n}$. The variety $\widetilde{X}$ is called an *n-extension* of X. If $n = 1$ we call X *extendable* and $\widetilde{X}$ an *extension* of Y.

We refer to [21] for a beautiful tour on this subject.

If X is a Fano 3-fold of genus $g = -\frac{1}{2}K_X^3 + 1$, then a general $S \in |-K_X|$ is a $K3$ surface naturally endowed with the ample divisor $-K_X|_S$ which makes $(S, -K_X|_S)$ an extendable polarized $K3$ surface of genus g. Suppose that X has index $i_X > 1$, i.e., $-K_X = i_X H$ for an ample divisor H, indivisible in $\mathrm{Pic}(X)$. Then $(S, -K_X|_S)$ is non-primitive because $-K_X|_S = i_X\, H|_S$ is at least i_X-divisible. Therefore, by considering Fano 3-folds of index > 1, we naturally land in the world of extendable non-primitively polarized $K3$ surfaces.

Notation 3.2 We denote by $\mathcal{K}_g^k$ the moduli stack of polarized $K3$ surfaces of genus g and index k, i.e., pairs (S, L) such that S is a $K3$ surface, possibly with ADE singularities, and L is an ample and globally generated line bundle on S with $L^2 = 2g - 2$, such that $L = kL_1$ with L_1 a primitive line bundle on S; note that (S, L_1) belongs to $\mathcal{K}_{g_1}^1$, which we usually denote by $\mathcal{K}_{g_1}^{\mathrm{prim}}$, where $2g_1 - 2 = L_1^2$ and $g = 1 + k^2(g_1 - 1)$.

We have the following necessary condition for the extendability of a projective variety:

Theorem 3.3 ([22]) *Let $X \subset \mathbf{P}^n$ be a smooth, projective, irreducible, non-degenerate variety, not a quadric, and write $L = \mathcal{O}_X(1)$. Set*

$$\alpha(X, L) = h^0(N_{X/\mathbf{P}^n} \otimes L^{-1}) - n - 1.$$

If $\alpha(X, L) < n$, then X is at most $\alpha(X, L)$-extendable.

When the polarization of X is clear from the context, we write $\alpha(X)$ instead of $\alpha(X, L)$. Note that if X is a smooth $K3$ surface or Fano variety (resp. a canonical curve, hence $L = K_X$) then

$$\alpha(X) = H^1(X, T_X \otimes L^{-1}) \quad (\text{resp. } \operatorname{cork}(\Phi_{\omega_X}),$$

with Φ_{ω_X} the Gauss–Wahl map of X, see for instance [8, Sect. 3].

For $K3$ surfaces and canonical curves, the converse to Theorem 3.3 also holds, under some conditions. Precisely we have:

Theorem 3.4 ([8], Theorems 2.1 and 2.17) *Let (X, L) be a smooth polarized $K3$ surface (resp. $(X, L) = (C, K_C)$ a canonical curve) of genus g. Assume that $g \geq 11$ and* $\operatorname{Cliff}(S, L) \geq 3$. *Then (X, L) is $\alpha(X, L)$-extendable.*

More precisely, every non-zero $e \in H^1(S, T_S \otimes L^{-1})$ (resp. $e \in \ker({}^{\mathsf{T}}\Phi_{\omega_C})$) defines an extension X_e of X which is unique up to projective automorphisms of $\mathbf{P}^{g+1}$ (resp. $\mathbf{P}^g$) fixing every point of $\mathbf{P}^g$ (resp. $\mathbf{P}^{g-1}$), and there exists a universal extension $\tilde{X} \subset \mathbf{P}^{g+\alpha(X,L)}$ *(resp. $\mathbf{P}^{g-1+\alpha(X,L)}$) of X having each X_e as a linear section containing X.*

We denoted by $\operatorname{Cliff}(S, L)$ the *Clifford index* of any nonsingular curve $C \in |L|$; by [12, 14, 31], this does not depend on the choice of C. Note that in case (X, L) is a $K3$ surface the extension X_e in the theorem is an arithmetically Gorenstein Fano variety of dimension 3 with canonical singularities.

Unfortunately $H^1(S, T_S \otimes L^{-1})$ is not easy to compute in general, but in the non-primitive setting, we can reduce to a more amenable case.

Lemma 3.5 *Let $S \subset \mathbf{P}^g$ be a smooth $K3$ surface. Then:*

$$H^1(S, T_S \otimes L^{-j}) = \begin{cases} \operatorname{coker}\big[H^0(S, L)^\vee \to H^0(S, N_{S/\mathbf{P}^g}(-1))\big], & \text{if } j = 1 \\ H^0(S, N_{S/\mathbf{P}^g} \otimes L^{-j}), & \text{if } j \geq 2. \end{cases}$$

Proof See [9] (2.8). □

This lemma, applied to a smooth $(S, L_1) \in \mathcal{K}_{g_1}$ with L_1 very ample, tells us that $(S, L) = (S, jL_1)$ with $j \geq 2$ is extendable if and only if $H^0(S, N_{S/\mathbf{P}^{g_1}} \otimes L_1^{-j}) \neq 0$. The possibilities for the pair (S, L_1) and j are then very limited. In particular,

$H^0(S, N_{S/\mathbf{P}^{g_1}} \otimes L_1^{-j}) = 0$ for all $j \geq 2$ as soon as S satisfies the property N_2 (see Definition 4.7), see [18, Lemma 1.1] and the references therein. In fact the possibilities have been completely classified by Knutsen [18], see also [6, 9]. The result is the following:

Theorem 3.6 ([18]) *Let* $(S, L_1) \in \mathcal{K}_{g_1}^{\mathrm{prim}}$ *with* S *smooth and* L_1 *very ample. Then* $H^1(S, T_S \otimes L_1^{-j}) = 0$ *for all* $j \geq 2$ *except in the following cases:*

g_1	j	$g(L_1^j)$	$h^1(S, T_S \otimes L_1^{-j})$	Notes
3	2	9	10	any (S, L_1)
3	3	19	4	any (S, L_1)
3	4	33	1	any (S, L_1)
4	2	13	5	any (S, L_1)
4	3	28	1	any (S, L_1)
5	2	17	3	any (S, L_1)
6	2	21	1	any (S, L_1)
7	2	25	1	(1)
8	2	29	1	(2)
9	2	33	1	(3)
10	2	37	1	(4)

where

(1) S is one of the following:

  (I) a divisor in the linear system $|3H - 3F|$ on the quintic rational normal scroll $T \subset \mathbf{P}^7$ of type $(3, 1, 1)$, with H a hyperplane section and F a fiber of the scroll.
  (II) a quadratic section of the sextic Del Pezzo threefold $\mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1 \subset \mathbf{P}^7$ embedded by Segre.
  (III) the section of $\mathbf{P}^2 \times \mathbf{P}^2 \subset \mathbf{P}^8$, embedded by Segre, with a hyperplane and a quadric.

(2) S is an anticanonical divisor in a septic Del Pezzo 3-fold (the blow-up of $\mathbf{P}^3$ at a point).
(3) S is one of the following:

  (I) the 2-Veronese embedding of a quartic of $\mathbf{P}^3$; equivalently a quadratic section of the Veronese variety $v_2(\mathbf{P}^3) \subset \mathbf{P}^9$.
  (II) a quadratic section of the cone over the anticanonical embedding of the Hirzebruch surface $\mathbf{F}_1 \subset \mathbf{P}^8$.

(4) S is a quadratic section of the cone over the Veronese surface $v_3(\mathbf{P}^2) \subset \mathbf{P}^9$.

In all cases, except $g_1 = 3$ and $j = 2$, we have $\mathrm{Cliff}(S, L^j) \geq 3$.

Proof Using the identification given by Lemma 3.5, the proof reduces to list all possible cases described by Proposition 1.4 of [18]. The final statement is an easy calculation. □

The case $g_1 = 3$ and $j = 2$ is not liable to Theorem 3.4 as the Clifford index in this case is too small, but it has been studied by hand in [5, 6].

Theorem 3.6 does not cover the cases of (S, L_1) hyperelliptic. We shall only consider the case $g(L_1) = 2$, which will be sufficient for our purposes. The following result has been obtained in [5] by geometric means; we give here a cohomological proof.

