Università degli Studi Roma Tre
Corso di Laurea in Matematica, a.a. 2014/2015
AL440 - Group Theory
Exercises (May 15^{th}, 2015)

Exercise 1. (Schur's theorem) In a group G if $Z(G)$ has finite index, then the derived group G' is finite.

Cheng Teorema E2.2

Exercise 2. If G is a torsion free group and $Aut(G)$ is finite, the G is abelian.

Machì pag 91, Esempio dopo il Teorema di Schur

Exercise 3. Let be given two finite nilpotent groups G_1 and G_2 with nilpotence class respectively c_1 and c_2. Prove that the direct product $G_1 \times G_2$ is a nilpotent group of nilpotence class less or equal than the maximum between c_1 and c_2.

Exercise 4. A finite group is nilpotent if and only if $xy = yx$ for any two elements x, y with orders relatively coprime.

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Exercise 5. A group of order p^n admits a central series of length $n + 1$, that is the quotiens have order p.

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Exercise 6. The dihedral group D_n is nilpotent if and only if $n = 2^k$.

Exercise 7. Let G be a nonabelian group of order pq, where p and q are distinct primes. Show that G is solvable, but not nilpotent.

solvable: there exists either a p-Sylow or a q-Sylow that are normal. Suppose that a p-Sylow P is normal. Then $P \cong \mathbb{Z}_p$ is abelian. So, the chain is given by $\{1\} \triangleleft P \triangleleft G$. G is not nilpotent because none element of a q-Sylow commutes with an element of a p-Sylow, otherwise we would have an element of order pq and G would be cyclic. Thus $Z(G) = \{1\}$. It can also be used exercice 4.

Exercise 8. Let G be a finite solvable group, all of whose Sylow subgroups are abelian. Prove that $Z(G) \cap G' = \{1\}$.

Exercise 9. Let G be a group. Show that G is nilpotent if and only if $G/Z(G)$ is nilpotent.

Exercise 10. Let G be a finite nonabelian nilpotent group of order n. If p is prime and $p \mid n$, show that $p^3 \mid n$.