Università degli Studi Roma Tre
Corso di Laurea in Matematica, a.a. 2014/2015
AL440 - Group Theory
Exercises (March 20^{th}, 2015)

Exercise 1. Let G be a group and $a, b \in G$ conjugate elements. Is it true that the centralizers $C_G(a)$ and $C_G(b)$ are conjugate subgroups of G? If not, give an example to deny.

Exercise 2. Show that D_6 contains two subgroups ismorphic to S_3.

Exercise 3. Show that $Aut(\mathbb{R}, +, \cdot) = \{1\}$ and that $Aut(\mathbb{C}, +, \cdot)$ is infinite.

Exercise 4. Let H be the unique subgroup of order 2 in a given group G. Then, show that $H \subseteq Z(G)$.

More generally, if G is finite and H the unique subgroup of order p, where p is the minimum divisor of $|G|$, then again $H \subseteq Z(G)$.

Exercise 5. (1) Construct an homomorphism $\theta : \mathbb{Z}_2 \to Aut(\mathbb{Z}_5)$.

(2) Consider the semidirect product $G = \mathbb{Z}_2 \rtimes_\theta \mathbb{Z}_5$ and detemine the order, the subgroups and the center of G.

Exercise 6. Let $G_1, \cdots, G_n$ groups and let, for $1 \leq i \leq n$, H_i be a subset of G_i.

(1) Show that $H = H_1 \times \cdots \times H_n$ è un sottogruppo normale di $G = G_1 \times \cdots \times G_n$ se e soltanto se, per ogni i, H_i è un sottogruppo normale di G_i.

(2) Se, per ogni i, H_i è un sottogruppo normale di G_i, provare che $G/H \cong G_1/H_1 \times \cdots \times G_n/H_n$.

Exercise 7. Find the classes of isomorphic groups among the following:

(1) $A := \langle a, b, c \mid b + c = 0 \rangle$;

(2) $B := \langle a, b, c \mid a + b + c = 0 \rangle$;

(3) $C := \langle a, b, c \mid 3a = 0 \rangle$;

(4) $D := \langle a, b, c \mid a = 0, b = 0 \rangle$;

(5) $\mathbb{Z}$;

(6) $\mathbb{Z} \times \mathbb{Z}$;

(7) $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$;

(8) $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$;

(9) $\mathbb{Z}/3\mathbb{Z}$.

Exercise 8. Show that for any $n > 3$ the alternating group A_n contains a subgroup isomorphic to S_{n-2}.

Exercise 9. Let G be a group of order p^3 with p prime. Show that if G is not abelian then $o(Z(G)) = p$.

Exercise 10. Use the argument of Cayley Theorem to identify the following groups as subgroups of suitable symmetric groups:

$$(\mathbb{Z}_4, +); \quad (V, \cdot); \quad (S_3, \circ).$$

Exercise 11. Let H and K be normal subgroups of a group G.

1. Show that $G/(H \cap K)$ is isomorphic to a subgroup of $G/H \times G/K$.
2. Show (using an example) that in general $G/(H \cap K)$ is not isomorphic to $G/H \times G/K$.

Exercise 12. Show that each group of order 255 has a normal subgroup of order 17.