Lemma 3.7 *Let $(S, L_1) \in \mathcal{K}_2^{\mathrm{prim}}$. Then the dimension of $H^1(S, T_S \otimes L_1^{-j})$ takes the values given by the following table:*

j	$g(L_1^j)$	$\mathrm{Cliff}(L_1^j)$	$h^1(S, T_S \otimes L_1^{-j})$
1	2	0	18
2	5	0	15
3	10	2	10
4	17	>2	6
5	26	>2	3
6	37	>2	1
≥7		>2	0

Recalling Theorem 3.4, this table implies that (S, L_1^j) is extendable for $4 \leq j \leq 6$, since $\mathrm{Cliff}(S, L_1^j) \geq 3$. In particular $(S, L_1^6) \in \mathcal{K}_{37}$ and is precisely 1-extendable; in fact it is hyperplane section of $\mathbf{P}(1, 1, 1, 3)$. Lemma 3.7 also tells us that the surfaces (S, L_1^4) and (S, L_1^5) are 6-extendable and 3-extendable, respectively, in agreement with the results in [5], (4.8). There, also the situation in case $j = 3$ (to which Theorem 3.4 does not apply), is completely described.

Proof The elementary computation of $\mathrm{Cliff}(L_1^j)$ is left to the reader. The surface S is a double plane $\pi : S \longrightarrow \mathbf{P}^2$ branched along a sextic Γ and $L_1 = \pi^*\mathcal{O}_{\mathbf{P}^2}(1)$. Denote by R the ramification curve of π. We have $\mathcal{O}_S(R) = L_1^3$. The cotangent sequence of π is

$$0 \to \pi^*\Omega^1_{\mathbf{P}^2} \longrightarrow \Omega^1_S \longrightarrow \Omega^1_{S/\mathbf{P}^2} \to 0$$

where $\Omega^1_{S/\mathbf{P}^2} = \mathcal{O}_R(-R) = L_1^{-3} \otimes \mathcal{O}_R = \omega_R^{-1}$. Therefore, for every j we have the following diagram, where the vertical sequence is the twisted Euler sequence restricted to S:

$$\begin{tikzcd}
 & 0 \arrow[d] & & \\
0 \arrow[r] & \pi^*\Omega^1_{\mathbf{P}^2} \otimes L_1^j \arrow[r] \arrow[d] & \Omega^1_S \otimes L_1^j \arrow[r] & L_1^{j-3} \otimes \mathcal{O}_R \to 0 \\
 & L_1^{j-1} \otimes H^0(S, L_1) \arrow[d] & & \\
 & L_1^j \arrow[d] & & \\
 & 0 & &
\end{tikzcd}$$

For $j \geq 7$, this diagram gives $H^1(S, \Omega^1_S \otimes L_1^j) = 0$. If $j = 1$ we get the following exact sequence:

$$\begin{tikzcd}
0 \to H^1(S, \Omega^1_S \otimes L_1) \arrow[r] & H^1(R, L_1^{-2} \otimes \mathcal{O}_R) \arrow[r] & H^2(S, \pi^*\Omega^1_{\mathbf{P}^2} \otimes L_1) \arrow[r] \arrow[d, equal] & 0 \\
 & & H^2(S, \mathcal{O}_S) \otimes H^0(S, L_1) &
\end{tikzcd}$$

which gives $h^1(S, \Omega^1_S \otimes L_1) = h^1(R, L_1^{-2} \otimes \mathcal{O}_R) - 3 = 18$.

If $j = 2$ we have

$$h^1(S, \pi^*\Omega^1_{\mathbf{P}^2} \otimes L_1^2) = \operatorname{corank}\big[\mathrm{Sym}^2 H^0(S, L_1) \to H^0(S, L_1^2)\big] = 0$$

and the following exact sequence:

$$0 \to H^1(S, \pi^*\Omega^1_{\mathbf{P}^2} \otimes L_1^2) \longrightarrow H^1(S, \Omega^1_S \otimes L_1^2) \longrightarrow H^1(R, L_1^{-1} \otimes \mathcal{O}_R) \to 0$$

which gives $h^1(S, \Omega^1_S \otimes L_1^2) = 0 + h^1(R, L_1^{-1} \otimes \mathcal{O}_R) = 15$.

If $3 \leq j \leq 6$ then

$$h^1(S, \pi^*\Omega^1_{\mathbf{P}^2} \otimes L_1^j) = 0$$

thus $H^1(S, \Omega^1_S \otimes L_1^j) \cong H^1(R, L_1^{j-3} \otimes \mathcal{O}_R)$, and the conclusion is clear. $\square$

4 Extendability and Graded Deformations of Cones

Consider a projective scheme $X \subset \mathbf{P}^r$ and let $A = R/I_X$ be its homogeneous coordinate ring, where $R = \mathbf{C}[X_0, \dots, X_r]$ and I_X is the saturated homogeneous ideal of X in $\mathbf{P}^r$. The *affine cone* over X is

$$CX := \mathrm{Spec}(A) \subset \mathbf{A}^{r+1}$$

and the *projective cone* over X is

$$\overline{CX} := \mathrm{Proj}(A[t]) \subset \mathbf{P}^{r+1}.$$

Recall the following standard definitions. The scheme X is *projectively normal*, resp. *arithmetically Cohen–Macaulay*, resp. *arithmetically Gorenstein* if the local ring of CX at the vertex is integrally closed, resp. Cohen–Macaulay, resp. Gorenstein. Also recall that if X is normal and arithmetically Cohen–Macaulay then it is projectively normal.

The deformation theory of CX is controlled by the cotangent modules T^1_{CX} and T^2_{CX}, which are graded because of the $\mathbf{C}^*$-action on A. We will only need the explicit description of the first one.

Proposition 4.1 *Let $X \subset \mathbf{P}^r$ be a non-degenerate scheme of pure dimension $d \geq 1$. Consider the following conditions:*

(a) *X is arithmetically Cohen–Macaulay (aCM for short).*
(b) *X is projectively normal.*

If either (a) or (b) holds then we have an exact sequence of graded modules:

$$\bigoplus_{k \in \mathbf{Z}} H^0(X, T_{\mathbf{P}^r}|_X(k)) \longrightarrow \bigoplus_{k \in \mathbf{Z}} H^0(X, N_{X/\mathbf{P}^r}(k)) \longrightarrow T^1_{CX} \to 0. \tag{1}$$

Proof Let $v \in CX$ be the vertex, $W = CX \setminus \{v\}$ and let $\pi : W \longrightarrow X$ be the projection. By definition we have an exact sequence:

$$H^0(CX, T_{\mathbf{A}^{r+1}}|_{CX}) \longrightarrow H^0(CX, N_{CX/\mathbf{A}^{r+1}}) \longrightarrow T^1_{CX} \to 0. \tag{2}$$

We assume that (a) or (b) holds. Then CX verifies Serre's condition S_2 at the vertex. The two sheaves F respectively involved in the two first terms of (2) are reflexive, each being the dual of a coherent sheaf, hence they have depth ≥ 2 at v as well by [16, Proposition 1.3] (for the implication we use, it is enough that the X from the notation of ibid. be S_2, as the proof given there shows). Therefore

$$H^0(CX, F) \cong H^0(W, F|_W).$$

Thus (2) induces an exact sequence

$$H^0(W,\, T_{\mathbf{A}^{r+1}}|_W) \longrightarrow H^0(W,\, N_{W/\mathbf{A}^{r+1}}) \longrightarrow T^1_{CX} \to 0.$$

As in the proof of [34, Lemma 1], one sees that

$$H^0(W,\, T_{\mathbf{A}^{r+1}}|_W) = \bigoplus_{k\in\mathbf{Z}} H^0(X, \mathcal{O}_X(k+1))^{r+1}$$

and

$$H^0(W,\, N_{W/\mathbf{A}^{r+1}}) = \bigoplus_{k\in\mathbf{Z}} H^0(X,\, N_{X/\mathbf{P}^r}(k)).$$

Then we have a commutative diagram

$$\begin{tikzcd}
H^0(W,\, T_{\mathbf{A}^{r+1}}|_W) \arrow[rr] \arrow[d, equal] & & H^0(W,\, N_{W/\mathbf{A}^{r+1}}) \arrow[d, equal] \\
\bigoplus_{k\in\mathbf{Z}} H^0(X, \mathcal{O}_X(k+1))^{r+1} \arrow[r, "\phi"] & \bigoplus_{k\in\mathbf{Z}} H^0(X,\, T_{\mathbf{P}^r}|_X(k)) \arrow[r] & \bigoplus_{k\in\mathbf{Z}} H^0(X,\, N_{X/\mathbf{P}^r}(k))
\end{tikzcd}$$

(in which the map ϕ comes from the Euler exact sequence), and (1) is proved. □

Considering the degree -1 pieces of the exact sequences (1), we get:

Corollary 4.2 *In the notation of Proposition 4.1, if (a) or (b) holds, then there is an exact sequence:*

$$H^0(X,\, T_{\mathbf{P}^r}|_X(-1)) \longrightarrow H^0(X, N_{X/\mathbf{P}^r}(-1)) \longrightarrow T^1_{CX,-1} \to 0. \tag{3}$$

The following corollary will be important in our applications. It applies in the cases under consideration in this article, because embedded $K3$ surfaces, Gorenstein weighted projective spaces of dimension 3 in their anticanonical embedding, and their linear curve sections are arithmetically Gorenstein.

Corollary 4.3 *Let $X \subset \mathbf{P}^r$ be either aCM of pure dimension ≥ 2 or arithmetically Gorenstein of pure dimension 1 and positive arithmetic genus. Then $\alpha(X) \leq$* $\dim(T^1_{CX,-1})$.

(See Theorem 3.3 for the definition of α).

Proof The twisted Euler exact sequence

$$0 \to \mathcal{O}_X(-1) \to H^0(X, \mathcal{O}(1)) \otimes \mathcal{O}_X \to T_{\mathbf{P}^r}|_X(-1) \to 0$$

induces the exact sequence

$$0 \to H^0(X, \mathcal{O}(1)) \to H^0(T_{\mathbf{P}^r}|_X(-1)) \\ \to H^1(\mathcal{O}_X(-1)) \to H^0(\mathcal{O}_X(1)) \otimes H^1(\mathcal{O}_X). \quad (4)$$

When $\dim(X) > 1$, since X is arithmetically Cohen–Macaulay we have that $H^1(\mathcal{O}_X(-1)) = 0$, and therefore by (4),

$$H^0(T_{\mathbf{P}^r}|_X(-1)) \cong H^0(X, \mathcal{O}(1)).$$

Then (3) gives a presentation

$$H^0(X, \mathcal{O}(1)) \longrightarrow H^0(X, N_{X/\mathbf{P}^r}(-1)) \longrightarrow T^1_{CX,-1} \to 0,$$

from which the desired inequality follows at once.

When $\dim(X) = 1$, (4) gives the following exact sequence of vector spaces,

$$0 \to H^0(X, \mathcal{O}(1)) \to H^0(T_{\mathbf{P}^r}|_X(-1)) \to \ker({}^{\mathsf{T}}\mu) \to 0,$$

where μ is the multiplication map

$$\mu : H^0(\mathcal{O}_X(1)) \otimes H^0(\omega_X) \to H^0(\omega_X(1)).$$

If X is arithmetically Gorenstein of positive genus we have $\omega_X = \mathcal{O}_X(\nu)$ for some $\nu \geq 0$, hence μ is the multiplication map

$$H^0(\mathcal{O}_X(1)) \otimes H^0(\mathcal{O}_X(\nu)) \to H^0(\mathcal{O}_X(\nu + 1))$$

which is surjective, and we conclude as before. □

If X is smooth, in most cases the leftmost map of (3) is injective, so that in fact $\alpha(X) = \dim(T^1_{CX,-1})$. The same holds in the cases under investigation in this article:

Corollary 4.4 *Let $X \subset \mathbf{P}^r$ be either a $K3$ surface with at most canonical singularities, or a Gorenstein weighted projective 3-space in its canonical embedding. Then $\alpha(X) = \dim(T^1_{CX,-1})$.*

Proof The kernel of the leftmost map of (3) is contained in $H^0(X, T_X(-1))$. If X is a $K3$ surface then $H^0(X, T_X) = 0$, hence $H^0(X, T_X(-1)) = 0$ as well. If X is a Gorenstein weighted projective space, then $H^0(X, T_X(-1)) = H^3(X, \Omega^1_X)^\vee$ by Serre duality, hence it is zero in this case as well. It follows that the leftmost map of (3) is injective.

On the other hand, it follows from the Euler exact sequence and the vanishing of $H^1(X, \mathcal{O}_X(-1))$ that $H^0(X, T_{\mathbf{P}^r}|_X(-1)) \cong H^0(X, \mathcal{O}_X(1))^\vee$ which has dimension $r + 1$, hence the result. □

The following will also be of fundamental importance for us.

Proposition 4.5 *Let $X \subset \mathbf{P}^g$ be a $K3$ surface with at worst canonical singularities. Then $T^1_{CX,-1} = \mathrm{Ext}^1_X(\Omega^1_X, \mathcal{O}_X(-1))$.*

Proof Taking $\mathrm{Hom}(\ .\ , \mathcal{O}_X(-1))$ of the conormal exact sequence of X in $\mathbf{P}^g$, and using the fact that the conormal sheaf of X in $\mathbf{P}^g$ and $\Omega^1_{\mathbf{P}^g}\big|_X$ are locally free, we obtain the exact sequence

$$H^0(X, T_{\mathbf{P}^g}|_X(-1)) \longrightarrow H^0(X, N_{X/\mathbf{P}^g}(-1)) \longrightarrow \mathrm{Ext}^1_X(\Omega^1_X, \mathcal{O}_X(-1)) \longrightarrow H^1(X, T_{\mathbf{P}^g}|_X(-1)). \qquad (5)$$

From the restricted and twisted Euler sequence

$$0 \longrightarrow \mathcal{O}_X(-1) \longrightarrow H^0(X, \mathcal{O}_X(1))^\vee \otimes \mathcal{O}_X \longrightarrow T_{\mathbf{P}^g}|_X(-1) \longrightarrow 0$$

we deduce that $H^1(X, T_{\mathbf{P}^g}|_X(-1)) = 0$. Therefore comparing the two exact sequences (5) and (3) gives the assertion. □

4.6 Consider now an extension $\widetilde{X}$ of a projectively normal $X \subset \mathbf{P}^r$. In such a situation, we let $e_{X/\tilde{X}} \in \mathrm{Ext}^1(\Omega^1_X, \mathcal{O}_X(-1))$ be the class of the conormal exact sequence

$$0 \longrightarrow \mathcal{O}_X(-1) \longrightarrow \Omega^1_{\tilde{X}}|_X \longrightarrow \Omega^1_X \longrightarrow 0.$$

If the extension $\tilde{X}$ is non-trivial, i.e., it is not a cone over X, then we can also associate to it a family of deformations of $\overline{CX}$, the projective cone over X, as follows. Let $X = \widetilde{X} \cap H$, where $H \cong \mathbf{P}^r \subset \mathbf{P}^{r+1}$ is a hyperplane. Consider in $\mathbf{P}^{r+2}$ the projective cone $\overline{C\widetilde{X}}$ and the pencil of hyperplanes H_t with center H. Let H_o be the hyperplane containing the vertex v of $\overline{C\widetilde{X}}$. Then $H_o \cap \overline{C\widetilde{X}} = \overline{CX}$, while $H_t \cap \overline{C\widetilde{X}} \cong \widetilde{X}$ for all $t \neq o$. After blowing up X we obtain a family

$$f : \mathrm{Bl}_X(\overline{C\widetilde{X}}) \longrightarrow \mathbf{P}^1$$

which is flat because $\widetilde{X}$ is projectively normal, with $f^{-1}(t) = H_t \cap \overline{C\widetilde{X}}$. By restriction we get a deformation of the affine cone CX. If $\widetilde{X}$ is smooth then this deformation is a smoothing of $\overline{CX} = f^{-1}(o)$. This is a classical construction called *sweeping out the cone* (see, e.g., [28, (7.6) (iii)]). Algebraically, the above construction has the following description. Let $\widetilde{X} = \mathrm{Proj}(\mathcal{A})$, where $\mathcal{A} = \mathbf{C}[X_0, \ldots, X_r, t]/J$. Then

$$A = \mathcal{A}/t\mathcal{A} = \mathbf{C}[X_0, \ldots, X_r]/I$$

where $I = J/tJ$. Consider $C\widetilde{X} = \mathrm{Spec}(\mathcal{A}) \subset \mathbf{A}^{r+2}$. The pencil of parallel hyperplanes $V(t) \subset \mathbf{A}^{r+2}$ has as projective closure the pencil $\{H_t\}$ considered before. Therefore the morphism

$$\phi : \mathrm{Spec}(\mathcal{A}) \longrightarrow \mathrm{Spec}(\mathbf{C}[t])$$

is the corresponding family of deformations of CX. It is clear that if $e_{X/\widetilde{X}} \in T^1_{CX,-1}$ (e.g., X is nonsingular or is a singular $K3$ surface) then the first-order deformation of X associated to ϕ is $e_{X/\widetilde{X}}$. Note that, by construction, $e_{X/\widetilde{X}}$ *is unobstructed both as a first-order deformation of* CX *and of* $\overline{CX}$.

The upshot of the above construction is that the datum of an extension $\tilde{X}$ of X gives a deformation of the cone over X. In fact the two objects correspond to the same ring $\mathcal{A}$: the former is $\mathrm{Proj}(\mathcal{A})$ and the latter is $\mathrm{Spec}(\mathcal{A})$. We shall now state a result of Wahl which will be crucial in what follows. It may be considered as a reverse sweeping out the cone, in that it produces an extension of X from a first-order deformation of the cone over X. We first need the following standard definition.

Definition 4.7 Let $X \subset \mathbf{P}^r$ be a non-degenerate projectively normal scheme of pure dimension ≥ 1. we say that X *has the property* N_2 or *satisfies* N_2 if its homogeneous coordinate ring $A = R/I_X$ has a minimal graded presentation over $R := \mathbf{C}[X_0, \dots, X_r]$ of the form:

$$R(-3)^a \xrightarrow{\psi} R(-2)^b \xrightarrow{\phi} R \longrightarrow A \to 0. \qquad (6)$$

Theorem 4.8 ([35], Proof of Theorem 7.1 and Remark 7.2) *Let* $X \subset \mathbf{P}^r$ *be a non-degenerate projectively normal scheme of pure dimension* ≥ 1 *and let* A *be its homogeneous coordinate ring. Consider the following two conditions:*

(a) X *has the property* N_2*;*
(b) $T^2_{A,k} = 0$ *for all* $k \leq -2$.

If (a) holds then any first-order deformation of CX *of degree* -1 *lifts to at most one graded deformation* $\mathcal{A}$ *over* $\mathbf{C}[t]$, *with* $\deg(t) = 1$. *Moreover* $Y := \mathrm{Proj}(\mathcal{A}) \subset \mathbf{P}^{r+1} = \mathrm{Proj}[t, X_0, \dots, X_r]$ *is an extension of* $X := \mathrm{Proj}(A) = \mathrm{Proj}(\mathcal{A}) \cap \{t = 0\} \subset \mathbf{P}^r$ *which is unique up to projective automorphisms of* $\mathbf{P}^{r+1}$ *fixing every point of* $\mathbf{P}^r = \{t = 0\}$.

If both (a) and (b) hold then every first-order deformation of CX *of degree* -1 *lifts to a graded deformation* $\mathcal{A}$ *as above.*

It is one of the main results of [2] that condition (b) above holds when X is a canonical curve. We shall use this and Theorem 3.4 to prove that the same holds when X is a Gorenstein weighted projective space of dimension 3, see Corollary 6.4.

For more details on the unicity statement, we refer to [8, Remark 4.8]. Note that assumption (a) implies that $H^0(N_{X/\mathbf{P}^r}(-k)) = 0$ for all $k \geq 2$, as we have already observed.

5 The Deformation Argument

We now come to our main technical result, and its application to deformations of weighted projective spaces.

5.1 Let (S, L) be a polarized $K3$ surface with canonical singularities and $g = h^0(L) - 1$. A *smoothing* of (S, L) is a pair $\big(p : \mathcal{S} \to (\Delta, o), \mathcal{L}\big)$, where p is a smoothing of S over an affine nonsingular pointed curve (Δ, o) and $\mathcal{L}$ extends L, i.e., $L = \mathcal{L}(o) := \mathcal{L}|_{p^{-1}(o)}$. There is a flat family of surfaces in $\mathbf{P}^g$ associated to such a smoothing:

$$\begin{tikzcd} \mathcal{S} \arrow[r, hook, "j"] \arrow[dr, "p"'] & \mathbf{P}^g \times \Delta \arrow[d, "pr_2"] \\ & \Delta \end{tikzcd} \tag{7}$$

where j is defined by the sections of $\mathcal{L}$.

We shall use the following notation: $\Delta^\circ := \Delta \setminus \{o\}$; $\mathcal{S}^\circ = \mathcal{S} \setminus p^{-1}(o)$; $p^\circ = p|_{\mathcal{S}^\circ}$; $\mathcal{L}^\circ = \mathcal{L}|_{\mathcal{S}^\circ}$.

A *relative extension* of $\mathcal{S} \subset \mathbf{P}^g \times \Delta$ consists of an $\mathcal{X} \subset \mathbf{P}^{g+1} \times \Delta$, flat over Δ, together with a relative hyperplane $\mathcal{H} \cong \mathbf{P}^g \times \Delta \subset \mathbf{P}^{g+1} \times \Delta$ such that $\mathcal{X} \cap \mathcal{H} = \mathcal{S}$ and $\mathcal{X}(t)$ is not a cone over $\mathcal{S}(t)$ for all $t \in \Delta$. Similarly, one defines relative extensions of $\mathcal{S}^\circ \subset \mathbf{P}^g \times \Delta^\circ$.

Theorem 5.2 *Let $S_0 \subset \mathbf{P}^g$ be a $K3$ surface, possibly with canonical singularities, and $V_0 \subset \mathbf{P}^{g+1}$ be an extension of S_0. Let $p : \mathcal{S} \to \Delta$ be a smoothing of S_0 in $\mathbf{P}^g$ as above, and assume that the following conditions hold:*

(a) $g \geq 11$, *and for all* $t \in \Delta^\circ$ *we have* $\mathrm{Cliff}(S_t) > 2$;
(b) S_0 *has the* N_2 *property;*
(c) $t \in \Delta \mapsto \alpha(S_t)$ *is constant.*

Then there exists a deformation of V_0 in $\mathbf{P}^{g+1}$ which is a relative extension of $\mathcal{S} \subset \mathbf{P}^g \times \Delta$.

Proof We have a base change map [20]:

$$\tau(o) : \mathcal{E}xt^1_p(\Omega^1_{\mathcal{S}/\Delta}, \mathcal{L}^{-1})_o \otimes k(o) \longrightarrow \mathrm{Ext}^1_{S_0}\big(\Omega^1_{S_0}, L_0^{-1}\big)$$

with $L_0 = \mathcal{L}(o) = \mathcal{O}_{S_0}(1)$ (note that $\Omega^1_{\mathcal{S}/\Delta}$ is Δ-flat because p is flat and has reduced fibers). By our assumption (c) and the results in Sect. 4, the function

$$t \in \Delta \longmapsto \dim\big[\mathrm{Ext}^1_{\mathcal{S}(t)}(\Omega^1_{\mathcal{S}(t)}, \mathcal{L}(t)^{-1})\big]$$

is constant, hence $\mathcal{E}xt^1_p(\Omega^1_{\mathcal{S}/\Delta}, \mathcal{L}^{-1})$ is locally free and $\tau(o)$ is an isomorphism. It follows that there exists a section $E \in \mathcal{E}xt^1_p(\Omega^1_{\mathcal{S}/\Delta}, \mathcal{L}^{-1})$ such that $\tau(o)(E) = e_{S_0/V_0}$ (see 4.6 for the definition of e_{S_0/V_0}).

For all $t \in \Delta^\circ$, the smooth $K3$ surface $\mathcal{S}(t) \subset \mathbf{P}^g$ satisfies the assumptions of Theorem 3.4 by (a), and therefore there exists a unique extension $\mathcal{S}(t) \subset \mathcal{X}(t) \subset \mathbf{P}^{g+1}$ such that $e_{\mathcal{S}(t)/\mathcal{X}(t)} = E(t)$. We then consider $\mathcal{X}^\circ := \bigcup_{t\in\Delta^\circ} \mathcal{X}(t)$: it is a relative extension of $\mathcal{S}^\circ$, and its Zariski closure $\mathcal{X} = \overline{\mathcal{X}^\circ} \subset \mathbf{P}^{g+1} \times \Delta$ is a relative extension of $\mathcal{S}$.

Let $X_0 = \mathcal{X}(o)$. One has $e_{S_0/X_0} = E(o) = e_{S_0/V_0}$, so assumption (b) and Theorem 4.8 imply that $X_0 = V_0$, which ends the proof. $\square$

We now set up the situation in which we will apply the above Theorem 5.2. The notation is the same as in 2.3.

5.3 Consider $\mathbf{P} = \mathbf{P}(a_0, a_1, a_2, a_3)$ a weighted projective space with Gorenstein canonical singularities, and (S, L) a general anticanonical divisor of $\mathbf{P}$, so $L = -K_{\mathbf{P}}|_S$. Let i_S be the divisibility of L in $\mathrm{Pic}(S)$, and L_1 be the primitive line bundle on S such that $L = i_S L_1$. Thus $(S, L_1) \in \mathcal{K}_{g_1}^{\mathrm{prim}}$, where $g_1 = h^0(L_1) - 1$.

We may then consider a deformation $\big(p : \mathcal{S} \to (\Delta, o), \mathcal{L}_1\big)$ of (S, L_1) to general primitive polarized smooth $K3$ surfaces of genus g_1. To such a smoothing, there is associated a flat family of surfaces in $\mathbf{P}^{g_1}$ as in (7) and also an analogous family in $\mathbf{P}^g$ defined by the sections of $\mathcal{L}_1^{i_S}$:

$$\begin{tikzcd} \mathcal{S} \arrow[r, hook, "j_{i_S}"] \arrow[dr, "p"'] & \mathbf{P}^{g_{\mathbf{P}}} \times \Delta \arrow[d, "pr_2"] \\ & \Delta \end{tikzcd} \tag{8}$$

where $g = h^0(S, L) - 1$ (j_{i_S} is the i_S-uple Veronese re-embedding of $\mathcal{S}$).

We shall apply Theorem 5.2 to $S_0 = S \subset \mathbf{P}^g$, and $V_0 = \mathbf{P} \subset \mathbf{P}^{g+1}$ in its anticanonical embedding. In this case, assumption (a) is always satisfied, as a direct computation shows. Assumption (b) is always satisfied as well, because $\mathbf{P}$ has the property N_2 by Proposition 6.1 below. Assumption (c), however, does not hold in all cases: we compute $\alpha(S, L)$ in Proposition 6.2 below, and compare it with $\alpha(S', L')$ for a general $(S', L') \in \mathcal{K}_g^{i_S}$, in other words with $\alpha(S', (L_1')^{i_S})$ for a general $(S', L_1') \in \mathcal{K}_{g_1}^{\mathrm{prim}}$.

When $\alpha(S, L) = \alpha(S', L')$ holds, we conclude that $\mathbf{P}$ deforms to a threefold extension of a general $K3$ surface $(S', L') \in \mathcal{K}_g^{i_S}$. This happens exactly for cases $\#i$, $i \in \{1, \dots, 7, 9\}$, see Table 3. We refer to Sect. 7 for a more precise description of the output in each of these cases.

6 Explicit Computations on WPS

The main object of this section is to analyze which Gorenstein projective spaces enjoy the required properties for Theorem 5.2 to apply, as described in 5.3 above. We carry this out by explicit computations using the software Macaulay2 [23]. As a

Table 2 First Betti numbers of Gorenstein weighted projective spaces

#	Weights	g_1	i_S	g	β_1	β_2
1	(1, 1, 1, 3)	2	6	37	595	13056
2	(1, 1, 4, 6)	2	6	37	595	13056
3	(1, 2, 2, 5)	2	5	26	276	4025
4	(1, 1, 1, 1)	3	4	33	465	8960
5	(1, 1, 2, 4)	3	4	33	465	8960
6	(1, 3, 4, 4)	3	3	19	136	1344
7	(1, 1, 2, 2)	4	3	28	325	5175
8	(1, 2, 6, 9)	4	3	28	325	5175
9	(2, 3, 3, 4)	4	2	13	55	320
10	(1, 4, 5, 10)	6	2	21	171	1920
11	(1, 2, 3, 6)	7	2	25	253	3520
12	(1, 3, 8, 12)	7	2	25	253	3520
13	(2, 3, 10, 15)	16	1	16	91	715
14	(1, 6, 14, 21)	22	1	22	190	2261

bonus, we obtain the number of times each Gorenstein weighted projective 3-space is extendable.

Proposition 6.1 *Let* $\mathbf{P} \subset \mathbf{P}^{g+1}$ *be a* 3*-dimensional Gorenstein weighted projective space, considered in its anticanonical embedding. Then* $\mathbf{P}$ *is projectively normal, and its homogeneous coordinate ring* $A = R/I_{\mathbf{P}}$ *has a minimal resolution of the form*

$$\cdots \longrightarrow R(-3)^{\beta_2} \longrightarrow R(-2)^{\beta_1} \longrightarrow R \longrightarrow A \longrightarrow 0$$

with β_1, β_2 *as indicated in Table 2. In particular* $\mathbf{P}$ *has the* N_2 *property.*

Of course $\beta_1 = \binom{g-2}{2}$, since curve linear sections of $\mathbf{P}$ are canonical curves of genus g.

Proof The projective normality follows from the fact that $\mathbf{P}$ has canonical curves as linear sections, see [8, Theorem 5.1]. For property N_2, we explicitly compute the ideal of $\mathbf{P}$ in $\mathbf{P}^{g+1}$ using Macaulay2, then compute the first syzygies of this ideal, and eventually check that they are of the asserted shape. This computation goes as follows.

Let $\mathbf{P} = \mathbf{P}(a_0, a_1, a_2, a_3)$ endowed with weighted homogeneous coordinates $\mathbf{x} = (x_0, x_1, x_2, x_3)$. First, one writes down the list $(M_0, \ldots, M_{g+1})$ of all monomials in $\mathbf{x}$ of weighted degree $s = a_0 + a_1 + a_2 + a_3$, which form a basis of $H^0(\mathbf{P}, -K_{\mathbf{P}})$. Then the ideal of the graph $\Gamma \subset \mathbf{P} \times \mathbf{P}^{g+1}$ of the embedding $\mathbf{P} \subset \mathbf{P}^{g+1}$ is

$$I_\Gamma = \big(y_i - M_i(\mathbf{x}), \quad i = 0, \ldots, g+1\big),$$

with $(y_0, \dots, y_{g+1})$ homogeneous coordinates on $\mathbf{P}^{g+1}$. One obtains the ideal $I_{\mathbf{P}}$ of $\mathbf{P} \subset \mathbf{P}^{g+1}$ by eliminating $\mathbf{x}$ from I_Γ, which may be performed efficiently using a Gröbner basis algorithm. Eventually, there is a Macaulay2 function which computes step by step the syzygies of this ideal. We provide the explicit Macaulay2 commands implementing this procedure at the end of the arXiv version of this article. □

In principle, one may use any basis of $H^0(\mathbf{P}, -K_{\mathbf{P}})$ to compute the ideal, but the computations turn out to work faster with a monomial basis. In fact doing so one takes advantage of $\mathbf{P}$ being a toric variety. There is also a Macaulay2 function computing the whole resolution of a graded ideal, but we have not been able to run these computations successfully for $I_{\mathbf{P}}$ (apart for #13) because the complexity was too large.

In principle, it is possible to compute all Betti numbers of any lattice ideal I_Λ as the dimensions of the reduced homology groups of a simplicial complex explicitly construct from the lattice Λ, see, e.g., [25, Theorem 9.2] or [26, Chap. 5]. It seems to us however that this leaves non-trivial computations to be performed, which we haven't tried to carry out.

Proposition 6.2 *Let $\mathbf{P} \subset \mathbf{P}^{g+1}$ be a 3-dimensional Gorenstein weighted projective space in its anticanonical embedding, and (S, L) be a general hyperplane section of $\mathbf{P}$. We write i_S for the divisibility of $L = -K_{\mathbf{P}}|_S$ in $\mathrm{Pic}(S)$. Let (S', L') be a general member of $\mathcal{K}_g^{i_S}$. Then the values of $\alpha(S)$ and $\alpha(S')$ are as indicated in Table 3. Moreover, $\alpha(\mathbf{P}) = \alpha(S) - 1$, and $\alpha(C) = \alpha(S) + 1$ for C a general curve linear section of $\mathbf{P}$.*

Table 3 Dimension of the weight -1 piece of T^1

#	Weights	g_1	i_S	$\alpha(S)$	$\alpha(S')$	3-fold
1	(1, 1, 1, 3)	2	6	1	1	$\mathbf{P}(1^3, 3)$
2	(1, 1, 4, 6)	2	6	1	1	$\mathbf{P}(1^3, 3)$
3	(1, 2, 2, 5)	2	5	3	3	$H_6 \subset \mathbf{P}(1^3, 3, 5)$
4	(1, 1, 1, 1)	3	4	1	1	$\mathbf{P}^3$
5	(1, 1, 2, 4)	3	4	1	1	$\mathbf{P}^3$
6	(1, 3, 4, 4)	3	3	4	4	$H_4 \subset \mathbf{P}(1^4, 3)$
7	(1, 1, 2, 2)	4	3	1	1	$\mathbf{Q}$
8	(1, 2, 6, 9)	4	3	2	1	$\mathbf{Q}$
9	(2, 3, 3, 4)	4	2	6	6	$H_3 \subset \mathbf{P}^4$
10	(1, 4, 5, 10)	6	2	3	1	V_5
11	(1, 2, 3, 6)	7	2	1	0	Does not exist
12	(1, 3, 8, 12)	7	2	2	0	Does not exist
13	(2, 3, 10, 15)	16	1	3	0	Does not exist
14	(1, 6, 14, 21)	22	1	2	0	Does not exist

In the table, we also indicate the general 3-fold extension of S', with the following notation: $\mathbf{Q}$ denotes the smooth 3-dimensional quadric in $\mathbf{P}^4$; H_d denotes a general degree d hypersurface in the specified projective space; V_5 denotes the degree 5 Del Pezzo threefold, i.e., the section of the Grassmannian $\mathbf{G}(2, 5)$ by a general $\mathbf{P}^6$ in the Plücker embedding. We refer to [5, 6] for these matters.

We will need the following lemma for the proof, which is a generalization of a well-known fact when all involved varieties are smooth.

Lemma 6.3 *Let $\mathbf{P} \subset \mathbf{P}^{g+1}$ be a Gorenstein weighted projective space, S a hyperplane section of $\mathbf{P}$, and C a hyperplane section of S. Then one has*

$$\alpha(C) \geq \alpha(S) + 1 \geq \alpha(\mathbf{P}) + 2.$$

Proof We first compare $\alpha(\mathbf{P})$ and $\alpha(S)$. Since S is a hyperplane section of $\mathbf{P}$, one has $N_{\mathbf{P}/\mathbf{P}^{g+1}}\big|_S = N_{S/\mathbf{P}^g}$. We thus have the following exact sequence, where the rightmost map is the restriction map:

$$0 \to N_{\mathbf{P}/\mathbf{P}^{g+1}}(-2) \longrightarrow N_{\mathbf{P}/\mathbf{P}^{g+1}}(-1) \longrightarrow N_{S/\mathbf{P}^g}(-1) \to 0 \tag{9}$$

with $\mathcal{O}(1)$ the line bundle induced by the embedding in $\mathbf{P}^{g+1}$. By Proposition 6.1, $\mathbf{P} \subset \mathbf{P}^{g+1}$ has the property N_2, hence $H^0(N_{\mathbf{P}/\mathbf{P}^{g+1}}(-2)) = 0$ (see [18, Lemma 1.1] and the references therein). So the long exact sequence induced by (9) shows the inequality

$$h^0\big(N_{\mathbf{P}/\mathbf{P}^{g+1}}(-1)\big) \leq h^0\big(N_{S/\mathbf{P}^g}(-1)\big).$$

By the definition of α in Theorem 3.3, this ends the proof. The inequality between $\alpha(S)$ and $\alpha(C)$ is obtained in the same way. □

Proof of Proposition 6.2 We know the ideal $I_{\mathbf{P}}$ of $\mathbf{P} \subset \mathbf{P}^{g+1}$ from the proof of Proposition 6.1. Using the Macaulay2 package "VersalDeformations" [17] one can then compute $\dim(T^1_{C\mathbf{P},-1})$, and this equals $\alpha(\mathbf{P})$ by Corollary 4.4.

Next, we choose two explicit (see below) linear functionals l_0 and l_1 defining hyperplanes H_0 and H_1 in $\mathbf{P}^{g+1}$, and consider $S_0 = \mathbf{P} \cap H_0 \subset \mathbf{P}^g$ and $C_0 = S_0 \cap H_1 \subset \mathbf{P}^{g-1}$. Using the same procedure we compute $\dim(T^1_{CS_0,-1})$ and $\dim(T^1_{CC_0,-1})$, and find out that

$$\dim(T^1_{CS_0,-1}) = \alpha(\mathbf{P}) + 1 \quad \text{and} \quad \dim(T^1_{CC_0,-1}) = \alpha(\mathbf{P}) + 2.$$

Again, the explicit Macaulay2 commands implementing this procedure are given at the end of the arXiv version of this article.

Let S be a general hyperplane section of $\mathbf{P}$. Then on the one hand one has $\alpha(S) \geq \alpha(\mathbf{P}) + 1$ by Lemma 6.3 above, and on the other hand one has $\alpha(S) \leq \alpha(S_0)$ by semicontinuity since $\alpha(S) = h^0(N_{S/\mathbf{P}^g}(-1)) - g - 1$ by definition, and $\alpha(S_0) \leq \dim(T^1_{CS_0,-1})$ by Corollary 4.3. Hence $\alpha(S) = \alpha(\mathbf{P}) + 1$. Similar reasoning yields $\alpha(C) = \alpha(\mathbf{P}) + 2$ for a general curve linear section of $\mathbf{P}$. □

In practice, if one chooses random linear functionals l_0 and l_1 then the complexity of the computation of the weight -1 piece of T^1 is too high and one cannot get an answer. We chose

$$l_0 = x_7 + x_{g+1} \quad \text{and} \quad l_1 = x_3 + x_g,$$

so that the corresponding linear sections are again toric; in particular S_0 is not a $K3$ surface and C_0 is singular. This is the reason why we have to resort to Corollary 4.3 in the proof; note that we cannot guarantee either that $\dim(T^1_{CS,-1})$ is semi-continuous as S approaches S_0. In principle, Macaulay2 can compute $h^0(N_{S_0/\mathbf{P}^g}(-1))$ directly, but in practice, it is not able to return an answer.

Corollary 6.4 *Let $\mathbf{P} \subset \mathbf{P}^{g+1}$ be a 3-dimensional Gorenstein weighted projective space in its anticanonical embedding. Then $\mathbf{P}$ is extendable exactly $\alpha(\mathbf{P})$ times.*

(Recall that $\alpha(\mathbf{P}) = \alpha(S) - 1$ with $\alpha(S)$ as in Table 3).

Proof First note that by Lvovski's Theorem 3.3, applied to a general (smooth) curve linear section of $\mathbf{P}$, $\mathbf{P} \subset \mathbf{P}^{g+1}$ is at most $\alpha(\mathbf{P})$-extendable. To prove the converse, let us consider C a general curve linear section of $\mathbf{P}$. It is a smooth canonical curve of genus $g \geq 11$ and Clifford index strictly larger than 2, hence liable to Theorem 3.4. So there exists a universal extension of C, which is an $(\alpha(C) + 1)$-dimensional variety $X \subset \mathbf{P}^{g-1+\alpha(C)}$, i.e., an $(\alpha(\mathbf{P}) + 3)$-dimensional variety $X \subset \mathbf{P}^{g+1+\alpha(\mathbf{P})}$.

The pencil of hyperplanes in $\mathbf{P}^{g+1}$ containing C cuts out on $\mathbf{P}$ a pencil of $K3$ surfaces, which are not all isomorphic by [27, Proposition 1.7] (as observed in [7], the latter statement in fact applies to all varieties different from cones). By the universality of X, this implies that $\mathbf{P}$ is a linear section of X, hence it is $\alpha(\mathbf{P})$-extendable. $\square$

7 Examples

In this section, we describe explicitly the output of Theorem 5.2 and make additional remarks. We first list the cases to which Theorem 5.2 applies; see also Remarks 7.7 and 7.8 for another point of view on these examples. The notation is that of Table 3.

Example 7.1 (#1 and #2) The general member of $\mathcal{K}^6_{37}$ extends to $\mathbf{P}(1^3, 3)$, hence the application of Theorem 5.2 to #1 is trivial. On the other hand, the application to #2 tells us that there exists a deformation of $\mathbf{P}(1, 1, 4, 6)$ to $\mathbf{P}(1^3, 3)$. Note that these are the only Fano varieties with canonical Gorenstein singularities of genus 37, the maximal possible value, by [30]. $\mathbf{P}(1^3, 3) \subset \mathbf{P}^{38}$ is the 2-Veronese reembedding of the cone in $\mathbf{P}^{10}$ over the Veronese variety $v_2(\mathbf{P}^3)$; in particular it is rigid. Thus the deformation of $\mathbf{P}(1, 1, 4, 6)$ to $\mathbf{P}(1^3, 3)$ exhibits a jump phenomenon.

Example 7.2 (#3) Theorem 5.2 tells us in this case that $\mathbf{P}(1, 2, 2, 5) \subset \mathbf{P}^{27}$ deforms to a general 6-ic hypersurface $H_6 \subset \mathbf{P}(1^3, 3, 5)$ in its anticanonical embedding by $\mathcal{O}(5)$. Such an H_6 is singular, and its singularities may be listed following [13]; in particular as $5 \nmid 6$, H_6 passes through the point $P_4 = (0:0:0:0:1)$ and one finds it has a quotient singularity of type $\frac{1}{5}(1,1,3)$ there.

Corollary 6.4 tells us that $\mathbf{P}(1,2,2,5) \subset \mathbf{P}^{27}$ is 2-extendable, as is $H_6 \subset \mathbf{P}(1^3,3,5)$. The same deformation argument as that given to prove Theorem 5.2 shows that the 2-extension of $\mathbf{P}(1,2,2,5)$ deforms to that of H_6, which is a sextic hypersurface $\tilde{H}_6 \subset \mathbf{P}(1^3,3,5^3)$ embedded by $\mathcal{O}(5)$, that is $-\frac{1}{3}K_{\tilde{H}_6}$, see [5].

Example 7.3 (#4 and #5) #4 is of course the Veronese variety $v_4(\mathbf{P}^3)$, which is rigid and extends the general member of $\mathcal{K}^4_{33}$; Theorem 5.2 is trivial in this case. The application to #5 however tells us that $\mathbf{P}(1,1,2,4) \subset \mathbf{P}^{34}$ smoothes to $v_4(\mathbf{P}^3)$, which may be seen elementarily as follows.

Spelling out a monomial basis of $H^0(\mathbf{P}(1,1,2,4),\mathcal{O}(4))$, one sees that $\mathcal{O}(4)$ induces an embedding of $\mathbf{P}(1,1,2,4)$ as a cone over $\mathbf{P}(1,1,2)$ embedded by its own $\mathcal{O}(4)$, with vertex a point, in $\mathbf{P}^9$. In turn $\mathbf{P}(1,1,2,4) \subset \mathbf{P}^{34}$ is embedded by $\mathcal{O}(8)$, hence it is the 2-Veronese reembedding of the latter cone in $\mathbf{P}^9$. In the same way, the embedding of $\mathbf{P}(1,1,2)$ by $\mathcal{O}(4)$ is the 2-Veronese reembedding of a quadric cone (of rank 3) in $\mathbf{P}^3$.

Thus in the embedding by $\mathcal{O}(4)$, $\mathbf{P}(1,1,2,4)$ is the cone over a section of the Veronese variety $v_2(\mathbf{P}^3)$ by a tangent hyperplane. This deforms to the cone over a section by a transverse hyperplane (this corresponds to smoothing the quadric in $\mathbf{P}^3$ image of $\mathbf{P}(1,1,2)$ by $\mathcal{O}(2)$). In turn, this deforms to the Veronese variety $v_2(\mathbf{P}^3)$ itself by "sweeping out the cone" (see 4.6). In its anticanonical embedding, $\mathbf{P}(1,1,2,4)$ correspondingly deforms to the 2-Veronese re-embedding of $v_2(\mathbf{P}^3)$, which is the Veronese variety $v_4(\mathbf{P}^3)$.

Example 7.4 (#6) This case is similar to #3 and we will be brief. Theorem 5.2 provides a deformation of $\mathbf{P}(1,3,4,4) \subset \mathbf{P}^{20}$ to the anticanonical embedding by $\mathcal{O}(3)$ of a general 4-ic $H_4 \subset \mathbf{P}(1^4,3)$. The latter is singular; in particular as $3 \nmid 4$, H_4 always passes through the coordinate point P_4 and has a quotient singularity of type $\frac{1}{3}(1,1,1)$ there, i.e., , it is locally isomorphic to the cone over the Veronese variety $v_3(\mathbf{P}^2)$.

The argument of Theorem 5.2 shows that the 3-extension of $\mathbf{P}(1,3,4,4) \subset \mathbf{P}^{20}$ deforms to that of H_4, which is a 4-ic hypersurface $\tilde{H}_4 \subset \mathbf{P}(1^4,3^4)$ embedded by $\mathcal{O}(3)$, see [6, Sect. 3].

Example 7.5 (#7) Theorem 5.2 provides a smoothing of $\mathbf{P}(1,1,2,2) \subset \mathbf{P}^{29}$ to a smooth quadric $\mathbf{Q}$, in its canonical embedding. This smoothing may be elementarily found, noting (as we did for case #5) that $\mathcal{O}(2)$ realizes $\mathbf{P}(1,1,2,2)$ as a rank 3 quadric in $\mathbf{P}^4$.

Example 7.6 (#9) This case is similar to #3 and #6, and in fact easier, so we will be very brief. Theorem 5.2 proves that $\mathbf{P}(2,3,3,4)$ deforms to a general cubic hypersurface H_3 in $\mathbf{P}^4$, in particular this is a smoothing. The 5-extension of $\mathbf{P}(2,3,3,4)$ deforms to that of H_3, which is a complete intersection $\tilde{H}_2 \cap \tilde{H}_3$ in $\mathbf{P}(1^5,2^6)$, see [6, Sect. 3].

Remark 7.7 The degeneration of $\mathbf{P}(1^3, 3)$ to $\mathbf{P}(1^2, 4, 6)$ may be seen explicitly as follows; this has been shown to us by the referee, inspired by [15, Sect. 11]. The weighted projective space $\mathbf{P}(1^2, 4, 6)$ is $\mathrm{Proj}(R)$ with R the graded algebra $\mathbf{C}[x, y, z, w]$ in which x, y, z, w have respective weights 1, 1, 4, 6. It is isomorphic to $\mathrm{Proj}(R^{(2)})$ where $R^{(2)}$ is the algebra determined by $\mathcal{O}(2)$ on $\mathbf{P}(1^2, 4, 6)$, i.e., the graded piece $R_n^{(2)}$ is R_{2n} for all $n \in \mathbf{Z}$ by definition.

We claim that $\mathrm{Proj}(R^{(2)})$ is naturally a quadric in $P(1^3, 2, 3)$. To see this we note that $R^{(2)}$ is generated as a $\mathbf{C}$-algebra by

$$x^2, xy, y^2, z, w,$$

which have weights 2, 2, 2, 4, 6 in R, hence 1, 1, 1, 2, 3 in $R^{(2)}$. The only relation between them is $x^2 \cdot y^2 = (xy)^2$, so

$$R^{(2)} \cong \frac{\mathbf{C}[a, b, c, u, v]}{(ac - b^2)},$$

the isomorphism being given by mapping x^2, xy, y^2, z, w to a, b, c, u, v, respectively. Therefore, $\mathrm{Proj}(R^{(2)})$ is the quadric $ac = b^2$ in $\mathrm{Proj}(\mathbf{C}[a, b, c, u, v]) = \mathbf{P}(1^3, 2, 3)$.

The degeneration is then gotten by noting that $\mathbf{P}(1^3, 3)$ is the quadric $u = 0$ in $\mathbf{P}(1^3, 2, 3)$. In the pencil of quadrics

$$ac - b^2 + \lambda u = 0,$$

the member given by $\lambda = 0$ is $\mathbf{P}(1^2, 4, 6)$ and all the others are isomorphic to $\mathbf{P}(1^3, 3)$.

Remark 7.8 In fact, following a suggestion of the referee, we have found that all our examples above may be understood as in the previous Remark 7.7. In general, we shall consider the d-Veronese embedding, i.e., the graded ring $R^{(d)}$, with $d = s/i_S$, where s is the sum of the weights so that $\omega_{\mathbf{P}}^{-1} = \mathcal{O}(s)$, and i_S is as in Table 3. Let us briefly indicate the explicit computations. We take $R = \mathbf{C}[x, y, z, w]$ the graded algebra giving the weighted projective space under consideration, as in Remark 7.7.

Example 7.2: $\mathbf{P}(1, 2, 2, 5)$ is itself a sextic hypersurface in $P(1^3, 3, 5)$. Indeed the algebra $R^{(2)}$ is generated by

$$x^2, y, z, xw, w^2$$

which have weights 2, 2, 2, 6, 10 in R, hence 1, 1, 1, 3, 5 in $R^{(2)}$. The only relation between these generators is $x^2 \cdot w^2 = (xw)^2$, which is in weight 6 in $R^{(2)}$, so that $\mathrm{Proj}(R^{(2)})$ is naturally a sextic hypersurface in $P(1^3, 3, 5)$.

Example 7.3: $\mathbf{P}(1^2, 2, 4)$ is isomorphic to a quadric in $\mathbf{P}(1^4, 2)$, hence it is a degeneration of $\mathbf{P}^3$. Indeed the algebra $R^{(2)}$ is generated by

$$x^2, xy, y^2, z, w$$

which have weights $2, 2, 2, 2, 4$ in R, hence $1, 1, 1, 1, 2$ in $R^{(2)}$. The only relation between these generators is $x^2 \cdot y^2 = (xy)^2$, which has weight 2 in $R^{(2)}$.

Example 7.4*:* $\mathbf{P}(1, 3, 4, 4)$ is itself a quartic in $\mathbf{P}(1^4, 3)$. Indeed the algebra $R^{(4)}$ is generated by

$$x^4, xy, y^4, v, w$$

which have weights $4, 4, 12, 4, 4$ in R, hence $1, 1, 3, 1, 1$ in $R^{(4)}$. The only relation between these generators is $x^4 \cdot y^4 = (xy)^4$, which has weight 4 in $R^{(2)}$.

Example 7.5*:* we have already noted that $\mathcal{O}(2)$ realizes $\mathbf{P}(1, 1, 2, 2)$ as a rank 3 quadric in $\mathbf{P}^4$.

Example 7.6*:* $\mathbf{P}(2, 3, 3, 4)$ is the complete intersection of a quadric and a cubic in $\mathbf{P}(1^5, 2)$, and thus it is a degeneration of a cubic in $\mathbf{P}^4$. The algebra $R^{(6)}$ is generated by

$$x^3, xw, w^3, y^2, yz, z^2$$

which have weights $6, 6, 12, 6, 6, 6$ in R, hence $1, 1, 2, 1, 1, 1$ in $R^{(6)}$. This time there are two relations, namely

$$x^3 \cdot w^3 = (xw)^3 \quad \text{and} \quad y^2 \cdot z^2 = (yz)^2,$$

which have respectively degrees 3 and 2 in $R^{(6)}$.

We conclude with some remarks on the cases in which Theorem 5.2 does not apply because $\alpha(S) > \alpha(S')$; the notation is still that of Table 3.

Proposition 7.9 *Let $\mathbf{P} = \mathbf{P}(a_0, a_1, a_2, a_3)$ be a Gorenstein weighted projective* 3-*space of type #i with $i \in \{8, 10, 11, 12, 13, 14\}$. Then an anticanonical divisor of $\mathbf{P}$ is a double cover of the weighted projective plane $\mathbf{P}(a_0, a_1, a_2)$, branched over a bianticanonical divisor $B \in |-2K_{\mathbf{P}(a_0,a_1,a_2)}|$. In all cases $2K_{\mathbf{P}(a_0,a_1,a_2)}$ is invertible, whereas in all cases but #11 the canonical sheaf $K_{\mathbf{P}(a_0,a_1,a_2)}$ is not invertible.*

Proof The key fact is that in all cases one has $a_3 = a_0 + a_1 + a_2$. Then in homogeneous coordinates $(x_0 : \cdots : x_3)$, a degree $s = a_0 + a_1 + a_2 + a_3$ homogeneous polynomial is of the form

$$x_3^2 + x_3 \cdot f_{a_3}(x_0, x_1, x_2) + f_s(x_0, x_1, x_2),$$

with f_d homogeneous of degree d. We may change weighted homogeneous coordinates by setting $x_3' = x_3 + \frac{1}{2} f_{a_3}$. This gives the polynomial

$$(x_3')^2 + f_s'(x_0, x_1, x_2)$$

where $f_s' = f_s - f_{a_3}^2$, which defines a double cover of $\mathbf{P}(a_0, a_1, a_2)$ as asserted. The last affirmation is readily checked using the statements of Sect. 2. □

Example 7.10 (#11) The anticanonical divisors in $\mathbf{P}(1, 2, 3, 6)$ are double covers of $\mathbf{P}(1, 2, 3)$, which is in fact a Del Pezzo surface of degree 6, with one A_1 and one A_2 double points, which may be constructed by blowing up the plane $\mathbf{P}^2$ along three aligned infinitely near points.

In the embedding by $\mathcal{O}(6) = -\frac{1}{2}K_{\mathbf{P}}$, $\mathbf{P}(1, 2, 3, 6)$ is the cone over this toric Del Pezzo surface, in its anticanonical embedding in $\mathbf{P}^6$. It follows that $\mathbf{P}(1, 2, 3, 6) \subset \mathbf{P}^7$ is a limit of cones over smooth Del Pezzo surfaces of degree 6. Every such cone T is obstructed in $\mathrm{Hilb}^{\mathbf{P}^7}$, being in the closure of two components, one parametrizing embedded $\mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1$'s and the other hyperplane sections of $\mathbf{P}^2 \times \mathbf{P}^2$, as observed in [10, Example 4.5]. Therefore $[\mathbf{P}(1, 2, 3, 6)] \in \mathrm{Hilb}^{\mathbf{P}^7}$ is obstructed as well. In fact the embedded versal deformation of $\mathbf{P}(1, 2, 3, 6) \subset \mathbf{P}^7$ has been explicitly computed, see [1, 4].

On the other hand, $\mathcal{O}(12) = \mathcal{O}(K_{\mathbf{P}})$ embeds $\mathbf{P}(1, 2, 3, 6) \subset \mathbf{P}^{26}$ and $\alpha(\mathbf{P}, K_{\mathbf{P}}) = 0$, while its general anticanonical divisor S satisfies $\alpha(S, -K_{\mathbf{P}}|_S) = 1$ (see Table 3). Therefore, by Theorem 4.8, $\overline{CS}$ has a unique 1-parameter deformation to $\mathbf{P}$.

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